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Angle Trisection by Straightedge and Compass Only

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Abstract: *For many centuries since ancient times the angle trisection has remained one of the three geometric problems that could not have a clear solution by straightedge and compass. Should such a solution had been, no one would have spent time on it. Moreover, no need would have been to prove that its solution is impossible due to the third degree equation. No need would have been to prove that it is impossible to construct a great number of regular polygons which now by this solution is absolutely possible. Furthermore, this solution by putting an end to all this, should also make us wonder whether Algebra can substitute Geometry especially when in an equation exist arithmetic factors behind which, a possibly different structure of it, is well hidden.*

I.The Solution

Let BOE be an angle greater than or equal to 0 degrees and less than or equal to 180 degrees, with sides, the equal line segments OB and OE (Figures 1, 2).

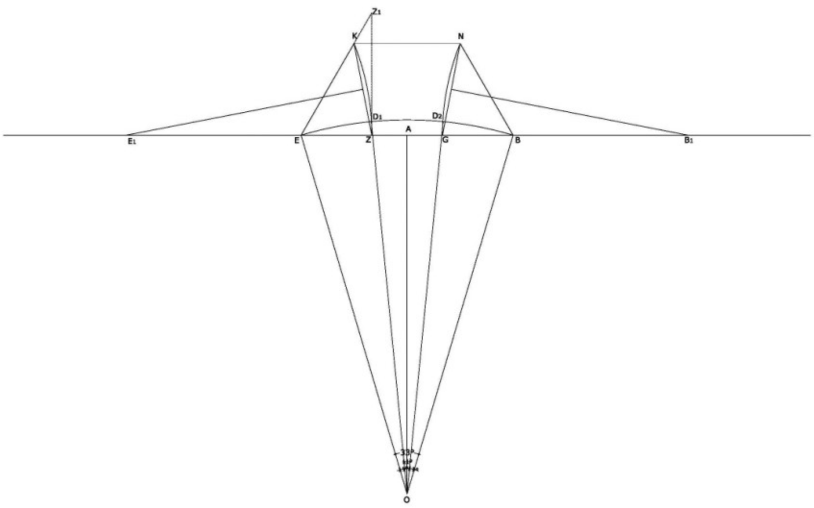

Figure 1

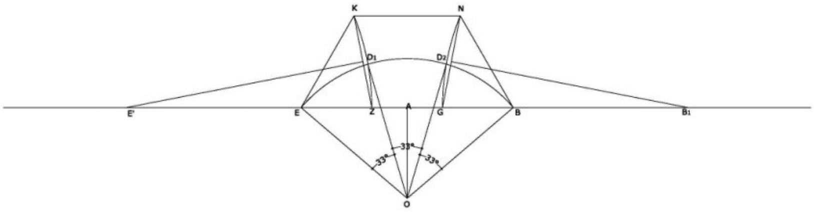

Figure 2

With center of the circle the point O and radius OE, we write the arc EB and EB chord, the latter being
divided into two equal segments EA and AB and into three equal segments EZ, ZG and GB.

With center A and radius AB we write a circle and the equilateral triangles KEA and NAB and thus, half upper EKNB regular hexagon inscribed in circle (A,AB) (Figure 1).

From the middle of the line segments KZ and NG we raise perpendicular lines to either side extensions of the line segment EB which will intersect it at points E1 and B1.

With center E1 and radius E1Z we write circle and respectively with center B1 and

radius B1G we write again a circle; the circumferences of these circles will intersect the circle circumference (A,AB) at points K and N which are the vertices of the regular hexagon, whose upper half part is shown in Figure 1. The points of intersection of these two circles (E1,E1Z) and (B1,B1G) to the arc EB of the circle (O,OE) are the points D1
and D2.

We will prove that the chords ED1, D1D2 and D2B included in the arc EB of the circle (O, OE) are equal and therefore, the corresponding arcs ED1, D1D2 and D2B are equal. Consequently the angles D1OE, D2OD1and BOD2 are equal, that is, random angle BOE has been trisected.

II.Way of Proving

We will prove that the equal sides EK, KN and NB of the regular hexagon, compared one by one and in order with the segments ED1, D1D2 and D2B (that is EK with ED1, KN with D1D2 and NB with D2B) are reduced by the same segment length ie that ED1 = D1D2 = D2B.

But since D1, D2 and B are points of the same arc EB of the circle (O, OE) and ED1, D1D2 and D2B are chords of arcs of the same circle, then respectively the angles BOD2, D2OD1 and D1OE will also be equal, as going through equal arcs.

2.1 Steps of Proof

1. We prove that K is the midpoint of the segment S1Z1.
2. We prove that the triangles Z1F X1, SS1V are equal so Z1F = SS1 and therefore because the first step proved that K is the midpoint of S1Z1, it becomes clear that SK = KF and SK = SY1 = DD1.
3. Because ES = ED1 and because ES = (EK) − (SK) where SK represents the reduction of the length of the side of the original regular hexagon in order to be ED1 = ES, we prove that ED1 = D1D2 and that D1D2 is equal to KN (which is equal to EK) reduced by the length (MD1) + (D2M1) = DD1, i.e. ED1 = D1D2.

III.Proof

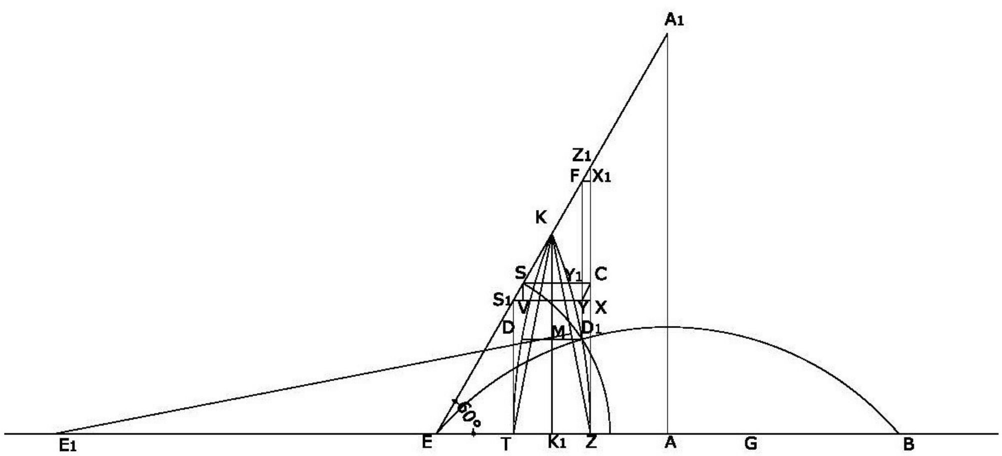

Figure 3

In Figure 3 we see the left part of the upper hexagon EKNB which we created.

Whatever we will prove about the random point D1 on the arc KZ, of the circumference of circle (E1,E1Z),

similarly applies for its symmetric arc NG to the right of line AA1.

From point Z rises a perpendicular that will intersect the upward extension of side EK at point Z1.This

creates the right triangle ZZ1E with a 60° degree angle at the vertex E and a 30° degree angle at the vertex

Z1. So, side EZ = 1/2 (Z1E) and S1 the midpoint of Z1E. With center the point E and radius

EZ we write the arc ZS1 and from point S1 we draw a parallel line to EZ which will intersect ZZ1 at point X. In the right triangle XZ1S1, the side S1X = 1/2 (Z1S1). (Angle of 60° degrees at vertex S1)
By definition of the line segments EA and EZ, their difference is equal to 1/6 of EB. But since EK = EA and EZ = ES1, the segment S1K = 1/6 (EB). In addition EZ = 1/2 (EZ1) = ES1 = S1Z1 and by definition EZ = 1/3 (EB) so S1Z1 = 2/6 (EB). However because S1K = 1/6 (EB), the point K is the midpoint of S1Z1.

From the point K we bring a perpendicular to EZ which will intersect it at point K1.

From the point S1 also a perpendicular to EZ which will intersect it at point T.
The segments TK1 and K1Z are equal because S1K and KZ1 are equal between the parallel lines S1T, KK1 and Z1Z. From the point S1 we bring a perpendicular to ZZ1 which it will intersect at point X.

We create the arc KT symmetrical to the arc KZ as to the axis KK1. Any any point D1 is equidistant with its symmetric D, from the axis KK1 and the two symmetrical points are also equidistant from the perpendiculars to EZ, i.e. Z1Z and S1T. That is, points D and D1 equidistant from S1T and ZZ1 are on a straight line, parallel to EA.
From point D1 we rise a perpendicular to intersect S1X at point Y and extended upwardly it will intersect EZ1 at point F. Any point D1 of the arc KZ of the circle (E1, E1Z) is one and only one fixed length of distance D1Y from YX and one and only one fixed length of distance D1L from ZZ1 and vice versa, i.e. the size of D1L implies the size D1Y and vice versa. The same applies to the corresponding points symmetrical to the axis KK1.
We will prove that SK = SY1 = DD1.

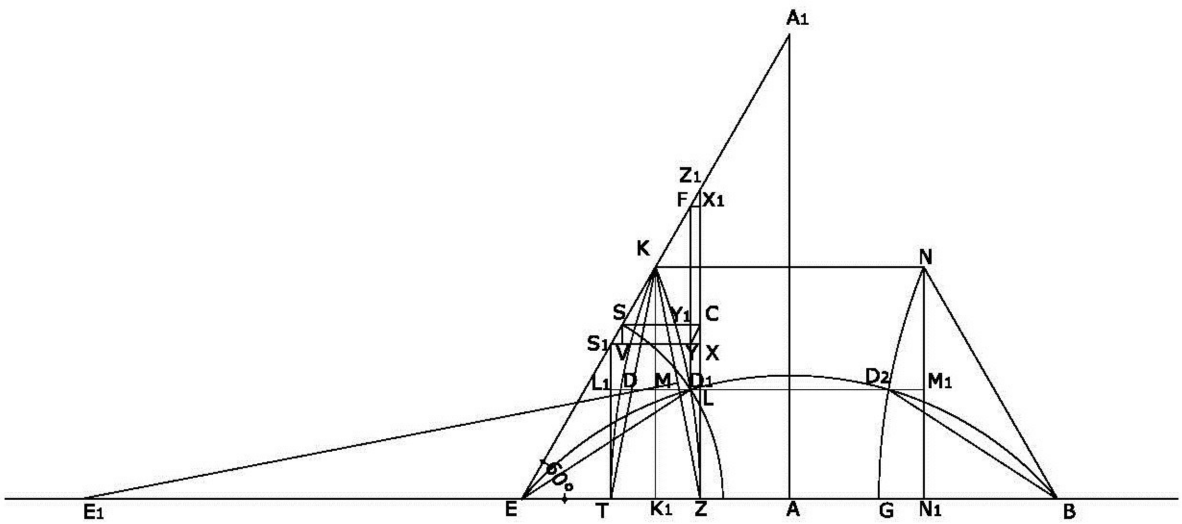

Figure 4

We join the vertex K of the regular hexagon with the point A to form the equilateral triangle KEA (Figure 4). The line KA will intersect D1F at the point Y1. From Y1 we draw a parallel to YX, which will intersect KA at point C. This creates the rectangle YXCY1 with diagonal Y1X

From the point D1 we draw a parallel to EZ and perpendicular to ZZ1, which will intersect segments S1T,

KK1 and ZZ1 respectively at the points L1, M and L. It will also intersect the arc KT at point D.
Because in the right triangle XZ1S1 the angles of vertices Z1 and S1 are 30° degrees and 60° degrees respectively and K is the midpoint of S1Z1, the triangle .KXZ1 is isosceles with base angles of 30° degrees each. So, the right triangles CY1X and X1Z1F will be equal, as similar with equal vertical sides F X1 and Y1C. Respectively, Y1C = Y X = D1L= L1D =S1 V (point D symmetric to D1).And because the right triangles
CY1X and XCY are equal (Y1X and Y C diagonals of the rectangle Y XCY1) due to their similarity and
equality of their vertical sides Y X and S1V and given that the lengths of the line segments D1L, D1Y, L1D
and DV are fixed and unique for point D1 and in relation to it, it follows that:
First, the points S, Y1 and C will lie on a straight line because SV = Y1Y = CX are the continuations of equal straight line segments DV, D1Y and LX.
Second, the triangles X1Z1F and V SS1 are equal as equal to a third party (the triangle XCY) so, S1S = FZ1 and therefore SK = KF .
Third, that the straight line segment SK as a side of the equilateral triangle KSY1 is equal to SY1 and the latter equal to DD1.
Essentially however, SK is the length representing the reduction of the sideEK of the regular hexagon up to point S, which lies on a circle with center point E and radius ED1, i.e. ES = ED1, which means that from the equality ES = (EK) − (SK) and by the replacement of equal lengths we have ED1 = (EK) − (DD1). If from point N of the side KN of the regular hexagon we draw a perpendicular NN1 to EB and extend L1L to intersect NN1 at point M1, then, we make the segment MM1 = KN where M is the midpoint of DD1.
Now, if the same procedure is repeated exactly on the symmetrical with axis AA1, right part of the regular hexagon on the equilateral triangle NAB at first we take the point D2 which is the respective intersection of two arcs as defined in the beginning and the segment D2M1 = 1/2 (DD1) = MD1. But
since MM1 = (MD1) + (D1D2) + (D2M1), it follows that D1D2 = (MM1) − (DD1) = (EK) − (SK) = ED1 which means that the random point D1, taken as defined in the beginning, creates the equal chords ED1 and D1D2 and the respective equal arcs and the respective equal angles D1OE and D2OD1, i.e. the trisection of the random angle BOE has been accomplished.

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