IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 10, Issue 6 Ver. IV (Nov - Dec. 2014), PP 07-17
www.iosrjournals.org

Ev – Dominating Sets and *Ev* – Domination Polynomials of Paths

A.Vijayan[1], J. Sherin Beula[2]
[1]*Assistant Professor, Department of Mathematics, Nesamony Memorial Christian College, Marthandam, Kanyakumari District, Tamil Nadu, South India*
[2]*Assistant Professor, Department of Mathematics, Mar Ephraem College of Engineering and Technology, Marthandam, Kanyakumari District, Tamil Nadu, South India*

Abstract: *Let G = (V, E) be a simple graph. A set S ⊆ E(G) is a edge-vertex dominating set of G (or simply an ev - dominating set), if for all vertices v∈V(G), there exists an edge e∈S such that e dominates v. Let $D_{ev}(P_n, i)$ denote the family of all ev - dominating sets of P_n with cardinality i. Let $d_{ev}(P_n, i) = |D_{ev}(P_n, i)|$. In this paper, we obtain a recursive formula for $d_{ev}(P_n, i)$. Using this recursive formula, we construct the polynomial, $D_{ev}(P_n, x) = \sum_{i=\lceil \frac{n}{4} \rceil}^{n} d_{ev}(P_n, i) x^i$, which we call ev - domination polynomial of P_n and obtain some properties of this polynomial.*
Keywords: *ev - Domination Set, ev - Domination Number, ev - Domination Polynomials*

I. Introduction

Let G = (V, E) be a simple graph of order |V| = n. For any vertex v ∈ V, the open neighbourhood of v is the set $N(v) = \{u \in V / uv \in E\}$ and the closed neighbourhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set S ⊆ V, the open neighbourhood of S is $N(S) = U_{v\in S} N(v)$ and the closed neighbourhood of S is $N[S] = N(S) \cup S$. A set S ⊆ E(G) is a edge-vertex dominating set of G (or simply an ev - dominating set), if for all vertices v∈V(G), there exists an edge e∈S such that e dominates v. The ev - domination number of a graph G is defined as the minimum size of an ev - dominating set of edges in G and it is denoted as $\gamma_{ev}(G)$. A simple path is a path in which all its internal vertices have degree two, and the end vertices have degree one and is denoted by P_n.

1.1 Definition

Let $D_{ev}(G,i)$ be the family of ev-dominating sets of a graph G with cardinality i and let $d_{ev}(G,i) = |D_{ev}(G,i)|$. Then the ev-domination polynomial $D_{ev}(G,x)$ of G is defined as $D_{ev}(G,x) = \sum_{i=\gamma_{ev}(G)}^{|V(G)|} d_{ev}(G,i) x^i$, where $\gamma_{ev}(G)$ is the ev-domination number of G.

Let $D_{ev}(P_n, i)$ be the family of ev-dominating sets of the graph P_n with cardinality i and let $d_{ev}(P_n, i) = |D_{ev}(P_n, i)|$. We call the polynomial $D_{ev}(P_n, x) = \sum_{i=\lceil \frac{n}{4} \rceil}^{n} d_{ev}(P_n, i) x^i$ the ev- domination polynomial of the graph P_n [2].

In the next section, we construct the families of the ev-dominating sets of paths by recursive method.

As usual we use $\lfloor x \rfloor$ for the largest integer less than or equal to x and $\lceil x \rceil$ for the smallest integer greater than or equal to x. Also, we denote the set $\{e_1, e_2, \cdots, e_{n-1}\}$ by [e_{n-1}] and the set $\{1, 2, \cdots, n\}$ by [n], throughout this paper.

II. Ev-Dominating Sets Of Paths

Let $D_{ev}(P_n, i)$ be the family of ev-dominating sets of P_n with cardinality i. We investigate the ev-dominating sets of P_n. We need the following lemma to prove our main results in this section.

Lemma 2.1. [3]: $\gamma_{ev}(P_n) = \left\lceil \frac{n}{4} \right\rceil$.

By Lemma 2.1 and the definition of ev-domination number, one has the following Lemma:

Lemma 2.2. $D_{ev}\left(P_n, i\right) = \Phi$ if and only if $i \geq n$ or $i < \left\lceil \frac{n}{4} \right\rceil$.

Lemma 2.3.[2]: If a graph G contains a simple path of length 4k – 1, then every ev-dominating set of G must contain at least k edges of the path.

Proof: The path has 4k vertices. As every edge dominates at most 4 vertices, the 4k vertices are covered by at least k edges.

Lemma 2.4. If $Y \in D_{ev}\left(P_{n-5}, i-1\right)$, and there exists x $\in$[e_{n-1}] such that $Y \bigcup \{x\} \in D_{ev}\left(P_n, i\right)$ then $Y \in D_{ev}\left(P_{n-4}, i-1\right)$.

Proof: Since $Y \in D_{ev}\left(P_{n-5}, i-1\right)$, Y contains at least one edge labeled e_{n-6} or e_{n-7}.

If $e_{n-6} \in Y$, then $Y \in D_{ev}\left(P_{n-4}, i-1\right)$. Suppose $e_{n-7} \in Y$ and $e_{n-6} \notin Y$, then Y covers the vertices upto n - 5.

If we take any other edge x in P_n, it will cover at most 4 vertices. Hence $Y \bigcup \{x\}$ will cover at most n - 5 + 4 = n – 1 vertices, a contradiction to $Y \bigcup \{x\} \in D_{ev}\left(P_n, i\right)$. Therefore, our assumption is wrong. Hence, $e_{n-6} \in Y$. Therefore, $Y \in D_{ev}\left(P_{n-4}, i-1\right)$.

Lemma 2.5.

a) If $D_{ev}\left(P_{n-1}, i-1\right) = D_{ev}\left(P_{n-3}, i-1\right) = \Phi$ then $D_{ev}\left(P_{n-2}, i-1\right) = \Phi$

b) If $D_{ev}\left(P_{n-1}, i-1\right) \neq \Phi$, $D_{ev}\left(P_{n-3}, i-1\right) \neq \Phi$ then $D_{ev}\left(P_{n-2}, i-1\right) \neq \Phi$

c) If $D_{ev}\left(P_{n-1}, i-1\right) = \Phi$, $D_{ev}\left(P_{n-2}, i-1\right) = \Phi$, $D_{ev}\left(P_{n-3}, i-1\right) = \Phi$, $D_{ev}\left(P_{n-4}, i-1\right) = \Phi$, then $D_{ev}\left(P_n, i\right) = \Phi$

Proof: a) Since $D_{ev}\left(P_{n-1}, i-1\right) = D_{ev}\left(P_{n-3}, i-1\right) = \Phi$, by Lemma 2.2, $i-1 \geq n-1$ or $i-1 < \left\lceil \frac{n-1}{4} \right\rceil$ and $i-1 \geq n-3$ or $i-1 < \left\lceil \frac{n-3}{4} \right\rceil$. Therefore, $i-1 \geq n-1$ or $i-1 < \left\lceil \frac{n-3}{4} \right\rceil$. Hence $i-1 \geq n-1 \geq n-2$ or $i-1 < \left\lceil \frac{n-3}{4} \right\rceil < \left\lceil \frac{n-2}{4} \right\rceil$. In either case, we have $D_{ev}\left(P_{n-2}, i-1\right) = \Phi$.

b) Suppose that $D_{ev}\left(P_{n-2}, i-1\right) = \Phi$, so by Lemma 2.2, we have $i-1 \geq n-2$ or $i-1 < \left\lceil \frac{n-2}{4} \right\rceil$. If $i-1 \geq n-2$, then $i-1 \geq n-3$. Therefore, $D_{ev}\left(P_{n-3}, i-1\right) = \Phi$, a contradiction. If $i-1 < \left\lceil \frac{n-2}{4} \right\rceil$, then $i-1 < \left\lceil \frac{n-1}{4} \right\rceil$. Therefore, $D_{ev}\left(P_{n-1}, i-1\right) = \Phi$, a contradiction. Hence, $D_{ev}\left(P_{n-2}, i-1\right) = \Phi$.

c) Suppose that $D_{ev}\left(P_n, i\right) \neq \Phi$. Let $Y \in D_{ev}\left(P_n, i\right)$. Then, there exists at least one edge labelled e_{n-1} or e_{n-2} is in Y. If $e_{n-1} \in Y$, then by Lemma (2.3), at least one edge labelled $e_{n-2}, e_{n-3}, e_{n-4}$ or e_{n-5} is in Y. If $e_{n-2} \in Y$ or $e_{n-3} \in Y$ then $Y - \{e_{n-1}\} \in D_{ev}\left(P_{n-1}, i-1\right)$, a contradiction. If $e_{n-4} \in Y$, then $Y - \{e_{n-1}\} \in D_{ev}\left(P_{n-2}, i-1\right)$, a contradiction. If $e_{n-5} \in Y$, then $Y - \{e_{n-1}\} \in D_{ev}\left(P_{n-3}, i-1\right)$, a contradiction. Therefore $e_{n-1} \notin Y$. Now suppose that $e_{n-2} \in Y$. Then, by lemma 2.3, at least one edge labelled $e_{n-3}, e_{n-4}, e_{n-5}$ or e_{n-6} is in Y. If $e_{n-3} \in Y$ or $e_{n-4} \in Y$, then $Y - \{e_{n-2}\} \in D_{ev}\left(P_{n-2}, i-1\right)$, a contradiction. If $e_{n-5} \in Y$, then $Y - \{e_{n-2}\} \in D_{ev}\left(P_{n-3}, i-1\right)$, a contradiction. If $e_{n-6} \in Y$, then $Y - \{e_{n-2}\} \in D_{ev}\left(P_{n-4}, i-1\right)$, a contradiction. Therefore, $D_{ev}\left(P_n, i\right) = \Phi$.

Lemma 2.6. If $D_{ev}\left(P_n, i\right) \neq \Phi$, then

a) $D_{ev}\left(P_{n-1}, i-1\right) = D_{ev}\left(P_{n-2}, i-1\right) = D_{ev}\left(P_{n-3}, i-1\right) = \Phi$ and $D_{ev}\left(P_{n-4}, i-1\right) \neq \Phi$ if and only if n = 4k and i = k for some k $\in$ N;

b) $D_{ev}\left(P_{n-2}, i-1\right) = D_{ev}\left(P_{n-3}, i-1\right) = D_{ev}\left(P_{n-4}, i-1\right) = \Phi$ and $D_{ev}\left(P_{n-1}, i-1\right) \neq \Phi$ if and only if i = n-1;

c) $D_{ev}\left(P_{n-1}, i-1\right) = \Phi$, $D_{ev}\left(P_{n-2}, i-1\right) \neq \Phi$, $D_{ev}\left(P_{n-3}, i-1\right) \neq \Phi$, $D_{ev}\left(P_{n-4}, i-1\right) \neq \Phi$, if and only if n = 4k + 2 and $i = \left\lceil \frac{4k+2}{4} \right\rceil$ for some k$\in$N;

d) $D_{ev}\left(P_{n-1}, i-1\right) \neq \Phi$, $D_{ev}\left(P_{n-2}, i-1\right) \neq \Phi$, $D_{ev}\left(P_{n-3}, i-1\right) \neq \Phi$ and $D_{ev}\left(P_{n-4}, i-1\right) = \Phi$ if and only if i = n – 3;

e) $D_{ev}\left(P_{n-1}, i-1\right) \neq \Phi$, $D_{ev}\left(P_{n-2}, i-1\right) \neq \Phi$, $D_{ev}\left(P_{n-3}, i-1\right) = \Phi$ and $D_{ev}\left(P_{n-4}, i-1\right) = \Phi$ if and only if i = n – 2;

f) $D_{ev}\left(P_{n-1}, i-1\right) \neq \Phi$, $D_{ev}\left(P_{n-2}, i-1\right) \neq \Phi$, $D_{ev}\left(P_{n-3}, i-1\right) \neq \Phi$ and $D_{ev}\left(P_{n-4}, i-1\right) \neq \Phi$ if and only if $\left\lceil \frac{n-1}{4} \right\rceil + 1 \leq i \leq n-3$.

Proof: a) ($\Rightarrow$) Since $D_{ev}\left(P_{n-1}, i-1\right) = D_{ev}\left(P_{n-2}, i-1\right) = D_{ev}\left(P_{n-3}, i-1\right) = \Phi$, by Lemma 2.2, $i-1 \geq n-1$ or $i-1 < \left\lceil \frac{n-3}{4} \right\rceil$. If $i-1 \geq n-1$, then $i \geq n$ and by Lemma 2.2, $D_{ev}\left(P_n, i\right) = \Phi$, a contradiction.

So $i-1 < \left\lceil \frac{n-3}{4} \right\rceil$, (2.1)

and since $D_{ev}\left(P_{n-4}, i-1\right) \neq \Phi$, we have $\left\lceil \frac{n-4}{4} \right\rceil \leq i-1 < n-4$. (2.2)

From (2.1) and (2.2), $\left\lceil \frac{n-4}{4} \right\rceil \leq i-1 < \left\lceil \frac{n-3}{4} \right\rceil$. (2.3)

When n is a multiple of 4, $\left\lceil \frac{n-4}{4} \right\rceil = \frac{n}{4} - 1$ and $\left\lceil \frac{n-3}{4} \right\rceil = \frac{n}{4}$. Therefore, $\frac{n}{4} - 1 \leq i-1 < \frac{n}{4}$. Therefore, $i-1 = \frac{n}{4} - 1$, we get $i = \frac{n}{4}$. Thus, when $n = 4k$, (2.3) holds good and $i = \frac{n}{4} = k$. When $n \neq 4k$, $\left\lceil \frac{n-4}{4} \right\rceil = \left\lceil \frac{n}{4} \right\rceil - 1$ and $\left\lceil \frac{n-3}{4} \right\rceil = \left\lceil \frac{n}{4} \right\rceil - 1$. Therefore, $\left\lceil \frac{n}{4} \right\rceil - 1 \leq i-1 < \left\lceil \frac{n}{4} \right\rceil - 1$, which is not possible. Hence $n = 4k$ and $i = k$.

($\Leftarrow$) If n = 4k and i = k for some k$\in$N, then by Lemma 2.2, $\left\lceil \frac{n-1}{4} \right\rceil = \left\lceil \frac{4k-1}{4} \right\rceil = k = i > i-1$. Therefore, $i-1 < \left\lceil \frac{n-1}{4} \right\rceil$, which implies $D_{ev}(P_{n-1}, i-1) = \Phi$. Similarly, $D_{ev}(P_{n-2}, i-1) = \Phi = D_{ev}(P_{n-3}, i-1)$. Now $\left\lceil \frac{n-4}{4} \right\rceil = k-1 = i-1$. Therefore, $\left\lceil \frac{n-4}{4} \right\rceil \leq i-1$, which implies $D_{ev}(P_{n-4}, i-1) \neq \Phi$.

b)($\Rightarrow$) Since $D_{ev}(P_{n-2}, i-1) = D_{ev}(P_{n-3}, i-1) = D_{ev}(P_{n-4}, i-1) = \Phi$, by Lemma 2.2, $i-1 \geq n-2$ or $i-1 < \left\lceil \frac{n-4}{4} \right\rceil$. If $i-1 < \left\lceil \frac{n-4}{4} \right\rceil$, then $i-1 < \left\lceil \frac{n-1}{4} \right\rceil$ and by lemma 2.2, $D_{ev}\left(P_{n-1}, i-1\right) = \Phi$, a contradiction.

So $i-1 \geq n-2$ (2.4)

Since, $D_{ev}\left(P_{n-1}, i-1\right) \neq \Phi$, $\left\lceil \frac{n-1}{4} \right\rceil \leq i-1 < n-1$ (2.5)

From (2.4) and (2.5), we have $n-1 > i-1 \geq n-2$. Therefore, $i-1 = n-2$. Therefore, $i = n-1$

($\Leftarrow$) If $i = n-1$, then by lemma 2.2, $\left\lceil \frac{n-1}{4} \right\rceil \leq i-1 < n-1$. Therefore, $D_{ev}(P_{n-1}, i-1) \neq \Phi$.

c) ($\Rightarrow$) Since $D_{ev}\left(P_{n-1}, i-1\right) = \Phi$, by Lemma 2.2, $i-1 \geq n-1$ or $i-1 < \left\lceil \frac{n-1}{4} \right\rceil$. If $i-1 \geq n-1$, then $i-1 \geq n-2 \geq n-3 \geq n-4$, by Lemma 2.2, $D_{ev}\left(P_{n-2}, i-1\right) = D_{ev}\left(P_{n-3}, i-1\right) = D_{ev}\left(P_{n-4}, i-1\right) = \Phi$, a contradiction. Therefore, $i-1 < \left\lceil \frac{n-1}{4} \right\rceil$. Which implies, $i < \left\lceil \frac{n-1}{4} \right\rceil + 1$ (2.6)

Since, $D_{ev}\left(P_{n-2}, i-1\right) \neq \Phi$, $\left\lceil \frac{n-2}{4} \right\rceil \leq i-1 < n-2$. Hence, $\left\lceil \frac{n-2}{4} \right\rceil + 1 \leq i < n-1$ (2.7)

Similarly, $\left\lceil \frac{n-3}{4} \right\rceil +1 \le i < n-2$ (2.8)

and $\left\lceil \frac{n-4}{4} \right\rceil +1 \le i < n-3$ (2.9)

From (2.6) and (2.7), $\left\lceil \frac{n-2}{4} \right\rceil +1 \le i < \left\lceil \frac{n-1}{4} \right\rceil +1$ (2.10)

Therefore, (2.10) hold when $k = \frac{n-2}{4}$ or $n = 4k+2$ and $i = k+1 = \left\lceil \frac{4k+2}{4} \right\rceil$, for some $k \in N$.

Suppose $n = 4k+2$, then $\left\lceil \frac{n-2}{4} \right\rceil +1 = k+1$ and $\left\lceil \frac{n-1}{4} \right\rceil +1 = k+2$. Therefore, from (2.10), we have, $k+1 \le i < k+2$, which implies $i = k+1$. Suppose $n \ne 4k+2$, i.e., $n = 4k,\ 4k+1,\ 4k+3$.

Case(i) When $n = 4k$. From (2.10), we get $\left\lceil \frac{4k-2}{4} \right\rceil +1 = k+1$ and $\left\lceil \frac{4k-1}{4} \right\rceil +1 = k+1$. Therefore, $k+1 \le i < k+1$, which is not possible.

Case(ii) When $n = 4k+1$. From (2.10), we get $\left\lceil \frac{4k+1-2}{4} \right\rceil +1 = k+1$ and $\left\lceil \frac{4k+1-1}{4} \right\rceil +1 = k+1$. Therefore, $k+1 \le i < k+1$, which is not possible.

Case(iii) When $n = 4k+3$. From (2.10), we get $\left\lceil \frac{4k+3-2}{4} \right\rceil +1 = k+2$ and $\left\lceil \frac{4k+3-1}{4} \right\rceil +1 = k+2$ Therefore, $k+2 \le i < k+2$, which is not possible. Therefore, $n = 4k+2$

(⇐) If n = 4k + 2 and $i = \left\lceil \frac{4k+2}{4} \right\rceil$ for some k ∈ N, and $D_{ev}(P_n, i) \ne \Phi$, then by Lemma 2.2, $\left\lceil \frac{n}{4} \right\rceil \le i < n$, $\left\lceil \frac{n}{4} \right\rceil = \left\lceil \frac{4k+2}{4} \right\rceil = i > i-1$. Therefore, $i-1 < \left\lceil \frac{n-1}{4} \right\rceil$. Therefore, $D_{ev}(P_{n-1}, i-1) = \Phi$. Also, $\left\lceil \frac{n-2}{4} \right\rceil = k$. Therefore, $\left\lceil \frac{n-2}{4} \right\rceil \le i-1 < n-2$ and $\left\lceil \frac{n-3}{4} \right\rceil \le i-1 < n-3$ and $\left\lceil \frac{n-4}{4} \right\rceil \le i-1 < n-4$. Hence $D_{ev}\left(P_{n-2}, i-1\right) \ne \Phi$, $D_{ev}\left(P_{n-3}, i-1\right) \ne \Phi$, $D_{ev}\left(P_{n-4}, i-1\right) \ne \Phi$.

d) (⇒) Since $D_{ev}\left(P_{n-4}, i-1\right) = \Phi$, by Lemma 2.2, $i-1 \ge n-4$ or $i-1 < \left\lceil \frac{n-4}{4} \right\rceil$ (2.11)

Since $D_{ev}\left(P_{n-3}, i-1\right) \ne \Phi$, by Lemma 2.2, $\left\lceil \frac{n-3}{4} \right\rceil \le i-1 < n-3$ (2.12)

Similarly, $D_{ev}(P_{n-2}, i-1) \ne \Phi$ and $D_{ev}(P_{n-1}, i-1) \ne \Phi$, by lemma 2.2 $\left\lceil \frac{n-2}{4} \right\rceil \le i-1 < n-2$ (2.13)

$\left\lceil \frac{n-1}{4} \right\rceil \le i-1 < n-1$ (2.14)

From (2.11) and (2.13), we get $i-1 < \left\lceil \frac{n-4}{4} \right\rceil$ is not possible. Therefore, $i-1 \ge n-4 \Rightarrow i \ge n-3$ (2.15)

From (2.12), $i-1 < n-3 \Rightarrow i \le n-3$ (2.16)

From (2.15) and (2.16), $i = n-3$

(⇐) If $i = n-3$, $i-1 = n-4$ then by Lemma 2.2, Therefore, $D_{ev}\left(P_{n-4}, i-1\right) = \Phi$. Also $\left\lceil \frac{n-1}{4} \right\rceil \le i-1 < n-1$, therefore, $D_{ev}\left(P_{n-1}, i-1\right) \ne \Phi$; $\left\lceil \frac{n-2}{4} \right\rceil \le i-1 < n-2$, therefore, $D_{ev}\left(P_{n-2}, i-1\right) \ne \Phi$ and $\left\lceil \frac{n-3}{4} \right\rceil \le i-1 < n-3$, therefore, $D_{ev}\left(P_{n-3}, i-1\right) \ne \Phi$.

e) (⇒) Since $D_{ev}(P_{n-4}, i-1) = \Phi$, by Lemma 2.2, $i-1 \ge n-4$ or $i-1 < \left\lceil \frac{n-4}{4} \right\rceil$ (2.17)

Since $D_{ev}(P_{n-3}, i-1) = \Phi$, by Lemma 2.2, $i-1 \geq n-3$ or $i-1 < \left\lceil \frac{n-3}{4} \right\rceil$ (2.18)

Since $D_{ev}(P_{n-2}, i-1) \neq \Phi$ and $D_{ev}(P_{n-1}, i-1) \neq \Phi$, by lemma 2.2 $\left\lceil \frac{n-2}{4} \right\rceil \leq i-1 < n-2$ and (2.19)

$\left\lceil \frac{n-1}{4} \right\rceil \leq i-1 < n-1$ (2.20)

From (2.18) and (2.19), we get $i-1 < \left\lceil \frac{n-3}{4} \right\rceil$ is not possible. Therefore, $i-1 \geq n-3 \Rightarrow i \geq n-2$ (2.21)

From (2.19), $i-1 < n-2 \Rightarrow i \leq n-2$ (2.22)

From (2.21) and (2.22), $i = n-2$

($\Leftarrow$) If $i = n-2$, $i-1 = n-3$ then by Lemma 2.2, $D_{ev}\left(P_{n-3}, i-1\right) = \Phi$ and $i-1 \geq n-4$ therefore, $D_{ev}\left(P_{n-4}, i-1\right) = \Phi$ and $\left\lceil \frac{n-1}{4} \right\rceil \leq i-1 < n-1$, therefore, $D_{ev}\left(P_{n-1}, i-1\right) \neq \Phi$ and $\left\lceil \frac{n-2}{4} \right\rceil \leq i-2 < n-2$, therefore, $D_{ev}\left(P_{n-2}, i-1\right) \neq \Phi$.

f) ($\Rightarrow$) Since $D_{ev}\left(P_{n-1}, i-1\right) \neq \Phi$, $D_{ev}\left(P_{n-2}, i-1\right) \neq \Phi$, $D_{ev}\left(P_{n-3}, i-1\right) \neq \Phi$, and $D_{ev}\left(P_{n-4}, i-1\right) \neq \Phi$, then by applying Lemma (2.2), $\left\lceil \frac{n-1}{4} \right\rceil \leq i-1 < n-1$, $\left\lceil \frac{n-2}{4} \right\rceil \leq i-1 < n-2$, $\left\lceil \frac{n-3}{4} \right\rceil \leq i-1 < n-3$, $\left\lceil \frac{n-4}{4} \right\rceil \leq i-1 < n-4$. So $\left\lceil \frac{n-1}{4} \right\rceil \leq i-1 < n-4$ and hence $\left\lceil \frac{n-1}{4} \right\rceil + 1 \leq i < n-3$.

($\Leftarrow$) If $\left\lceil \frac{n-1}{4} \right\rceil + 1 \leq i < n-3$, then by lemma 2.2 we have, $\left\lceil \frac{n-1}{4} \right\rceil \leq i-1 < n-1$, $\left\lceil \frac{n-2}{4} \right\rceil \leq i-1 < n-2$, $\left\lceil \frac{n-3}{4} \right\rceil \leq i-1 < n-3$, $\left\lceil \frac{n-4}{4} \right\rceil \leq i-1 < n-4$. Therefore, $D_{ev}\left(P_{n-1}, i-1\right) \neq \Phi$, $D_{ev}\left(P_{n-2}, i-1\right) \neq \Phi$, $D_{ev}\left(P_{n-3}, i-1\right) \neq \Phi$, and $D_{ev}\left(P_{n-4}, i-1\right) \neq \Phi$.

Theorem 2.7. For every $n \geq 5$ and $i \geq \left\lceil \frac{n}{4} \right\rceil$,

a) If $D_{ev}\left(P_{n-1}, i-1\right) = D_{ev}\left(P_{n-2}, i-1\right) = D_{ev}\left(P_{n-3}, i-1\right) = \Phi$ and $D_{ev}\left(P_{n-4}, i-1\right) \neq \Phi$ then $D_{ev}\left(P_n, i\right) = \{e_2, e_6, \ldots, e_{n-14}, e_{n-10}, e_{n-6}, e_{n-2}\}$.

b) If $D_{ev}\left(P_{n-2}, i-1\right) = D_{ev}\left(P_{n-3}, i-1\right) = D_{ev}\left(P_{n-4}, i-1\right) = \Phi$ and $D_{ev}\left(P_{n-1}, i-1\right) \neq \Phi$ then $D_{ev}\left(P_n, i\right) = \left\{\left[e_{n-1}\right]\right\}$.

c) If $D_{ev}\left(P_{n-1}, i-1\right) = \Phi$, $D_{ev}\left(P_{n-2}, i-1\right) \neq \Phi$, $D_{ev}\left(P_{n-3}, i-1\right) \neq \Phi$ and $D_{ev}\left(P_{n-4}, i-1\right) \neq \Phi$ then

$$D_{ev}\left(P_n, i\right) = \left\{\begin{array}{l} \left\{e_2, e_6, \ldots, e_{n-8}, e_{n-4}\right\} \cup \\ \left\{X_1 \cup \left\{e_{n-1}\right\} / X_1 \in D_{ev}(P_{n-3}, i-1)\right\} \cup \\ \left\{X_2 \cup \left\{e_{n-2}\right\} / X_2 \in D_{ev}\left(P_{n-4}, i-1\right)\right\} \end{array}\right\}$$

d) If $D_{ev}\left(P_{n-3}, i-1\right) = \Phi$, $D_{ev}\left(P_{n-2}, i-1\right) \neq \Phi$, $D_{ev}\left(P_{n-1}, i-1\right) \neq \Phi$, then $D_{ev}\left(P_n, i\right) = \left\{\left[e_{n-1}\right] - \{x\} / x \in \left[e_{n-1}\right]\right\}$.

e) If $D_{ev}\left(P_{n-1}, i-1\right) \neq \Phi$, $D_{ev}\left(P_{n-2}, i-1\right) \neq \Phi$, $D_{ev}\left(P_{n-3}, i-1\right) \neq \Phi$, $D_{ev}\left(P_{n-4}, i-1\right) \neq \Phi$, then

$$D_{ev}\left(P_n, i\right) = \left\{\begin{array}{l} \left\{X_1 \cup \left\{e_{n-1}\right\} / X_1 \in D_{ev}\left(P_{n-1}, i-1\right)\right\} \cup \\ \left\{X_2 \cup \left\{e_{n-2}\right\} / X_2 \in D_{ev}\left(P_{n-2}, i-1\right)\right\} \cup \\ \left\{X_3 \cup \left\{e_{n-1}\right\} / X_3 \in D_{ev}\left(P_{n-3}, i-1\right)\right\} \cup \\ \left\{X_4 \cup \left\{e_{n-2}\right\} / X_4 \in D_{ev}\left(P_{n-4}, i-1\right)\right\} \end{array}\right\}$$

Proof:

a) Since $D_{ev}\left(P_{n-1},i-1\right)=D_{ev}\left(P_{n-2},i-1\right)=D_{ev}\left(P_{n-3},i-1\right)=\Phi$ and $D_{ev}\left(P_{n-4},i-1\right)\neq\Phi$ by Lemma 2.6 (i) n =4k, and i = k for some k ∈ N. Clearly, the set $\{e_2,e_6,...,e_{n-14},e_{n-10},e_{n-6},e_{n-2}\}$ has $\frac{n}{4}$ elements. By the definition of P_n , e_2 has joining with e_1 and e_3 also e_6 has joining with e_5 and e_7. Therefore, e_2 and e_6 dominated all the vertices from 1 to 8. Proceeding like this, we obtain that $\{e_2,e_6,...,e_{n-14},e_{n-10},e_{n-6},e_{n-2}\}$ dominates all vertices upto n. The other sets with cardinality $\frac{n}{4}$ are $\{e_1,e_5,e_9,...,e_{n-7},e_{n-3}\}$, $\{e_3,e_7,e_{11},...,e_{n-5},e_{n-1}\}$ etc. In the first set, e_{n-3} does not cover the vertex n. The second set does not cover the vertex 1 and so on. Therefore, $\{e_2,e_6,...,e_{n-14},e_{n-10},e_{n-6},e_{n-2}\}$ is the only ev-dominating set of cardinality $\frac{n}{4}=k=i$.

b) We have $D_{ev}\left(P_{n-2},i-1\right)=D_{ev}\left(P_{n-3},i-1\right)=D_{ev}\left(P_{n-4},i-1\right)=\Phi$ and $D_{ev}\left(P_{n-1},i-1\right)\neq\Phi$. By Lemma 2.6 (ii), we have i = n – 1. So, $D_{ev}\left(P_n,i\right)=\{[e_{n-1}]\}$.

c) We have $D_{ev}\left(P_{n-1},i-1\right)=\Phi$, $D_{ev}\left(P_{n-2},i-1\right)\neq\Phi$, $D_{ev}\left(P_{n-3},i-1\right)\neq\Phi$, and $D_{ev}\left(P_{n-4},i-1\right)\neq\Phi$. By Lemma 2.6 (iii), n = 4k + 2 and $i=\left\lceil\frac{4k+2}{4}\right\rceil=k+1$ for some k ∈ N. Since $X=\{e_2,e_6,...,e_{4k-6},e_{4k-2}\}\in D_{ev}\left(P_{4k},k\right)$, $X\cup\{e_{4k}\}\in D_{ev}\left(P_{4k+2},k+1\right)$. Also, if $X\in D_{ev}\left(P_{4k-1},k\right)$ then $X\cup\{e_{4k+1}\}\in D_{ev}\left(P_{4k+2},k+1\right)$. Also, if $X\in D_{ev}\left(P_{4k-2},k\right)$ then $X\cup\{e_{4k}\}\in D_{ev}\left(P_{4k+2},k+1\right)$. Therefore, we have

$$\left\{\begin{array}{l}\{e_2,e_6,...,e_{4k-6},e_{4k-2}\}\cup\\ \left\{X_1\cup\{e_{4k+1}\}/X_1\in D_{ev}\left(P_{4k-1},k\right)\right\}\cup\\ \left\{X_2\cup\{e_{4k}\}/X_2\in D_{ev}\left(P_{4k-2},k\right)\right\}\end{array}\right\}\subseteq D_{ev}\left(P_{4k+2},k+1\right) \quad (2.23)$$

Now let $Y\in D_{ev}\left(P_{4k+2},k+1\right)$. Then e_{4k+1} or e_{4k} is in Y. If e_{4k+1}∈Y, then by Lemma 2.3, at least one edge labelled e_{4k} , e_{4k-1} or e_{4k-2} is in Y. If e_{4k} or e_{4k-1} is in Y, then $Y-\{e_{4k+1}\}\in D_{ev}\left(P_{4k+1},k\right)$, a contradiction; because $D_{ev}\left(P_{4k+1},k\right)=\Phi$. Hence e_{4k-2} ∈ Y, e_{4k-1} ∉ Y and e_{4k} ∉ Y. Therefore, $Y=X\cup\{e_{4k+1}\}$ for some $X\in D_{ev}\left(P_{4k-1},k\right)$. Now, suppose that e_{4k} ∈Y and e_{4k+1} ∉ Y. By Lemma 2.3, at least one edge labelled e_{4k-1} , e_{4k-2} , e_{4k-3} is in Y. If e_{4k-1} ∈ Y , then $Y-\{e_{4k}\}\in D_{ev}\left(P_{4k-1},k\right)=\{e_2,e_6,...,e_{4k-6}\}$, a contradiction because e_{4k-1} ∉ X for all $X\in D_{ev}\left(P_{4k-1},k\right)$. Therefore, e_{4k-2} or e_{4k-3} is in Y, but e_{4k-1}∉Y. If e_{4k-2} ∈ Y, then $Y=X\cup\{e_{4k}\}$ for some $X\in D_{ev}\left(P_{4k},k\right)$. If e_{4k-3} ∈ Y, then $Y=X\cup\{e_{4k}\}$ for some $X\in D_{ev}\left(P_{4k-2},k\right)$. Thus $Y=X\cup\{e_{4k-2}\}$ for some $X\in D_{ev}\left(P_{4k-2},k\right)$.

$$\text{So, } D_{ev}\left(P_{4k+2},k+1\right)\subseteq\left\{\begin{array}{l}\{e_2,e_6,...,e_{4k-6},e_{4k-2}\}\cup\\ \left\{X_1\cup\{e_{4k+1}\}/X_1\in D_{ev}\left(P_{4k-1},k\right)\right\}\cup\\ \left\{X_2\cup\{e_{4k}\}/X_2\in D_{ev}\left(P_{4k-2},k\right)\right\}\end{array}\right\} \quad (2.24)$$

From (2.23) and (2.24), we have $D_{ev}\left(P_{4k+2},k+1\right)=\left\{\begin{array}{l}\{e_2,e_6,...,e_{4k-6},e_{4k-2}\}\cup\\ \left\{X_1\cup\{e_{4k+1}\}/X_1\in D_{ev}\left(P_{4k-1},k\right)\right\}\cup\\ \left\{X_2\cup\{e_{4k}\}/X_2\in D_{ev}\left(P_{4k-2},k\right)\right\}\end{array}\right\}$

d) If $D_{ev}\left(P_{n-3},i-1\right)=\Phi$, $D_{ev}\left(P_{n-2},i-1\right)\neq\Phi$, $D_{ev}\left(P_{n-1},i-1\right)\neq\Phi$, by Lemma 2.6 (iv), i = n – 2. Therefore, $D_{ev}\left(P_n,i\right)=\{[e_{n-1}]-\{x\}/x\in[e_{n-1}]\}$.

e) We have, $D_{ev}\left(P_{n-1},i-1\right)\neq\Phi$, $D_{ev}\left(P_{n-2},i-1\right)\neq\Phi$, $D_{ev}\left(P_{n-3},i-1\right)\neq\Phi$, $D_{ev}\left(P_{n-4},i-1\right)\neq\Phi$. Let

$X_1 \in D_{ev}\left(P_{n-1}, i-1\right)$, then $X_1 \cup \{e_{n-1}\} \in D_{ev}\left(P_n, i\right)$. Let $X_2 \in D_{ev}\left(P_{n-2}, i-1\right)$, then $X_2 \cup \{e_{n-2}\} \in D_{ev}\left(P_n, i\right)$. Now let $X_3 \in D_{ev}\left(P_{n-3}, i-1\right)$, then $X_3 \cup \{e_{n-1}\} \in D_{ev}\left(P_n, i\right)$. Now let $X_4 \in D_{ev}\left(P_{n-4}, i-1\right)$, then $X_4 \cup \{e_{n-2}\} \in D_{ev}\left(P_n, i\right)$. Thus, we have

$$\left\{\begin{array}{l} \left\{X_1 \cup \{e_{n-1}\} / X_1 \in D_{ev}\left(P_{n-1}, i-1\right)\right\} \cup \\ \left\{X_2 \cup \{e_{n-2}\} / X_2 \in D_{ev}\left(P_{n-2}, i-1\right)\right\} \cup \\ \left\{X_3 \cup \{e_{n-1}\} / X_3 \in D_{ev}\left(P_{n-3}, i-1\right)\right\} \cup \\ \left\{X_4 \cup \{e_{n-2}\} / X_4 \in D_{ev}\left(P_{n-4}, i-1\right)\right\} \end{array}\right\} \subseteq D_{ev}\left(P_n, i\right) \qquad (2.25)$$

Now $Y \in D_{ev}\left(P_n, i\right)$. Suppose that , $e_{n-1} \in Y$, $e_{n-2} \in Y$ then by Lemma 2.3, at least one edge labelled $e_{n-2}, e_{n-3}, e_{n-4}$ in Y. If $e_{n-2} \in Y$ or $e_{n-3} \in Y$ then, $Y = X_1 \cup \{e_{n-1}\}$ for some $X_1 \in D_{ev}\left(P_{n-1}, i-1\right)$. Now suppose that, $e_{n-3} \in Y$ or $e_{n-4} \in Y$, then by lemma 2.3 one edge labelled, e_{n-5}, e_{n-6} in Y. If $e_{n-3} \in Y$ or $e_{n-4} \in Y$ then $Y = X_2 \cup \{e_{n-2}\}$ for some $X_2 \in D_{ev}\left(P_{n-2}, i-1\right)$. If $e_{n-4} \in Y$ or $e_{n-5} \in Y$ then $Y = X_3 \cup \{e_{n-1}\}$ for some $X_3 \in D_{ev}\left(P_{n-3}, i-1\right)$. If $e_{n-6} \in Y$ or $e_{n-5} \in Y$ then $Y = X_4 \cup \{e_{n-2}\}$ for some $X_4 \in D_{ev}\left(P_{n-4}, i-1\right)$.

Therefore, $$D_{ev}\left(P_n, i\right) \subseteq \left\{\begin{array}{l} \left\{X_1 \cup \{e_{n-1}\} / X_1 \in D_{ev}\left(P_{n-1}, i-1\right)\right\} \cup \\ \left\{X_2 \cup \{e_{n-2}\} / X_2 \in D_{ev}\left(P_{n-2}, i-1\right)\right\} \cup \\ \left\{X_3 \cup \{e_{n-1}\} / X_3 \in D_{ev}\left(P_{n-3}, i-1\right)\right\} \cup \\ \left\{X_4 \cup \{e_{n-2}\} / X_4 \in D_{ev}\left(P_{n-4}, i-1\right)\right\} \end{array}\right\}$$

(2.26)

From (2.25) and (2.26), $$D_{ev}\left(P_n, i\right) = \left\{\begin{array}{l} \left\{X_1 \cup \{e_{n-1}\} / X_1 \in D_{ev}\left(P_{n-1}, i-1\right)\right\} \cup \\ \left\{X_2 \cup \{e_{n-2}\} / X_2 \in D_{ev}\left(P_{n-2}, i-1\right)\right\} \cup \\ \left\{X_3 \cup \{e_{n-1}\} / X_3 \in D_{ev}\left(P_{n-3}, i-1\right)\right\} \cup \\ \left\{X_4 \cup \{e_{n-2}\} / X_4 \in D_{ev}\left(P_{n-4}, i-1\right)\right\} \end{array}\right\}.$$

III. Ev-Domination Polynomials of P_n

Let $D_{ev}\left(P_n, x\right) = \sum_{i=\left\lceil \frac{n}{4} \right\rceil}^{n} d_{ev}\left(P_n, i\right) x^i$ be the ev-domination polynomial of a path P_n . In this section, we derive the expression for $D_{ev}\left(P_n, x\right)$.

Theorem 3.1.

i) If $D_{ev}\left(P_n, i\right)$ is the family of ev-dominating sets with cardinality i of P_n , then $d_{ev}\left(P_n, i\right) = d_{ev}\left(P_{n-1}, i-1\right) + d_{ev}\left(P_{n-2}, i-1\right) + d_{ev}\left(P_{n-3}, i-1\right) + d_{ev}\left(P_{n-4}, i-1\right)$, where $d_{ev}\left(P_n, i\right) = \left|D_{ev}\left(P_n, i\right)\right|$.

ii) For every $n \geq 6$, $D_{ev}\left(P_n, x\right) = x\left[D_{ev}\left(P_{n-1}, x\right) + D_{ev}\left(P_{n-2}, x\right) + D_{ev}\left(P_{n-3}, x\right) + D_{ev}\left(P_{n-4}, x\right)\right]$ with the initial values $D_{ev}\left(P_2, x\right) = x$,

$D_{ev}\left(P_3, x\right) = x^2 + 2x$,

$D_{ev}\left(P_4, x\right) = x^3 + 3x^2 + x$,

$D_{ev}\left(P_5, x\right) = x^4 + 4x^3 + 4x^2 + 0x$.

Proof: i) Using (a), (b), (c), (d) and (e) of Theorem 2.7, we prove (a) part.

Suppose (a) of Theorem 2.7 holds. From (e), we have $D_{ev}\left(P_n, i\right) = \left\{X_4 \cup \{e_{n-2}\} / X_4 \in D_{ev}\left(P_{n-4}, i-1\right)\right\}$.

Since, $D_{ev}\left(P_{n-1}, i-1\right) = D_{ev}\left(P_{n-2}, i-1\right) = D_{ev}\left(P_{n-3}, i-1\right) = \Phi$

$\left|D_{ev}\left(P_{n-1},i-1\right)\right|=\left|D_{ev}\left(P_{n-2},i-1\right)\right|=\left|D_{ev}\left(P_{n-3},i-1\right)\right|=0$. Therefore, $\left|D_{ev}\left(P_n,i\right)\right|=\left|D_{ev}\left(P_{n-4},i-1\right)\right|$.
Therefore, $d_{ev}\left(P_n,i\right)=d_{ev}\left(P_{n-4},i-1\right)$. Therefore, in this case $d_{ev}\left(P_n,i\right)=d_{ev}\left(P_{n-4},i-1\right)$ holds.
Suppose (b) of Theorem 2.7 holds. From (e), we have $D_{ev}\left(P_n,i\right)=\left\{X_1\cup\{e_{n-1}\}/\ X_1\in D_{ev}\left(P_{n-1},i-1\right)\right\}$. Since $D_{ev}\left(P_{n-2},i-1\right)=D_{ev}\left(P_{n-3},i-1\right)=D_{ev}(P_{n-4},i-1)=\Phi$, $\left|D_{ev}\left(P_{n-2},i-1\right)\right|=\left|D_{ev}\left(P_{n-3},i-1\right)\right|=\left|D_{ev}\left(P_{n-4},i-1\right)\right|=0$.
Therefore, $\left|D_{ev}\left(P_n,i\right)\right|=\left|D_{ev}\left(P_{n-1},i-1\right)\right|$.
Therefore, $d_{ev}\left(P_n,i\right)=d_{ev}\left(P_{n-1},i-1\right)$. Therefore, in this case $d_{ev}\left(P_n,i\right)=d_{ev}\left(P_{n-1},i-1\right)$ holds.

Suppose (c) of Theorem 2.7 holds. From (e), we have $D_{ev}\left(P_n,i\right)=\left\{\begin{array}{l}\left\{X_2\cup\{e_{n-2}\}/\ X_2\in D_{ev}\left(P_{n-2},i-1\right)\right\}\cup\\ \left\{X_3\cup\{e_{n-1}\}/\ X_3\in D_{ev}\left(P_{n-3},i-1\right)\right\}\cup\\ \left\{X_4\cup\{e_{n-2}\}/\ X_4\in D\left(P^2_{n-4},i-1\right)\right\}\end{array}\right\}$.

Since $D_{ev}\left(P_{n-1},i-1\right)=\Phi$. Therefore, $\left|D_{ev}\left(P_{n-1},i-1\right)\right|=0$.
Therefore, $\left|D_{ev}\left(P_n,i\right)\right|=\left|D_{ev}\left(P_{n-2},i-1\right)\right|\cup\left|D_{ev}\left(P_{n-3},i-1\right)\right|\cup\left|D_{ev}\left(P_{n-4},i-1\right)\right|$.
Therefore, $d_{ev}\left(P_n,i\right)=d_{ev}\left(P_{n-2},i-1\right)+d_{ev}\left(P_{n-3},i-1\right)+d_{ev}\left(P_{n-4},i-1\right)$.

Suppose (e) of Theorem 2.7 holds. From (e), we have $D_{ev}\left(P_n,i\right)=\left\{\begin{array}{l}\left\{X_1\cup\{e_{n-1}\}/\ X_1\in D_{ev}\left(P_{n-1},i-1\right)\right\}\cup\\ \left\{X_2\cup\{e_{n-2}\}/\ X_2\in D_{ev}\left(P_{n-2},i-1\right)\right\}\cup\\ \left\{X_3\cup\{e_{n-1}\}/\ X_3\in D_{ev}\left(P_{n-3},i-1\right)\right\}\cup\\ \left\{X_4\cup\{e_{n-2}\}/\ X_4\in D_{ev}\left(P_{n-4},i-1\right)\right\}\end{array}\right\}$.

Therefore, $\left|D_{ev}\left(P_n,i\right)\right|=\left|D_{ev}\left(P_{n-1},i-1\right)\right|\cup\left|D_{ev}\left(P_{n-2},i-1\right)\right|\cup\left|D_{ev}\left(P_{n-3},i-1\right)\right|\cup\left|D_{ev}\left(P_{n-4},i-1\right)\right|$. Therefore, $d_{ev}\left(P_n,i\right)=d_{ev}\left(P_{n-1},i-1\right)+d_{ev}\left(P_{n-2},i-1\right)+d_{ev}\left(P_{n-3},i-1\right)+d_{ev}\left(P_{n-4},i-1\right)$. Therefore, we have the theorem.

ii) $$d_{ev}\left(P_n,i\right)x^i=d_{ev}\left(P_{n-1},i-1\right)x^i+d_{ev}\left(P_{n-2},i-1\right)x^i+d_{ev}\left(P_{n-3},i-1\right)x^i+d_{ev}\left(P_{n-4},i-1\right)x^i$$
$$\sum d_{ev}\left(P_n,i\right)x^i=\sum d_{ev}\left(P_{n-1},i-1\right)x^i+\sum d_{ev}\left(P_{n-2},i-1\right)x^i+\sum d_{ev}\left(P_{n-3},i-1\right)x^i+\sum d_{ev}\left(P_{n-4},i-1\right)x^i$$
$$\sum d_{ev}\left(P_n,i\right)x^i=x\sum d_{ev}\left(P_{n-1},i-1\right)x^{i-1}+x\sum d_{ev}\left(P_{n-2},i-1\right)x^{i-1}+x\sum d_{ev}\left(P_{n-3},i-1\right)x^{i-1}$$
$$+x\sum d_{ev}\left(P_{n-4},i-1\right)x^{i-1}$$
$$\sum d_{ev}\left(P_n,i\right)x^i=x\Big[\sum d_{ev}\left(P_{n-1},i-1\right)x^{i-1}+\sum d_{ev}\left(P_{n-2},i-1\right)x^{i-1}+\sum d_{ev}\left(P_{n-3},i-1\right)x^{i-1}$$
$$+\sum d_{ev}\left(P_{n-4},i-1\right)x^{i-1}\Big]$$
$$D_{ev}\left(P_n,x\right)=x\left[D_{ev}\left(P_{n-1},x\right)+D_{ev}\left(P_{n-2},x\right)+D_{ev}\left(P_{n-3},x\right)+D_{ev}\left(P_{n-4},x\right)\right]$$
with the initial values
$D_{ev}\left(P_2,x\right)=x$,
$D_{ev}\left(P_3,x\right)=x^2+2x$,
$D_{ev}\left(P_4,x\right)=x^3+3x^2+x$,
$D_{ev}\left(P_5,x\right)=x^4+4x^3+4x^2+0x$.

Table 1. $d_{ev}\left(P_n, i\right)$**, the number of ev-dominating set of** P_n **with cardinality i.**

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
n															
2	1														
3	2	1													
4	1	3	1												
5	0	4	4	1											
6	0	4	8	5	1										
7	0	3	12	13	6	1									
8	0	1	14	25	19	7	1								
9	0	0	12	38	44	26	8	1							
10	0	0	8	46	81	70	34	9	1						
11	0	0	4	46	122	150	104	43	10	1					
12	0	0	1	38	155	266	253	147	53	11	1				
13	0	0	0	25	168	402	512	399	200	64	12	1			
14	0	0	0	13	155	526	888	903	598	264	76	13	1		
15	0	0	0	5	122	600	1344	1757	1492	861	340	89	14	1	
16	0	0	0	1	81	600	1794	2997	3206	2343	1200	429	103	15	1

In the following Theorem, we obtain some properties of $d_{ev}\left(P_n, i\right)$.

Theorem 3.2. The following properties hold for the coefficients of $D_{ev}\left(P_n, x\right)$;

1) $d_{ev}\left(P_{4n}, n\right) = 1$, for every n $\in$ N.

2) $d_{ev}\left(P_n, n-1\right) = 1$, for every $n \geq 2 \in$ N

3) $d_{ev}\left(P_n, n-2\right) = n-1$, for every $n \geq 3$

4) $d_{ev}\left(P_n, n-3\right) = (n-1)C_2 - 2 = \frac{n(n-3)}{2} - 1$, for every $n \geq 4$

5) $d_{ev}\left(P_n, n-4\right) = (n-1)C_3 - 2(n-3) = \frac{1}{6}[n^3 - 6n^2 - n + 30]$, for every $n \geq 5$

6) $d_{ev}\left(P_{4n-1}, n\right) = n+1$, for every $n \in N$.

Proof:

1) Since $D_{ev}\left(P_{4n}, n\right) = \{e_2, e_6, ..., e_{4k-2}\}$, we have $d_{ev}\left(P_{4n}, n\right) = 1$.

2) Since $D_{ev}\left(P_n, n-1\right) = \{[e_{n-1}]\}$, we have the result $d_{ev}\left(P_n, n-1\right) = 1$ for every $n \geq 2$.

3) Since $D_{ev}\left(P_n, n-2\right) = \{[e_{n-1}] - \{x\} / x \in [e_{n-1}]\}$, we have $d_{ev}\left(P_n, n-2\right) = n-1$ for $n \geq 3$.

4) By induction on n, the result is true for n = 4. L.H.S.= $d_{ev}\left(P_4 \ , \right) =$ (from table 1) R.H.S $= \left(\frac{3\times 2}{2}\right) - 2 = 1$.

Therefore, the result is true for n = 4. Now suppose that the result is true for all numbers less than n and we prove it for n. By Theorem 3.1,

$d_{ev}\left(P_n, n-3\right) = d_{ev}\left(P_{n-1}, n-4\right) + d_{ev}\left(P_{n-2}, n-4\right) + d_{ev}\left(P_{n-3}, n-4\right) + d_{ev}\left(P_{n-4}, n-4\right)$

$= \frac{(n-2)(n-3)}{2} - 2 + n - 3 + 1$

$= \frac{n^2 - 3n - 2n + 6}{2} - 4 + n$

$= \frac{n^2 - 5n + 6 - 8 + 2n}{2}$

$= \frac{n^2 - 3n - 2}{2}$

$= \frac{n(n-3) - 2}{2}$

$= \frac{n(n-3)}{2} - 1.$

5) By induction on n, the result is true for n = 5. $\text{L.H.S} = d_{ev}\left(P_5, 1\right) = 0$ (from table 1) $\text{R.H.S} = 4C_3 - 2(5-3) = 4 - 4 = 0$. Therefore, the result is true for n = 5. Now suppose the result is true for all natural numbers less than n and we prove it for n. By Theorem 3.1,

$d_{ev}\left(P_n, n-4\right) = d_{ev}\left(P_{n-1}, n-5\right) + d_{ev}\left(P_{n-2}, n-5\right) + d_{ev}\left(P_{n-3}, n-5\right) + d_{ev}\left(P_{n-4}, n-5\right)$

$= (n-2)C_3 - 2(n-4) + \frac{(n-2)(n-5)}{2} - 1 + n - 4 + 1$

$= (n-2)C_3 - 2(n-4) + \frac{(n-2)(n-5)}{2} - 1 + n - 4 + 1$

$= \frac{(n-2)!}{3!(n-5)!} - 2n + 8 + \frac{n^2 - 7n + 10}{2} + n - 4$

$= \frac{(n-2)(n-3)(n-4)(n-5)!}{6(n-5)!} + \frac{n^2 - 7n + 10}{2} - n + 4$

$= \frac{(n^2 - 5n + 6)(n-4)}{6} + \frac{n^2 - 7n + 10}{2} - n + 4$

$= \frac{n^3 - 9n^2 + 26n - 24 + 3n^2 - 21n + 30 - 6n + 24}{6}$

$= \frac{1}{6}[n^3 - 6n^2 - n + 30] = \frac{1}{6}[(n-5)(n-3)(n+2)]$

6) From the table it is true.

Theorem 3.3.

1) $\sum_{i=n}^{4n} d_{ev}\left(P_i, n\right) = 4\sum_{i=n-1}^{4n-4} d_{ev}\left(P_i, n-1\right),\ n \geq 2.$

2) For every $j \geq \left\lceil \frac{n}{4} \right\rceil$, $d_{ev}\left(P_{n+1}, j+1\right) - d_{ev}\left(P_n, j+1\right) = d_{ev}\left(P_n, j\right) - d_{ev}\left(P_{n-4}, j\right)$

3) If $S_n = \sum_{i=\left\lceil \frac{n}{4} \right\rceil}^{n} d_{ev}\left(P_n, i\right)$, then for every n ≥ 6, $S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4}$ with initial values S_2= 1, S_3 = 3, S_4=5 and S_5 = 9.

Proof: 1)We prove by induction on n. First suppose that n = 2 then,

$\sum_{i=2}^{8} d_{ev}\left(P_i, 2\right) = 4\sum_{i=2}^{4} d_{ev}\left(P_i, 1\right) = 16$.

Now suppose that the result is true for every $n < k$, and we prove for $n = k$.

$\sum_{i=k}^{4k} d_{ev}\left(P_i, k\right) = \sum_{i=k}^{4k} d_{ev}\left(P_{i-1}, k-1\right) + \sum_{i=k}^{4k} d_{ev}\left(P_{i-2}, k-1\right) + \sum_{i=k}^{4k} d_{ev}\left(P_{i-3}, k-1\right) + \sum_{i=k}^{4k} d_{ev}\left(P_{i-4}, k-1\right)$

$= 4\sum_{i=k-1}^{4(k-1)} d_{ev}\left(P_{i-1}, k-2\right) + 4\sum_{i=k-1}^{4(k-1)} d_{ev}\left(P_{i-2}, k-2\right) + 4\sum_{i=k-1}^{4(k-1)} d_{ev}\left(P_{i-3}, k-2\right) + 4\sum_{i=k-1}^{4(k-1)} d_{ev}\left(P_{i-4}, k-2\right)$

$= 4\sum_{i=k-1}^{4k-4} d_{ev}\left(P_i, k-1\right)$

We have the result.

2) By Theorem 3.1, we have

$$d_{ev}\left(P_{n+1}, j+1\right)-d_{ev}\left(P_{n}, j+1\right)=d_{ev}\left(P_{n}, j\right)+d_{ev}\left(P_{n-1}, j\right)+d_{ev}\left(P_{n-2}, j\right)+d_{ev}\left(P_{n-3}, j\right)$$
$$-d_{ev}\left(P_{n-1}, j\right)-d_{ev}\left(P_{n-2}, j\right)-d_{ev}\left(P_{n-3}, j\right)-d_{ev}\left(P_{n-4}, j\right).$$

$$d_{ev}\left(P_{n+1}, j+1\right)-d_{ev}\left(P_{n}, j+1\right)=d_{ev}\left(P_{n}, j\right)-d_{ev}\left(P_{n-4}, j\right)$$

Therefore, we have the result

3) By theorem (3.1), we have

$$S_n=\sum_{i=\left\lceil \frac{n}{4}\right\rceil}^{n} d_{ev}\left(P_n, i\right)$$

$$S_n=\sum_{i=\left\lceil \frac{n}{4}\right\rceil}^{n}\left[d_{ev}\left(P_{n-1}, i-1\right)+d_{ev}\left(P_{n-2}, i-1\right)+d_{ev}\left(P_{n-3}, i-1\right)+d_{ev}\left(P_{n-4}, i-1\right)\right]$$

$$=\sum_{i=\left\lceil \frac{n}{4}\right\rceil-1}^{n-1} d_{ev}\left(P_{n-1}, i\right)+\sum_{i=\left\lceil \frac{n}{4}\right\rceil-1}^{n-1} d_{ev}\left(P_{n-2}, i\right)+\sum_{i=\left\lceil \frac{n}{4}\right\rceil-1}^{n-1} d_{ev}\left(P_{n-3}, i\right)+\sum_{i=\left\lceil \frac{n}{4}\right\rceil-1}^{n-1} d_{ev}\left(P_{n-4}, i\right)$$

$$S_n=S_{n-1}+S_{n-2}+S_{n-3}+S_{n-4}.$$

IV. Concluding Remarks

In [2], the domination polynomial of path was studied and obtained the very important property, $d\left(P_n, i\right)=d\left(P_{n-1}, i-1\right)+d\left(P_{n-2}, i-1\right)+d\left(P_{n-3}, i-1\right)$. It is interesting that we have derived an analogues relation for the ev-domination of path of the form, $d_{ev}\left(P_n, i\right)=d_{ev}\left(P_{n-1}, i-1\right)+d_{ev}\left(P_{n-2}, i-1\right)+d_{ev}\left(P_{n-3}, i-1\right)+d_{ev}\left(P_{n-4}, i-1\right)$. One can characterise the roots of the polynomial $D_{ev}\left(P_n, x\right)$ and identify whether they are real or complex. Another interesting character to be investigated is whether $D_{ev}\left(P_n, x\right)$ is log concave or not.

References

[1]. S. Alikhani and Y.-H. Peng, "Introduction to Domination Polynomial of a Graph," arXiv:0905.2251v1[math.co], 2009.

[2]. S. Alikhani and Y.-H. Peng, "Dominating Sets and Domination Polynomials of Paths," International Journal of Mathematics and Mathematical Sciences, Vol. 2009, 2009, Article ID: 542040.

[3]. G. Chartand and P. Zhang, "Introduction to Graph Theory," McGraw-Hill, Boston, 2005.

[4]. A. Vijayan and K. Lal Gipson, "Dominating sets and Domination polynomials of square of paths", Open Journal of Discrete Mathematics, 3, 60 – 69, January-2013, USA.

[5]. E. SampathKumar and S.S. Kamath, "Mixed Domination in Graphs," Sankhya: The Indian Journal of Statistics, special volume54, pp. 399 – 402, 1992.