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Dynamics of Allelopathic Two Species Model Having Delay in Predation

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Abstract: *In this paper, we have considered an allelopathic model of two species and discussed the dynamics of the model when the effect of interaction on prey is based on the rate of consumption of the prey by the predator and the rate of release of toxicant by the predator. The interaction of the predator and prey results on the growth of predator after a time interval (τ). It is shown that the time delay can cause a switch from stable state to unstable state and there by Hopf-bifurcation occurs.*

Key words*: Hopf –bifurcation, Prey-Predator, Stability, Time-delay, Toxicant.*

I. Introduction

Lotka [1] and Volterra [2] initiated the research in the field of theoretical ecology. Since then many researchers studied the Predator-Prey or competitive model with Mutualism and commensalism. The dynamical relationship between predators & their prey is one of the important aspect in the population dynamics. There has been great interest in dynamical characteristics like stable, unstable & oscillatory behavior. The problem of harvesting two ecological independent & logistically growing fish species was studied by C.W. clark [3], Brauer and Soudack [4, 5] , Dai & Tang [6] , Myerscough [7].

Allelopathy can be defined as the direct or indirect harmful effect of one species on another through the production of a chemical released into the environment. In recent times many researchers are extensively studying the eco-toxicological effects of toxicants released by the marine biological species among themselves. An example of this kind is the toxicant produced by the Uni celluar green alga chlorella vulgaris. The toxicant limits the size of it's own population and also inhabits growth of the Planktonic algaes, Asterionella Formosa and Nitzschia frustulum.Maynardsmith[8],Chattopadhyay [9] studied a two species Lotka –Volterra competitive system, in which toxic substance produced by one species effects the other.They also studied the stability properties of the system.

Das [10,11] studied & analyzed the harvesting of fish species,as aPrey –Predator model in the presence of a toxin released by some other source. In their work a catch-rate function is defined in place of the usually catch-per unit effort hypothesis for the problem of non-selective harvesting. Kar and Chaudhuri(9) proposed a model for two competing fish species in the presence of toxin and Combined harvesting of the species, keeping in view the Maynard smith conjecture. As Maynard smith's conjecture is valid for the large classes of marine species, the model proposed by Kar and Chaudhuri (9) is applicable to large class of other marine species. R.P.Gupta etal . [13] extended the models proposed by Kar and Chaudari [12] for any two populations having competition and harvested by different agencies with different harvesting efforts.They discussed the existence of two saddle-node bifurcations using Sotomayor's theorem.

In nature there are large classes of marine species other than fish such as Algae and bacterial that produce toxic substance which effect the other species while competing for food [14]. But in some cases the toxic substance released by one species may not effect the other species immediately, but with some delay in time.

In this paper we propose a two species Prey-Predator model in which the two species having densities $z_1(t), z_2(t)$ are harvested by different agencies with harvesting efforts H_1 & H_2 respectively and the corresponding catchability coefficients of the two species being C_1, C_2. In this proposed dynamical model the two species obey law of logistic growth with intrinsic growth rates r, s and have carrying capacities K, L .Both the species release toxic substances which affect the other, the toxic coefficients of prey & predator are respectively η_1, η_2 .The predator population has food source other than the prey. The effect of interaction of the species on prey is based on the rate of consumption of the prey by the predator and the rate of release of toxicant by the predator. The net rate of effect of interaction on prey is denoted by a_1. Similarly the effect of interaction of the species on predator is based on the rate of its predation and the rate of release of toxicant by the prey. The net rate of effect of interaction on predator is denoted by a_2. All these parameters are assumed to be positive.

It is well known that in some prey-predator systems the rate of change in the predator depends on numbers of prey and predator at time t and in some other systems the rate of change in the predator depends on prey & predators population present at some previous times say(t- τ) .In this paper we studied the changes occurred in stability of the dynamical system when a delay (τ) is incorporated in predation term. The theoretical results are validated by the numerical simulations.

II. Mathematical Model

The mathematical formulation of the toxicant prey-predator dynamical problem with different harvesting efforts is

$$\left.\begin{aligned} \frac{dz_1}{dt} &= rz_1\left(1-\frac{z_1}{K}\right)-a_1z_1z_2-\eta_1z_1^2z_2-C_1H_1z_1 \\ \frac{dz_2}{dt} &= sz_2\left(1-\frac{z_2}{L}\right)+a_2z_1(t)z_2(t)-\eta_2z_1z_2^2-C_2H_2z_2 \end{aligned}\right\} -----(2.1.1)$$

It is well known that in some prey-predator systems the rate of change in the predator depends on numbers of prey and predator at time t and in some other systems the rate of change in the predator depends on prey & predators population present at some previous times say (t- τ).By incorporating time delay τ in predation term,the equation (2.1.1) becomes

$$\left.\begin{aligned} \frac{dz_1}{dt} &= rz_1\left(1-\frac{z_1}{K}\right)-a_1z_1z_2-\eta_1z_1^2z_2-C_1H_1z_1 \\ \frac{dz_2}{dt} &= sz_2\left(1-\frac{z_2}{L}\right)+a_2z_1(t-\tau)z_2(t-\tau)-\eta_2z_1z_2^2-C_2H_2z_2 \end{aligned}\right\} -----(2.1.2)$$

From biological point of view we only interested on the interior equilibrium E (z_1^*, z_2^*).

Let $Z_1 = z_1 - z_1^*, Z_2 = z_2 - z_2^*$ be the perturbed variables

After removing the non-linear terms we obtain linearaized system corresponding to (2.1.2) is

$$\left.\begin{aligned} \frac{dZ_1}{dt} &= \left[r-\frac{2r}{K}z^*_1-a_1z^*_2-2\eta_1z_1^*z_2^*-c_1H_1\right]Z_1+\left[-a_1z_1^*-\eta_1z^{*2}_1\right]Z_2 \\ \frac{dZ_2}{dt} &= \left[-\eta_2z^{*2}_2+a_2e^{-\mu\tau}z^*_2\right]Z_1+\left[s-\frac{2s}{L}z^*_2-2\eta_2z^*_1z^*_2-c_2H_2+a_2e^{-\mu\tau}z^*_1\right]Z_2 \end{aligned}\right\} -----(2.1.3)$$

The characteristic equation of the linear system is given by

$$\Delta(\mu,\tau) = X(\mu) + Y(\mu)e^{-\mu\tau} = 0 \qquad ----- \quad \textbf{(2.1.4)}$$

Where $X(\mu) = \mu^2+P\mu + R$,$Y(\mu) = Q\mu + S$

P = - A – D	A = $r-\frac{2r}{K}z_1^*-a_1z_2^*-2\eta_1z_1^*z_2^*-c_1H_1$
Q = $-a_2z_1^*$	B = $-a_1z_1^*-\eta_1z_1^{*2}$
R = AD – BC	C = $-\eta_2z^{*2}_2$
S = $(Az_1^*-Bz_2^*)a_2$	D = $s-\frac{2s}{L}z_2^*-2\eta_2z_1^*z_2^*-c_2H_2$

We discuss the stability of the system at the interior equilibrium E(z_1^*, z_2^*).

III. Stability Analysis

Case 1: when $\tau = 0$

In the absence of discrete time delay we investigate the stability of the system (2.1.2) around the interior equilibrium (E) . The system (2.1.2) becomes a system of ordinary differential Equations (2.1.1),then the corresponding characteristic equation is

X (μ) + Y(μ) = 0.

$$\text{i. e, } (\mu^2+P\mu+R) + (Q\mu+S) = 0$$

$$\mu^2 + (P+Q)\,\mu + (R+S) = 0 \qquad \text{-----} \quad \mathbf{(3.1.1)}$$

Sum of the roots = **-** (P+Q) < 0

Product of the roots = R+S

R+S > 0 if $\dfrac{a_1 L}{s} < \dfrac{\eta_1}{\eta_2} < \dfrac{r}{K a_2}$

We can say that both the roots of (3.1.1) are real & negative (or) complex conjugate with negative real part iff P+Q > 0 & R+S > 0 . ----- **(3.1.2)**

Hence, in the absence of time delay, the system is locally asymptotically stable when $\dfrac{a_1 L}{s} < \dfrac{\eta_1}{\eta_2} < \dfrac{r}{K a_2}$ is valid.

Theorem 3.1: In the absence of the delay, the system (2.1.1) is locally asymptotically stable at the $E\left(z_1^*, z_2^*\right)$)

iff $\dfrac{a_1 L}{s} < \dfrac{\eta_1}{\eta_2} < \dfrac{r}{K a_2}$

We augment our analytical findings through numerical simulations by the following example**.**

Example 1

The parameters in the model (2.1.1) are taken as

r =4, s=5, a_1 =0.1, a_2 =0.9, η_1 =0.008 , η_2 =0.005, c_1 =0.1 , c_2 =0.2, H_1 =30, H_2 =40,K=500,L=400.

Initial values of the species are $z_1 = 10, z_2 = 5$ and τ = 0

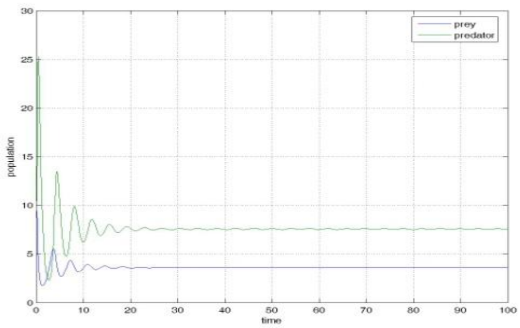

Fig.1 show the variation of populations against time.

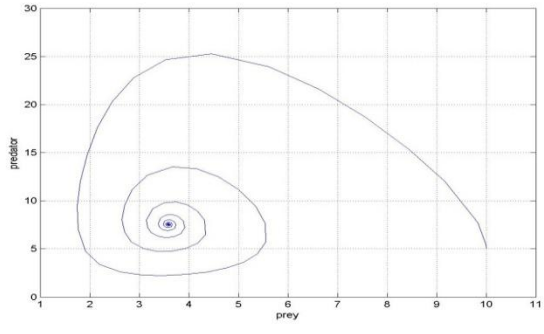

Fig.2 show the phase- portraits of prey and predator

Case 2: when $\tau > 0$,

Let $\mu(\tau) = \alpha(\tau) + i\theta(\tau)$ be a root of the characteristic equation (2.1.4).

Let τ be a particular value of the delay such that $\alpha(\tau)=0$, $\theta(\tau)>0$

Put $\mu = i\theta$ in (2.1.4) we get

$[(i\,\theta)^2 + P(i\,\theta)+R] + [Q\,(i\,\theta) + s]\, e^{-i\theta\tau} = 0$

$i^2\,\theta^2 + P\,(i\,\theta) + R + [Q(\,i\,\theta)+S\,]\{\cos\theta\tau - i\sin\theta\tau\} = 0$

$-\theta^2 + p\,(i\,\theta) + R + [\,iQ\,\theta + S\,]\{\cos\theta\tau - i\sin\theta\tau\}=0$

Separating the real & imaginary parts, we get

$\theta^2 - R = S\cos\theta\tau + Q\,\theta\sin\theta\tau$ ----- **(3.1.3)**

$P\,\theta = S\sin\theta\tau - Q\,\theta\cos\theta\tau$

Squaring & adding these two equations, we get the fourth order equation

$\theta^4 + \theta^2(P^2-2R-Q^2) + (R^2-S^2) = 0$ ----- **(3.1.4)**

Sub case 1: If $P^2 - 2R - Q^2 > 0\, and\, R^2 - S^2 > 0$ then the equation (3.1.4) does not have any real solutions. Hence this case is omitted as $\theta(\tau)$ is a real number.

Sub case 2: If $P^2 - 2R - Q^2 > 0$ and $R^2 - S^2 < 0$ then the equation (3.1.4) have a unique positive root, it is θ_0^2 and let the corresponding τ be τ_0 .

Sub case 3: If $P^2 - 2R - Q^2 < 0,\ R^2 - S^2 > 0\ and\ \left(P^2 - 2R - Q^2\right)^2 - 4\left(R^2 - S^2\right) > 0$ then the equation (3.1.4) have two positive roots. Let them be $\theta_\pm^2$ and the corresponding τ be $\tau_n^\pm$.

The above said positive roots, either from sub case 2 or from sub case 3,satisfy all the equations from (3.1.3) to (3.1.4).

Eliminating $\sin\theta\,\tau$ from (3.1.3), we get

$$\tau_k = \frac{1}{\theta}\cos^{-1}\left[\frac{\theta^2(S - PQ) - RS}{S^2 + Q^2\theta^2}\right] + \frac{2k\pi}{\theta}\, where k = 0,1,2.... \qquad \text{------} \quad \mathbf{(3.1.5)}$$

Now differentiating equation (2.1.4) w.r.t τ, we obtain

$$\left[2\mu + P + Qe^{-\mu\tau} - \tau(Q\mu + S)e^{-\mu\tau}\right]\frac{d\mu}{d\tau} = \mu e^{-\mu\tau}(Q\mu + S) \qquad \text{-----} \quad \mathbf{(3.1.6)}$$

$$\left[\frac{d\mu}{d\tau}\right]^{-1} = \frac{2\mu + P}{-\mu(\mu^2 + P\mu + R)} + \frac{Q}{\mu(Q\mu + S)} - \frac{\tau}{\mu} \qquad \text{-----} \quad \mathbf{(3.1.7)}$$

$$\mathrm{Re}\left[\frac{d\mu}{d\tau}\right]^{-1} = \frac{2(\theta^2 - R) + P^2}{\theta^4 + (P^2 - 2R)\theta^2 + R^2} - \frac{Q^2}{S^2 + Q^2\theta^2}$$

Thus,

$$Sign\left[\frac{d}{d\tau}(\mathrm{Re}\,\mu)\right] = Sign\left[\mathrm{Re}\left(\frac{d\mu}{d\tau}\right)^{-1}\right]_{\mu = i\theta}$$

$$= Sign\left[\frac{2\theta^2 + (P^2 - 2R - Q^2)}{Q^2\theta^2 + S^2}\right] \qquad \text{-----} \quad \mathbf{(3.1.8)}$$

Note that the θ may be $\theta_0\, or\, \theta_\pm$

Theorem 3.2: In the presence of the delay, the system (2.1.2) is locally asymptotically stable at $E\left(z_1^*, z_2^*\right)$ iff $R^2 - S^2 < 0$ for all $\tau < \tau_0$ .It is unstable for all $\tau > \tau_0$ and hopf - bifurcation occurs at $\tau = \tau_0$

Proof:

From equation (3.1.8) we have

$$Sign\left[\frac{d}{d\tau}(\mathrm{Re}\,\mu)\right]=Sign\left[\mathrm{Re}(\frac{d\mu}{d\tau})^{-1}\right]_{\mu=i\theta_0}=sign\left[\frac{2\theta_0^2+(P^2-2R-S^2)}{Q^2\theta_0^2+S^2}\right]$$

$$sign\left[\mathrm{Re}\left(\frac{d\mu}{d\tau}\right)\right]_{\mu=i\theta_0}=sign\left[\mathrm{Re}\left(\frac{d\mu}{d\tau}\right)^{-1}\right]_{\theta=\theta_0,\tau=\tau_0}>0$$

This signify that there exits Eigen values with negative real part for $\tau<\tau_0$ and there exits Eigen values with positive real part for $\tau>\tau_0$.Therefore, the transversality condition holds and hence hopf-bifurcation occurs at $\theta=\theta_0,\tau=\tau_0$

Theorem 3.3: In the presence of delay, the system is locally asymptotically stable at $E\left(z_1^*,z_2^*\right)$ iff

$$P^2-2R-Q^2<0,\ R^2-S^2>0 and$$

$$\left(P^2-2R-Q^2\right)^2-4\left(R^2-S^2\right)>0\quad forall\ \tau\varepsilon\left[0,\tau_0^+\right)\bigcup\left(\tau_0^-,\tau_1^+\right)\bigcup...\bigcup\left(\tau_{m-1}^-,\tau_m^+\right).$$

It is unstable $forall\ \tau\ \varepsilon\left[\tau_0^+,\tau_0^-\right)\bigcup\left(\tau_1^+,\tau_1^-\right)\bigcup...\bigcup\left(\tau_{m-1,}^+\tau_m^-\right)$ for some positive integer m.

Proof:

From (3.1.8) it follows that

$$sign\left\{\left[\frac{d}{d\tau}(\mathrm{Re}\,\mu)\right]_{\mu=i\theta_+}\right\}=sign\left\{+\frac{\sqrt{(\mathrm{P}^2-2R-Q^2)^2-4(R^2-S^2)}}{\left\{(-\theta_+^2+R)^2+P^2\theta_+^2\right\}\left\{S^2+Q^2\theta_+^2\right\}}\right\}$$

Therefore, $\left\{\frac{d}{d\tau}(\mathrm{Re}\,\mu)\right\}_{\theta=\theta_+,\tau=\tau_n^+}>0$.

Again, $sign\left\{\left[\frac{d}{d\tau}(\mathrm{Re}\,\mu)\right]_{\mu=i\theta-}\right\}=sign\left\{-\frac{\sqrt{(\mathrm{P}^2-2R-Q^2)^2-4(R^2-S^2)}}{\left\{(-\theta_-^2+R)^2+P^2\theta_-^2\right\}\left\{S^2+Q^2\theta_+^2\right\}}\right\}$

Therefore, $\left\{\frac{d}{d\tau}(\mathrm{Re}\,\mu)\right\}_{\theta=\theta_-,\tau=\tau_n^-}<0$.

Hence we have $\left[\frac{d}{d\tau}(\mathrm{Re}\,\mu)\right]_{\tau=\tau_n^+}>0$ and $\left[\frac{d}{d\tau}(\mathrm{Re}\,\mu)\right]_{\tau=\tau_n^-}<0$ Hence , the transversality conditions are satisfied . This completes the proof.

We augment our analytical findings through numerical simulations by the following examples.

Example 2

The parameters of the model (2.1.2) are taken as

r =4, s=5, a_1=0.1, a_2=0.9, η_1=0.008 , η_2=0.005, c_1=0.1 , c_2=0.2, H_1=30, H_2=40,K=500,L=400.

Initial values of the species are $z_1=10, z_2=5$ and τ =0.25< $\tau_0=0.29$

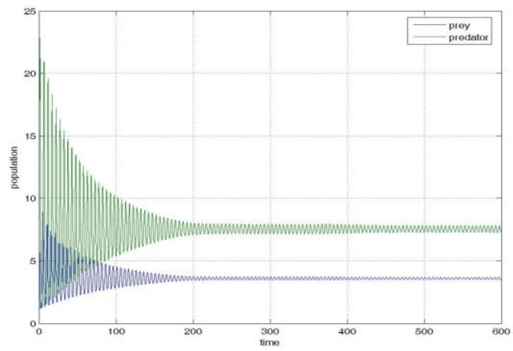

Fig.3 show the stable variation of populations against time

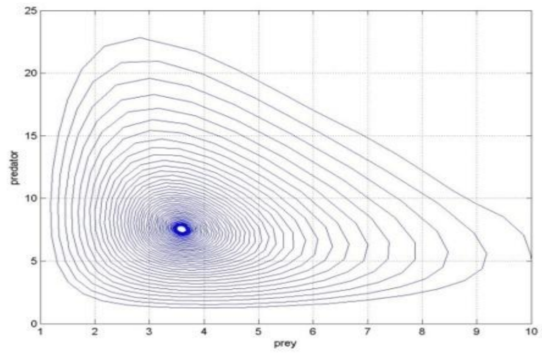

Fig.4 show the phase- portraits of prey and predator

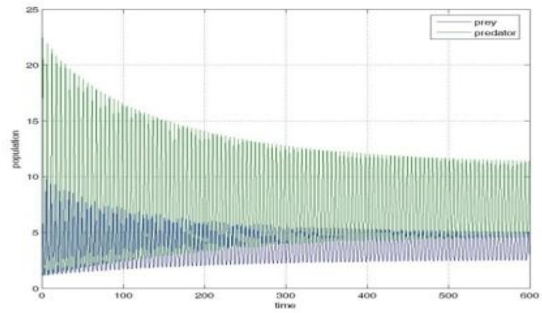

Fig.5 show the unstable variation of populations against time when $\tau = \tau_0 = 0.29$

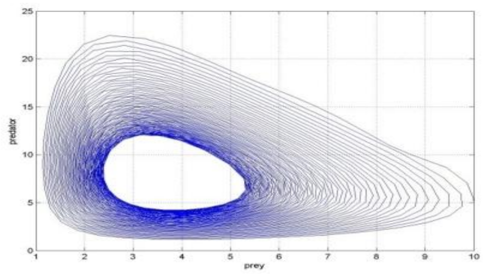

Fig.6 show the phase- portraits of prey and predator $\tau = \tau_0 = 0.29$

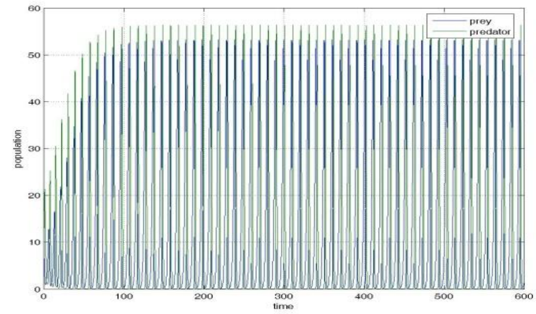

Fig.7 show the unstable variation of populations against time When τ = 0.5 > 0.29

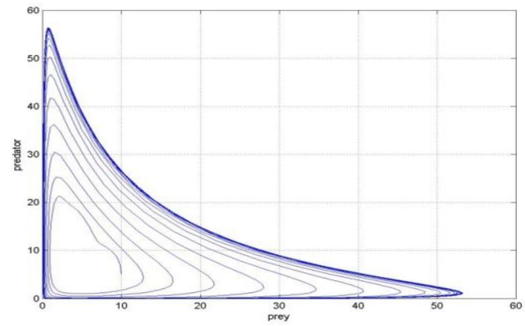

Fig.8: show the Phase portraits of prey and predator when τ = 0.5 > 0.29

IV. Concluding Remarks

The model discussed here has the following characteristics

(a) The prey population decreases not only by predation but also by the toxic substance released by the predator.

(b) The predator population also decreases by the toxic substance released by the prey population. Hence the model has not only the prey-predator relationship but also the two species have effects of toxicants released by them on each other.

It is observed that delay of all dimensions does not induce any instability; the delay of certain dimensions can induce instability oscillation via Hopf bifurcation. It is also observed that switching of stability occurs. The effect of delay incorporated in the predation term is clearly studied, and the theoretical results obtained are validated through the numerical simulations using MATLAB.

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