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On the Determinant of a Product of Two Polynomial Matrices

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Abstract: *The Cayley Hamilton theorem for an arbitrary matrix A is generalized to a polynomial matrix. It is proved that if h(λ) = det[f(λ)g(λ)] then h(A) = 0 iff det f(A) = 0 or det g(A) = 0.*
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I. Introduction

In England, J.J.Sylvester first introduced the term "Matrix" in the year 1848. Matrix algebra was nurtured by the work of Arthur Cayley in 1855. Matrices are vital and essential part of the area of mathematics. Applications of matrices are found in most of the scientific fields.

The Cayley Hamilton theorem is one of the most important theorem in Matrix analysis which is extremely versatile and useful. It was given by Cayley in his 1858 Memoir on the Theory of Matrices. It says that every square matrix satisfies its own characteristic equation $|A-\lambda I|=0.$

Early in the development the formula det(AB) = det(A)det(B) provide a connection between matrix algebra and determinants.

The Cayley-Hamilton theorem and its generalizations have been used in control systems [2] and also au-tomation and control in [3], electronics and circuit theory [4], time-systems with delays [5], singular 2-D linear systems [6], 2-D continuous discrete linear systems [7], automation and electrotechnics [8], etc.

In this paper a generalization of the Cayley Hamilton theorem for polynomial matrices is presented. The linear polynomial matrix (A – λ I) of det (A – λ I) in the classical Cayley Hamilton theorem is replaced by the general polynomial matrix $f(\lambda) = A_0 + A_1\lambda + ... + A_n\lambda^n$ where $A_i^{'}$s for i = 0, 1, 2,…,n are square matrices of the same order.

It is proved that if h(λ) = det [f(λ)g(λ)] then for a square matrix A, h(A) = 0 iff det f(A) = 0 or det g(A) = 0. It is illustrated with help of some examples.

II. Preliminaries

Definition 2.1

A matrix A whose entries are polynomials is said to be a polynomial matrix.

Example 2.2

A 3 x 3 polynomial matrix of degree 2 is given below.

$$A = \begin{pmatrix} 1 & \lambda^2 & \lambda \\ 0 & 2\lambda & 2 \\ 3\lambda + 2 & \lambda^2 - 1 & 0 \end{pmatrix} = A_0 + A_1\lambda + A_2\lambda^2$$

where $A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 2 & -1 & 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Example 2.3

A 3 x 3 polynomial matrix of degree 2 with coefficient matrices are symmetric is given below.

$$A = \begin{pmatrix} 1+2\lambda & 2+\lambda+2\lambda^2 & 6+5\lambda+\lambda^2 \\ 2+\lambda+2\lambda^2 & 4+4\lambda+4\lambda^2 & 3+2\lambda+3\lambda^2 \\ 6+5\lambda+\lambda^2 & 3+2\lambda+3\lambda^2 & 5+5\lambda+\lambda^2 \end{pmatrix} = A_0 + A_1\lambda + A_2\lambda^2$$

where $A_0 = \begin{pmatrix} 1 & 2 & 6 \\ 2 & 4 & 3 \\ 6 & 3 & 5 \end{pmatrix}$, $A_1 = \begin{pmatrix} 2 & 1 & 5 \\ 1 & 4 & 2 \\ 5 & 2 & 5 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 3 & 1 \end{pmatrix}$.

Example 2.4

A 3 x 3 polynomial matrix of degree 2 with coefficient matrices are skew symmetric is given below.

$$A = \begin{pmatrix} 0 & 2-\lambda^2 & 1+\lambda+5\lambda^2 \\ -2+\lambda^2 & 3\lambda & 3+2\lambda+2\lambda^2 \\ -1-\lambda-5\lambda^2 & -3-2\lambda-2\lambda^2 & 5\lambda \end{pmatrix} = A_0 + A_1\lambda + A_2\lambda^2$$

where $A_0 = \begin{pmatrix} 0 & 2 & 1 \\ -2 & 0 & 3 \\ -1 & -3 & 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 2 \\ -1 & -2 & 5 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & -1 & 5 \\ 1 & 0 & 2 \\ -5 & -2 & 0 \end{pmatrix}$.

Example 2.5

A 2 x 2 polynomial matrix of degree2 with coefficient matrices are Hermitian is as follows

$$A = \begin{pmatrix} 1+\lambda^2 & (1-i)-i\lambda+(2-i)\lambda^2 \\ (1+i)+i\lambda+(2+i)\lambda^2 & 3+\lambda+5\lambda^2 \end{pmatrix} = A_0 + A_1\lambda + A_2\lambda^2$$

where $A_0 = \begin{pmatrix} 0 & 1-i \\ 1+i & 3 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & -i \\ i & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 2-i \\ 2+i & 5 \end{pmatrix}$.

Example 2.6

A 2 x 2 polynomial matrix of degree 2 with coefficient matrices are skew Hermitian is as follows

$$A = \begin{pmatrix} i & (1-i)+i\lambda+(2-i)\lambda^2 \\ (-1-i)-i\lambda+(-2+i)\lambda^2 & 3i \end{pmatrix} = A_0 + A_1\lambda + A_2\lambda^2$$

where $A_0 = \begin{pmatrix} i & 1-i \\ -1-i & 3i \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 2-i \\ -2+i & 0 \end{pmatrix}$

III. Main Theorem

Lemma 3.1

If A and B are square matrices of order n having elements as polynomials in λ each of degree $\leq$m then the elements of the matrix AB are also polynomials in λ degree $\leq$mn.

Example 3.2

$$Let\ A = \begin{pmatrix} 1 & \lambda \\ \lambda^2 & 0 \end{pmatrix} and\ B = \begin{pmatrix} \lambda^2 & 0 \\ \lambda & 1 \end{pmatrix} then\ AB = \begin{pmatrix} 2\lambda^2 & \lambda \\ \lambda^4 & 0 \end{pmatrix}$$

Theorem:3.3

Let $f(\lambda) = A_0 + A_1\lambda + ... + A_m\lambda^m$ and $g(\lambda) = B_0 + B_1\lambda + ... + B_m\lambda^m$ be polynomial matrices for $f(\lambda), g(\lambda) \in M_n(F[\lambda])$ where $A_i^{'}s, B_i^{'}s \in M_n(F)$ for i = 1, 2, 3,….m are square matrices of order n over the field F. If h(λ) = det[f(λ)g(λ)], then h(A) = 0 if and only if det f(A) = 0 (or) det g(A) = 0.

Proof:

Let h(A) = 0. We have to prove that det f(A) = 0 (or) det g(A) = 0.
Assume the contrary that det f(A) $\neq$ 0 and det g(A) $\neq$ 0.

Since for any two square matrices A and B, we have

$$\det(AB) = \det A \det B \tag{3.1}$$

using (3.1), we have $\det(f(\lambda)g(\lambda)) = \det f(\lambda) \det g(\lambda)$.

$h(A) = \det[f(A)g(A)] = \det f(A) \det g(A) \neq 0$.

Which is a contradiction to our assumption that h(A) = 0.

Hence det f(A) = 0 (or) det g(A) = 0.

Conversely,

Let us assume that det f(A) = 0 (or) det g(A) = 0. We have to prove that h(A) = 0.

Since

$$f(\lambda) = A_0 + A_1\lambda + ... + A_m\lambda^m \;\; and \;\; g(\lambda) = B_0 + B_1\lambda + ... + B_m\lambda^m \tag{3.2}$$

are matrices of order n x n having elements as polynomials in λ each of degree $\leq$ m, therefore by lemma 3.1, we have

$$f(\lambda)g(\lambda) = C_0 + C_1\lambda + C_2\lambda^2 + ... + C_{mn}\lambda^{mn} \tag{3.3}$$

also det $f(\lambda)$ and det $g(\lambda)$ are polynomials in λ over F[λ] of degree $\leq mn$.

Using lemma 1 of [1], we have

$$\det f(\lambda) = Q_0 + Q_1\lambda + ... + Q_{mn}\lambda^{mn} \tag{3.4}$$

and

$$\det g(\lambda) = R_0 + R_1\lambda + ... + R_{mn}\lambda^{mn} \tag{3.5}$$

Also h(λ) = det [f(λ)g(λ)] is a polynomial in λ over F[λ] of degree $\leq 2mn$.

Using lemma 1 of [1], we have

$$h(\lambda) = \det[f(\lambda)g(\lambda)] = P_0 + P_1\lambda + P_2\lambda^2 + ... + P_{2mn}\lambda^{2mn} \tag{3.6}$$

Now using (3.1), we have

$$\det(f(\lambda)g(\lambda)) = \det f(\lambda) \det g(\lambda) \tag{3.7}$$

Using (3.3), (3.4) and (3.6) in (3.7)

$$(P_0 + P_1\lambda + P_2\lambda^2 + ... + P_{2mn}\lambda^{2mn}) = (Q_0 + Q_1\lambda + ... + Q_{mn}\lambda^{mn})(R_0 + R_1\lambda + ... + R_{mn}\lambda^{mn}) \tag{3.8}$$

Comparing coefficients of the like terms on both sides of equation (3.8), we get

$$\left.\begin{array}{l} Q_0R_0 = P_0 \\ Q_0R_1 + Q_1R_0 = P_1 \\ Q_0R_2 + Q_1R_1 + Q_2R_0 = P_2 \\ . \\ . \\ . \\ Q_{mn}R_{mn} = P_{2mn} \end{array}\right\}. \tag{3.9}$$

Multiplying the equations in (3.9) by the matrices I, A, A^2, A^3,……. A^m, A^{m+1},…… A^{2mn-1}, A^{2mn} respectively and adding, we get

$Q_0R_0I + (Q_0R_1 + Q_1R_0)A + (Q_0R_2 + Q_1R_1 + Q_2R_0)A^2 + ……$

$…..+ (Q_{mn}R_{mn})A^{2mn} = P_0I + P_1A + P_2A^2 + ... + P_{2mn}A^{2mn}$.

$$\Rightarrow (Q_0I + Q_1A + Q_2A^2 + ... + Q_{mn}A^{mn})(R_0I + R_1A + R_2A^2 + R_3A^3 + ... + R_{mn}A^{mn}) = h(A)$$

$\Rightarrow$h(A) = 0.

Example3.4

Consider the function $f(\lambda) = A_0 + A_1\lambda + A_2\lambda^2$;

where $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; $A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Rightarrow f(\lambda) = \begin{pmatrix} 1 & \lambda \\ \lambda^2 & 1+\lambda \end{pmatrix}$.

And $g(\lambda) = B_0 + B_1\lambda + B_2\lambda^2$;

where $B_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \textit{and } B_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow g(\lambda) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda^2 \end{pmatrix}.$

If h(λ) = det [f(λ)g(λ)] = $A_0 + A_1\lambda + A_2\lambda^2 + A_3\lambda^3 + A_4\lambda^4 + A_5\lambda^5 + A_6\lambda^6 = \lambda^3 + \lambda^4 - \lambda^6$.
det f(λ) = 1 + λ – λ^3 and det g(λ) = λ^3.

Let $h(A) = A^3 - A^4 + A^6 = 0 = A^3(\,I + A \;\; - A^3)$

$$\Rightarrow \det g(A) = 0 \; or \; \det f(A) = 0.$$

Conversely,

Let det f(A) = 0 = I + A – A^3 or det g(A) = A^3 = 0.

$$\Rightarrow h(A) = A^3 - A^4 + A^6 = A^3(\,I + A \;\; - A^3) = A^3(0) = 0.$$

IV. Conclusion

In this paper we have proved a theorem on the determinant of the product of two polynomial matrices. Similarly we can prove all results relating to matrices and their determinants.

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