

## Bäcklund Transformations of Some Nonlinear Evolution Equations VIA Painlevé Analysis

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**Abstract:** In this paper we present the Painlevé test for the (1+1) –dimensional travelling regularized long wave (TRLW) equation, the (2+1)-dimensional Boiti-Leon-Manna-Pempinelli (BLMP) equation, the modified improved Kadomtsev-Petviashvili equation (MIKP) and the variant shallow water wave equations. The associated Bäcklund transformations are obtained directly from the Painlevé test.

**Keywords:** the (1+1) –dimensional travelling regularized long wave (TRLW) equation, the (2+1)-dimensional Boiti-Leon-Manna-Pempinelli (BLMP) equation, the modified improved Kadomtsev-Petviashvili equation (MIKP), the variant shallow water wave equations and Painlevé analysis.

### I. Introduction

Painlevé analysis is a powerful tool in investigating the integrability properties of differential equations. For systems with the painlevé property, Bäcklund transformations can be defined. These appear as truncations of certain expansions of solutions about its singular manifold. Many methods have been established to study characteristic properties for integrable NLEEs and their interrelations. Some of the most important methods are the inverse scattering method (IST) [1-4], the Bäcklund transformations [5, 6] and the Painlevé analysis method [7-12]. If a PDE which has no points such as movable branch, algebraic and logarithmic then is called P-type. An ordinary differential equation (ODE) might still admit movable essential singularities without movable branch points. This method does not identify essential singularities and therefore it provides only necessary conditions for an ODE to be of P-type. Singularity structure analysis admitting the P-property advocated by Ablowitz et al. for ODEs and extended to PDE by Weiss, Tabor and Carnevale (WTC), plays a key role of investigating the integrability properties of many NLEEs. The well-known procedure of WTC requires,

- The determination of leading orders Laurent series,
- The identification of powers at which the arbitrary functions can enter into the Laurent series called resonances,
- Verifying that, at the resonance values, sufficient number of arbitrary functions exist without introducing the movable critical manifold.

According to the WTC method, the general solution of PDE is in the form

$$u(x, t) = \phi^\alpha(x, t) \sum_{j=0}^{\infty} u_j(x, t) \phi^j(x, t) \quad (1.1)$$

where  $\alpha$  is a negative integer determined by balancing the powers of  $\phi$  of dominant terms in the equation.  $\phi$  is a non-characteristic manifold,  $\phi(x, t) = 0$  is the equation of singular manifold. The functions  $u_j$  ( $j = 0, 1, 2, \dots$ ) have to be determined by substitution of expansion (1.1) into the PDE, so it becomes:

$$\sum_{j=0}^{\infty} E_j(u_0, \dots, u_j, \phi) \phi^{j+q}(x, t) = 0, \quad (1.2)$$

where  $q$  is some negative constant,  $E_j$  depends on  $\phi$  only by the derivatives of  $\phi$ . The successive practical steps of Painlevé' analysis are the following:

- Determine the possible leading orders  $p$  by balancing two or more terms of the PDE and expressing that they dominate the other terms.
- Solve equation  $E_0 = 0$  for non-zero values of  $u_0$ ; this may lead to several solutions, called branches.
- Find the resonances, i.e. the values of  $k$  for which  $u_k$  cannot be determined from equation  $E_j = 0$ .

This last equation has generally the form

$$E_j = (j+1)P(j)\phi_x^j \phi_i^{n-2} u_j + Q(u_0, \dots, u_{j-1}, \phi) = 0, \quad j > 0 \quad (1.3)$$

where  $n$  is the order of the PDE,  $0 \leq j \leq n$  and  $p$  is a polynomial of degree  $n - 1$ . The values of the resonances are the zeros of  $p$ .

- Determine whether the resonances are ‘compatible’ or not. At resonance, after substitution in (1.3) of the previously computed  $u_i$ ,  $i \leq j - 1$ , the function  $Q$  is either zero or non-zero then in the case  $u_j$  can arbitrarily be chosen and the expansion (1.1) does not exist for arbitrary  $\phi$ , so the resonance is called compatible.

## II. The (1+1) –dimensional travelling regularized long wave (TRLW) equation

Consider the (1+1) –dimensional travelling regularized long wave equation [13]:

$$u_t + u_x + \beta u u_x + u_{xxt} = 0 \tag{2.1}$$

We first present the Painlevé test of the TRLW equation. The leading order of solution of Eq. (2.1) is assumed as

$$u \approx u_0 \phi^\alpha \tag{2.2}$$

Substituting Eq. (2.2) into (2.1) and equating the most dominant terms, the following results are obtained

$$u_0 = \frac{-12}{\beta} \phi_t^2, \quad \alpha = -2 \tag{2.3}$$

For finding the resonances, the full Laurent series

$$u = u_0 \phi^{-2} + \sum_{j=1}^{\infty} u_j \phi^{j-2} \tag{2.4}$$

is substituted into Eq. (2.1) and by equating the coefficients of  $\phi^{j-5}$ , the polynomial equation in  $j$  is found as

$$j^3 - 9j^2 + 14j + 24 = 0 \tag{2.5}$$

Using Eq. (2.5), the resonances are found to be

$$j = -1, 4, 6 \tag{2.6}$$

As usual, the resonance at  $j = -1$  corresponds to the arbitrariness of singular manifold  $\varphi(x, t) = 0$ . In order to check the existence of sufficient number of arbitrary functions at the other resonance values, the full Laurent expansion (2.4) is substituted in Eq. (2.1). From the coefficient of  $\phi^{-5}$ , the explicit value of  $u_0$  is obtained as given in Eq. (2.3).

To construct the Bäcklund transformation of Eq. (2.1), let us truncate the Laurent series at the constant level term to give

$$u = u_0 \phi^{-2} + u_1 \phi^{-1} \tag{2.7}$$

Hence

$$u = \frac{-12\phi_t^2}{\beta\phi^2} + \frac{u_1}{\phi}, \tag{2.8}$$

where the pair of function  $(u, u_1)$  satisfy Eq. (2.1), and hence Eq. (2.8) is the associated Bäcklund transformation of Eq. (2.1) relating a solution  $u$  with a known solution  $u_1$  of the Eq. (2.1) which can be taken to be so.

We can also construct another Bäcklund transformation of Eq. (2.1) to be

$$u = \frac{u_0}{\phi^2} + \frac{u_1}{\phi} + u_2, \tag{2.9}$$

where  $u, u_1$  satisfy Eq. (2.1) while  $u_2$  is given by:

$$u_2 = \frac{1}{-24\phi_t^2\phi_x} (-2u_{0x}\phi_{tt} + \beta u_1 u_{0x} - \beta u_1^2 \phi_x - 12\phi_t^2 u_{1x} - 2u_0 \phi_{tx} + 2u_1 \phi_x \phi_{tt} - \tag{2.10}$$

$$2u_0 \phi_x - 4u_{0t} \phi_{xt} - 4u_{0xt} \phi_t - 2u_{0tt} \phi_x + 4u_{1t} \phi_x \phi_t - 2u_0 \phi_t + 4u_1 \phi_{xt} \phi_t + 2u_{1x} \phi_t^2),$$

and hence Eq. (2.10) is a Bäcklund transformation too. Also let's truncate the Laurent series again we get

$$u = \frac{6\phi_x^2}{\phi^2} + \frac{u_1}{\phi} + u_2 + u_3\phi, \tag{2.11}$$

where  $u, u_1$  satisfy Eq. (2.1) and  $u_2$  is given by Eq. (2.10)  $u_3$  is given by:

$$u_3 = \frac{1}{-12\phi_x\phi_t^2} (-u_1\phi_x + \beta u_{20x} - \beta u_{21}\phi_x - u_1\phi_{xt} + \beta u_{2x}u_0 - u_1\phi_t + \beta u_{1x}u_1 - u_{1t}\phi_x + u_{0t} - 2u_{1xt}\phi + u_{0tx} + u_{0x} - 2u_{1t}\phi_t - u_{1x}\phi_{tt}) \tag{2.12}$$

We can make more truncations to the Laurent series at the constant level term  $u_4$  and it will produce another Bäcklund transformation for Eq. (2.1).

### III. The (2+1)-dimensional Boiti-Leon-Manna-Pempinelli (BLMP) equation

Consider the (2+1)-dimensional Boiti-Leon-Manna-Pempinelli (BLMP) equation [14-17]:

$$u_{yt} - 3u_{xy}u_x - 3u_{xx}u_y + u_{xxx} = 0 \tag{3.1}$$

We first present the Painlevé test of the BLMP equation. The leading order of solution of Eq. (3.1) is assumed as

$$u \approx u_0\phi^\alpha \tag{3.2}$$

Substituting Eq. (3.2) into (3.1) and equating the most dominant terms, the following results are obtained

$$u_0 = -2\phi_x, \quad \alpha = -1 \tag{3.3}$$

For finding the resonances, the full Laurent series.

$$u = u_0\phi^{-1} + \sum_{j=1}^{\infty} u_j\phi^{j-1} \tag{3.4}$$

is substituted into Eq. (3.1) and by equating the coefficients of  $\phi^{j-5}$ , the polynomial equation in  $j$  is found as

$$j^4 - 10j^3 + 23j^2 + 10j - 24 = 0 \tag{3.5}$$

Using Eq. (3.5), the resonances are found to be

$$j = -1, 1, 4, 6 \tag{3.6}$$

As usual, the resonance at  $j = -1$  corresponds to the arbitrariness of singular manifold  $\phi(x, y, t) = 0$ . In order to check the existence of sufficient number of arbitrary functions at the other resonance values, the full Laurent expansion (3.4) is substituted in Eq. (3.1). From the coefficient of  $\phi^{-5}$ , the explicit value of  $u_0$  is obtained as given in Eq. (3.3).

To construct the Bäcklund transformation of Eq. (3.1), let us truncate the Laurent series at the constant level term to give

$$u = u_0\phi^{-1} + u_1 \tag{3.7}$$

Hence

$$u = \frac{-2\phi_x}{\phi} + u_1 \tag{3.8}$$

where the pair of function  $(u, u_1)$  satisfy Eq. (3.1) and hence Eq. (3.8) is the associated Bäcklund transformation of Eq. (3.1) relating a solution  $u$  with a known solution  $u_1$  of the Eq. (3.1) which can be taken to be so.

We can also construct another Bäcklund transformation of Eq. (3.1) to be

$$u = \frac{-2\phi_x}{\phi} + u_1 + u_2\phi, \tag{3.9}$$

where  $u, u_1$  satisfy Eq. (3.1) while  $u_2$  is given by:

$$u_2 = \frac{1}{-24\phi_x^3\phi_y} (-12\phi_x^2\phi_{xxy} + 6u_{0xx}\phi_x\phi_y + 12u_{0x}\phi_x\phi_{xy} + 6u_{0xy}\phi_x^2 + 12\phi_x^3u_{1y} - 4\phi_x\phi_y\phi_t - 6u_{0xx}\phi_x\phi_y + 9u_{0y}u_{0x}\phi_x - 12\phi_x\phi_{xy}\phi_{xx} + 6u_{0x}\phi_{xx}\phi_y + 12\phi_x^2\phi_yu_{1x} - 6\phi_x^2u_{0xy} - 6\phi_x\phi_{xy}u_{0x} + 3u_{0x}^2\phi_y - 4\phi_{xxx}\phi_x\phi_y), \tag{3.10}$$

and hence Eq. (3.9) is a Bäcklund transformation too. Also let's truncate the Laurent series again we get

$$u = \frac{-2\phi_x}{\phi} + u_1 + u_2\phi + u_3\phi^2, \tag{3.11}$$

where  $u, u_1$  satisfy Eq. (3.1) and  $u_2$  is given by Eq. (3.10), while  $u_3$  is given by:

$$u_3 = \frac{-1}{24\phi_x^3\phi_y} (3u_0\phi_{xx}u_{1y} - u_{0y}\phi_t + 6u_{0x}\phi_xu_{1y} + 3u_{0y}\phi_{xy}u_{1x} - 3u_0\phi_x^2u_{2y} - u_{0t}\phi_y - u_{0y}\phi_{yt} + 3u_{0y}\phi_xu_{1x} + 3u_{0x}\phi_yu_{1x} + 3u_0\phi_xu_{1xy} + 3u_{0y}\phi_x^2u_2 + 3u_0\phi_yu_{1xx} - u_{0y}\phi_{xxx} - 3u_{0xx}\phi_{xy} - u_{0xxx}\phi_y - 3u_{0x}\phi_{xxy} - 3u_{0xxy}\phi_x - 3u_{0xy}\phi_{xx} - u_{0y}\phi_{xxy} - 3u_{0xy}u_{0x} - 3u_{0xx}u_{0y} + 3u_0\phi_x\phi_yu_{2x} + 6u_0\phi_{xx}u_2\phi_y + 9u_{0x}\phi_xu_2\phi_y + 6u_0\phi_{xy}u_2\phi_x) \tag{3.12}$$

We can make more truncations to the Laurent series at the constant level term  $u_4$  and it will produce another Bäcklund transformation for Eq. (3.1).

#### IV. The modified improved Kadomtsev-Petviashvili equation (MIKP)

Consider the modified improved Kadomtsev-Petviashvili equation (MIKP) [18, 19]:

$$u_{tx} + 2uu_x^2 + u^2u_{xx} + au_{xxxx} + bu_{yy} = 0 \tag{4.1}$$

We first present the Painlevé test of the MIKP equation. The leading order of solution of Eq. (4.1) is assumed as:

$$u \approx u_0\phi^\alpha \tag{4.2}$$

Substituting Eq. (4.2) into (4.1) and equating the most dominant terms, the following results are obtained

$$u_0 = \sqrt{-6a}\phi, \quad \alpha = -1 \tag{4.3}$$

For finding the resonances, the full Laurent series.

$$u = u_0\phi^{-1} + \sum_{j=1}^{\infty} u_j\phi^{j-1} \tag{4.4}$$

is substituted into Eq. (4.1) and by equating the coefficients of  $\phi^{j-5}$ , the polynomial equation in  $j$  is found as:

$$j^4 - 10j^3 + 29j^2 - 8j - 48 = 0 \tag{4.5}$$

Using Eq. (4.5), the resonances are found to be

$$j = -1, 3, 4, 4 \tag{4.6}$$

As usual, the resonance at  $j = -1$  corresponds to the arbitrariness of singular manifold  $\varphi(x, y, t) = 0$ . In order to check the existence of sufficient number of arbitrary functions at the other resonance values, the full Laurent expansion (4.4) is substituted in Eq. (4.1). From the coefficient of  $\phi^{-5}$ , the explicit value of  $u_0$  is obtained as given in Eq. (4.3).

To construct the Bäcklund transformation of Eq. (4.1), let us truncate the Laurent series at the constant level term to give

$$u = u_0\phi^{-1} + u_1 \tag{4.7}$$

Hence

$$u = \frac{\sqrt{-6a}\phi_x}{\phi} + u_1, \tag{4.8}$$

where the pair of function  $(u, u_1)$  satisfy Eq. (4.1) and hence Eq. (4.8) is the associated Bäcklund transformation of Eq. (4.1) relating a solution  $u$  with a known solution  $u_1$  of the Eq. (4.1) which can be taken to be so.

We can also construct another Bäcklund transformation of Eq. (4.1) to be

$$u = \frac{\sqrt{-6a}\phi_x}{\phi} + u_1 + u_2\phi, \tag{4.9}$$

where  $u, u_1$  satisfy Eq. (4.1) while  $u_2$  is given by:

$$\begin{aligned} u_2 = & \frac{1}{12a\phi_x^4} (-6a\phi_x^2 u_{0xx} + 12a\phi_x^2 u_1 \phi_{xx} - 8\sqrt{-6a}\phi_x^2 u_1 u_{0x} + 2\sqrt{-6a}u_1^2 \phi_x^3 + 2\sqrt{-6a}\phi_x^2 \phi_t \\ & + 8a\sqrt{-6a}\phi_x^2 \phi_{xxx} + 12au_{0xx} \phi_x^2 + 6a\sqrt{-6a}\phi_x^3 + 24au_{0x} \phi_x \phi_{xx} + 2\sqrt{-6ab}\phi_x \phi_y^2 + 24a\phi_x^3 u_{1x} \\ & + 2\sqrt{-6a}\phi_x u_{0x}^2), \end{aligned} \tag{4.10}$$

and hence Eq. (4.10) is a Bäcklund transformation too.

### V. The variant shallow water wave equations

Consider the variant shallow water wave equations [20]:

$$\begin{aligned} u_t + v_x + uu_x - \varepsilon^2 u_{xxt} &= 0, \\ v_t + u_x + (uv)_x + \delta^2 u_{xxx} &= 0 \end{aligned} \tag{5.1}$$

We first present the Painlevé test of the variant shallow water wave equations. The leading order of solution of system (5.1) is assumed as:

$$u \approx u_0 \phi^\alpha, \quad v \approx v_0 \phi^\beta \tag{5.2}$$

Substituting Eq. (5.2) into (5.1) and equating the most dominant terms, the following results are obtained

$$u_0 = 12\varepsilon^2 \phi_x \phi_t, \quad v_0 = -6\delta^2 \phi_x^2, \quad \alpha = \beta = -2 \tag{5.3}$$

For finding the resonances, the full Laurent series.

$$\begin{aligned} u &= u_0 \phi^{-2} + \sum_{j=1}^{\infty} u_j \phi^{j-2}, \\ v &= v_0 \phi^{-2} + \sum_{j=1}^{\infty} v_j \phi^{j-2}, \end{aligned} \tag{5.4}$$

is substituted into system (5.1) and by equating the coefficients of  $\phi^{j-5}$ , the polynomial equation in  $j$  is found as:

$$j^3 - 9j^2 + 14j + 24 = 0 \tag{5.5}$$

Using Eq. (5.5), the resonances are found to be

$$j = -1, 4, 6 \tag{5.6}$$

As usual, the resonance at  $j = -1$  corresponds to the arbitrariness of singular manifold  $\phi(x, t) = 0$ . In order to check the existence of sufficient number of arbitrary functions at the other resonance values, the full Laurent expansion (5.4) is substituted in system (5.1). From the coefficient of  $\phi^{-5}$ , the explicit value of  $u_0, v_0$  is obtained as given in Eq. (5.3).

To construct the Bäcklund transformation of system (5.1), let us truncate the Laurent series at the constant level term to give

$$u = u_0 \phi^{-2} + u_1 \phi^{-1} + u_2, \quad v = v_0 \phi^{-2} + v_1 \phi^{-1} + v_2, \tag{5.7}$$

Hence

$$u = \frac{12\varepsilon^2 \phi_x \phi_t}{\phi^2} + \frac{u_1}{\phi} + u_2, \quad v = \frac{-6\delta^2 \phi_x^2}{\phi^2} + \frac{v_1}{\phi} + v_2, \tag{5.8}$$

where the pair of function  $(u, v)$  and  $(u_1, v_1)$  satisfy system (5.1), while  $u_2, v_2$  satisfy respectively the following equations:

$$u_2 = \frac{1}{24\varepsilon^2\phi_x^2\phi_t} (2\varepsilon^2u_{0xx}\phi_t + 24\varepsilon^4\phi_x\phi_t\phi_{txx} - 4\varepsilon^2u_{1x}\phi_x\phi_t - 4\varepsilon^2u_{1\phi_x}\phi_{tx} + 4\varepsilon^2u_{0x}\phi_{tx} + 2\varepsilon^2u_{0t}\phi_{xx} - 2\varepsilon^2u_{1\phi_{xx}}\phi_t - 2\varepsilon^2u_{1t}\phi_x^2 + 4\varepsilon^2u_{0xx}\phi_x - 24\varepsilon^2\phi_x\phi_t^2 + 12\varepsilon^2\phi_x\phi_tu_{1x} + u_1u_{0x} - u_1^2\phi_x + 12\delta^2\phi_x^3),$$

$$v_2 = \frac{-1}{24\varepsilon^2\phi_x^2\phi_t} (24\varepsilon^2\phi_x^2\phi_t + 24\delta^2\varepsilon^2\phi_x\phi_t\phi_{xxx} + 6\delta^2u_{0x}\phi_{xx} + 6\delta^2u_{0xx}\phi_x - 6\delta^2u_{1\phi_x}\phi_{xx} - 6\delta^2u_{1x}\phi_x^2 - 12\delta^2\phi_x^3u_2 + 2\phi_xv_{1u_1} - v_{1u_{0x}} - v_{0x}u_1 + 6\delta^2\phi_x^2u_{1x} - 12\varepsilon^2\phi_x\phi_tv_{1x} - 12\delta^2\phi_x^2\phi_t),$$

and hence Eq. (5.8) is the associated Bäcklund transformation of system (5.1).

We can also construct another Bäcklund transformation of system (5.1) to be

$$u = \frac{u_0}{\phi^2} + \frac{u_1}{\phi} + u_2 + u_3\phi + u_4\phi^2, \quad v = \frac{v_0}{\phi^2} + \frac{v_1}{\phi} + v_2 + v_3\phi + v_4\phi^2,$$

where the pair of function  $(u, v)$  and  $(u_4, v_4)$  satisfy system (5.1) while  $u_2$  and  $v_2$  is given by Eq. (5.9) and Eq. (5.10) respectively but  $u_3$  and  $v_3$  are given respectively by the following equations:

$$u_3 = \frac{1}{12\varepsilon^2\phi_x^2\phi_t} (u_{0t} - u_1\phi_t + \varepsilon^2u_{1t}\phi_{xx} + \varepsilon^2u_{1\phi_{xx}} + 2\varepsilon^2u_{1x}\phi_{tx} - \varepsilon^2u_{0txx} + \varepsilon^2u_{1xx}\phi_t + 2\varepsilon^2u_{1tx}\phi_x + v_{0x} - v_{1\phi_x} + u_1u_{1x} + u_2u_{0x} - u_2u_{1\phi_x} + 12\varepsilon^2\phi_x\phi_tu_{2x}),$$

$$v_3 = \frac{1}{12\varepsilon^2\phi_x^2\phi_t} (-u_1\phi_x - 3\delta^2u_{1x}\phi_{xx} + \delta^2u_{0xxx} - 3\delta^2u_{1xx}\phi_x - \delta^2u_{1\phi_{xxx}} + v_{0t} - v_{1\phi_t} + v_{0x}u_2 - 6\delta^2\phi_x^2u_{2x} + v_{1x}u_1 + v_{1u_{1x}} + 12\varepsilon^2\phi_x\phi_tv_{2x} + v_2u_{0x} - \phi_xv_{1u_2} + 6\delta^2\phi_x^3u_3 - \phi_xv_{2u_1} + u_{0x}),$$

hence Eq. (5.11) is a Bäcklund transformation too.

## VI. Conclusion

In this paper, the Painlevé test has been successfully applied to obtain new Bäcklund transformation of some of the most important nonlinear partial differential equations that have many applications in different fields. Thus the Painlevé test can be applied to different nonlinear partial differential equations to obtain new Bäcklund transformation of these partial differential equations.

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