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Oscillation Theorems For Second Order Neutral Difference Equations

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Abstract: *In this paper new oscillation criteria for the second order neutral difference equation of the form*

$$\Delta\left[r(n)\left(\Delta\left(x(n)+p(n)x(\tau(n))\right)\right)\right]+q(n)x(\sigma(n))+v(n)x(\eta(n))=0$$

are presented. Gained results are based on the new comparison theorems, that enable us to reduce the problem of the oscillation of the second order equation to the oscillation of the first order equation. Obtained comparison principles essentially simplify the examination of the studied equations. We cover all possible cases when arguments are delayed, advanced or mixed.

I. Introduction

This paper is concerned with the oscillation behavior of the solutions of the second order neutral difference equation

$$\Delta\left[r(n)\left(\Delta\left(x(n)+p(n)x(\tau(n))\right)\right)\right]+q(n)x(\sigma(n))+v(n)x(\eta(n))=0, \qquad (E)$$

where Δ is the forward difference operator defined by $\Delta x(n)=x(n+1)-x(n)$ and

$N(n_0)=\{n_0,n_0+1,n_0+2,.......\}$.

The following conditions are assumed to be hold.

- $r(n)>0, q(n)>0, v(n)>0$ *and* $0\le p(n)\le p_0<\infty$;
- $\{\sigma(n)\}$ *and* $\{\eta(n)\}$ are sequences of positive integers such that $\lim_{n\to\infty}\sigma(n)=\infty$ *and* $\lim_{n\to\infty}\eta(n)=\infty$;
- $\{\tau(n)\}$ is a sequence of positive integers and $\tau\circ\sigma=\sigma\circ\tau$ *and* $\tau\circ\eta=\eta\circ\tau$.

Throughout this paper we shall assume that

$$R(n)=\sum_{n_0}^{n-1}\frac{1}{r(n)}\to\infty \text{ as } n\to\infty. \qquad (1)$$

We set $z(n)=x(n)+p(n)x(\tau(n))$.

By a solution of equation (E), we mean a nontrivial sequence $\{x(n)\}$ satisfying the equation (E) for all $n\in N(n_0)$ where n_0 is some nonnegative integer. A solution $\{x(n)\}$ is said to be oscillatory if it is eventually positive nor eventually negative and it is nonoscillatory otherwise. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

Since the second order equations have the applied applications there is the permanent interest in obtaining new sufficient conditions for the oscillation or nonoscillation of the solutions of varietal types of the second order equations. We refer the reader to the papers [2,3,4,5,7,9,11,12,13,14,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32] and the books [1,6,15] and the references cited therein. The Authors mainly studied delay equations.

S.R.Grace et.al. [8] considered the non neutral equations

$$\Delta(C_{n-1}\Delta x_{n-1}) + q_n x_n = 0 \ \ and \ \ \Delta(C_{n-1}\Delta x_{n-1}) + q_n x_n = t_n$$

and obtained some new oscillation criteria which are discrete analogues of some known results in the continuous case.

Zhang and Zhou [33] considered the second order equation

$$\Delta^2 x_{n-1} + p_n x_n = 0$$

and established some new oscillation theorems.

Remark 1. The conditions $\tau \circ \sigma = \sigma \circ \tau \ \ and \ \ \tau \circ \eta = \eta \circ \tau$ .contained in the hypothesis (H_3) are satisfied for instance if $\tau(n), \sigma(n)$ and $\eta(n)$ are of the same form that is if e.g., $\tau(n) = n - \tau$, then at the same time $\sigma(n) = n - \tau$ and $\eta(n) = n - \eta$.

Remark 2. All the functional inequalities considered in this paper are assumed to hold eventually, that is they are satisfied for all n large enough.

Remark 3. Without loss of generality we can deal only with the positive solutions of (E).

II. Main Results

It follows from (1) that the positive solutions of (E) have the following property

Lemma 2.1 *If* $x(n)$ *is a positive solution of (E), then the corresponding function* $z(n) = x(n) + p(n)x(\tau(n))$ satisfies $z(n) > 0, \ r(n)\Delta z(n) > 0, \ \Delta(r(n)\Delta z(n)) < 0,$ eventually.

Proof. Assume that $x(n)$ is a positive solution of (E). Then it follows from (E) that

$$\Delta(r(n)\Delta z(n)) = -q(n)x(\sigma(n)) - v(n)x(\eta(n)) < 0.$$

Consequently, $r(n)\Delta z(n)$ is decreasing and thus either $\Delta z(n) > 0 \ or \ \Delta z(n) < 0$ eventually. If we let $\Delta z(n) < 0,$ then also $r(n)\Delta z(n) < -c < 0$ and summing this from n_1 to $n-1$, we obtain

$$z(n) \le z(n_1) - c\sum_{n_1}^{n-1} \frac{1}{r(s)} \to -\infty \ \ as \ \ n \to \infty.$$

This contradicts the positivity of $z(n)$ and the proof is complete.

For our intended references, let us denote

$$Q(n) = min\{q(n), q(\tau(n))\}, \ V(n) = min\{v(n), v(\tau(n))\} \tag{2}$$

and

$$Q_1(n) = Q(n)(R(\sigma(n)) - R(n_1)), \ V_1(n) = V(n)(R(\eta(n)) - R(n_1)), \tag{3}$$

where $n \ge n_1$ and n_1 is large enough.

Theorem 2.2 *Assume that the first order neutral difference inequality*

$$\Delta(y(n) + p_0 y(\tau(n))) + Q_1(n)y(\sigma(n)) + V_1(n)y(\eta(n)) \le 0 \tag{E_2}$$

has no positive solution. Then (E) is oscillatory.

Proof. Assume that $x(n)$ is a positive solution of (E). Then the corresponding function $z(n)$ satisfies

$$\begin{aligned} z(\sigma(n)) &= x(\sigma(n)) + p(\sigma(n))x(\tau(\sigma(n))) \\ &\le x(\sigma(n)) + p_0 x(\sigma(\tau(n))), \end{aligned} \tag{4}$$

where we have used hypothesis (H_3) and similarly

$$z(\eta(n)) \le x(\eta(n)) + p_0 x(\eta(\tau(n))). \tag{5}$$

On the other hand, it follows from (E) that

$$\Delta\big(r(n)\Delta(z(n))\big) + q(n)x(\sigma(n)) + v(n)x(\eta(n)) = 0 \tag{6}$$

and moreover taking (H_1) and (H_3) into account, we have

$$0 = p_0\Delta\big(r(\tau(n))\Delta(z(\tau(n)))\big) + p_0 q(\tau(n))x(\sigma(\tau(n))) + p_0 v(n)x(\eta(\tau(n))). \tag{7}$$

Combining (6) and (7), we are led to

$$\Delta\big(r(n)\Delta(z(n))\big) + p_0\Delta\big(r(\tau(n))\Delta z(\tau(n))\big) + q(n)x(\sigma(n)) + p_0 q(\tau(n))x(\sigma(\tau(n)))$$
$$+ v(n)x(\eta(n)) + p_0 v(\tau(n))x(\eta(\tau(n))) = 0.$$

which in view of (4), (5) and (2) provides

$$\Delta\big(r(n)\Delta z(n)\big) + p_0\Delta\big(r(\tau(n))\Delta z(\tau(n))\big) + Q(n)z(\sigma(n)) + V(n)z(\eta(n)) \le 0 \tag{8}$$

It follows from Lemma 2.1 that $y(n) = r(n)\Delta z(n) > 0$ is decreasing and then

$$\begin{aligned} z(n) &\ge \sum_{n_1}^{n-1} \frac{1}{r(s)}\big(r(s)\Delta z(s)\big) \\ &\ge y(n-1)\sum_{n_1}^{n-1}\frac{1}{r(s)} \\ &= y(n)\big(R(n) - R(n_1)\big). \end{aligned} \tag{9}$$

Therefore, setting $r(n)\Delta(z(n)) = y(n)$ in (8) and utilizing (9), one can see that $y(n)$ is a positive solution of (E_2). This contradicts our assumption and the proof is complete.

Remark 4. In the comparison principle, in Theorem 2.2 we do not stipulate whether (E) is equation with delay, advanced or mixed arguments, so that the obtained results are applicable to all three types of equations. Moreover, our results hold also for both cases where $\tau(n) \le n$ or $\tau(n) \ge n$. On the other hand, the comparison theorem established in Theorem 2.2 reduces oscillation of (E) to the research of the first order neutral difference inequality (E_2). Therefore, applying the conditions for (E_2) to have no positive solution, we immediately get oscillation criteria for (E.)

Employing the additional conditions on the coefficients of (E). We can deduce from Theorem 2.2 various oscillation criteria for (E). We shall discuss separately the following two cases

$$r(n) \ge n \tag{10}$$

$$r(n) \le n \tag{11}$$

Theorem 2.3 *Assume that (10) holds. If the first order difference inequality*

$$\Delta w(n) + \frac{1}{1+p_0}Q_1(n)w(\sigma(n)) + \frac{1}{1+p_0}V_1(n)w(\eta(n)) \le 0 \tag{E_3}$$

has no positive solution, then (E) is oscillatory.

Proof. We assume that $x(n)$ is a positive solution of (E). Then Lemma 2.1 and the proof of the Theorem 2.2 imply that $y(n) = r(n)\Delta z(n) > 0$ is decreasing and it satisfies (E_2). Let us denote $w(n) = y(n) + p_0 y(\tau(n))$. It follows from (10) that

$$w(n) \le y(n)(1+p_0).$$

Substituting these terms into (E_2), we get that w(n) is a positive solution of (E_3). A contradiction.

Adding the restriction that both $\sigma(n)$ and $\eta(n)$ are delay arguments, we get easily verifiable oscillation criterion for the delay equation (E).

Corollary 2.4 *Assume that (10) holds and*

$$\sigma(n) = n - \sigma < n, \ \eta(n) = n - \eta < n. \tag{12}$$

If $n - \sigma \le n - \eta$ and also

$$\liminf_{n\to\infty} \sum_{n-\eta}^{n-1} \big(V_1(s) + Q_1(s)\big) > (1 + p_0)\left(\frac{\eta}{\eta+1}\right)^{\eta+1} \tag{13}$$

If $n - \eta \ge n - \sigma$ and also

$$\liminf_{n\to\infty} \sum_{n-\sigma}^{n-1} \big(V_1(s) + Q_1(s)\big) > (1 + p_0)\left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1} \tag{14}$$

then (E) is oscillatory.

Proof. Theorem 2.3 ensures the oscillation of (E) provided that (E_3) has no positive solution. Assume that $w(n)$ is a positive solution of (E_3). Then $w(n)$ is decreasing and if $\sigma(n) \le \eta(n)$, then $w(\sigma(n)) \ge w(\eta(n))$. Setting the last inequality to (E_3), we see that $w(n)$ is a positive solution of the difference inequality

$$\Delta w(n) + \frac{1}{1+p_0}\big(Q_1(n) + V_1(n)\big)w(\eta(n)) \le 0 \tag{E_3^*}$$

But according to Theorem 7.6.1 in [10], the condition (13) guarantees that (E_3^*) has no positive solution. This contradiction finishes the proof of the first part of the corollary. The second part can be verified similarly and so the rest of the proof can be omitted.

For our incoming references, let us denote

$$Q_2(n) = Q(n)\big(R(n) - R(n_1)\big), V_2(n) = V(n)\big(R(n) - R(n_1)\big) \tag{15}$$

where $n > n_1$, n_1 is large enough and $Q(n)$ and $V(n)$ are defined as in (2).

Putting on the constraint that both $\sigma(n)$ and $\eta(n)$ are the advanced arguments, we get the following oscillation criterion for the advanced equation (E).

Theorem 2.5 *Assume that (10) holds and*

$$\sigma(n) > n, \ \eta(n) > n. \tag{16}$$

If the first order advanced difference inequality

$$\Delta w(n) - \frac{1}{1+p_0} Q_2(n) w(\sigma(n)) - \frac{1}{1+p_0} V_2(n) w(\eta(n)) \ge 0 \tag{E_4}$$

has no positive solution, then (E) is oscillatory.

Proof. We assume that $x(n)$ is a positive solution of (E). Then proceeding exactly as in the proof of Theorem 2.2, we verify that the corresponding $z(n)$ satisfies (8). Summing of (8) from n to ∞ provides

$$r(n)\Delta z(n) + p_0 r(\tau(n))\Delta z(\tau(n)) \ge \sum_{n}^{\infty} \big(Q(s) z(\sigma(s)) + V(s) z(\eta(s))\big) \tag{17}$$

Since $r(n)\Delta z(n)$ is decreasing and (10) holds, then

$$r(n)\Delta z(n) + p_0 r(\tau(n))\Delta z(\tau(n)) \le r(n)\Delta z(n)(1 + p_0) \tag{18}$$

Combining (17) together with (18), we are led to

$$r(n)\Delta z(n)(1 + p_0) \ge \sum_{n}^{\infty} \big(Q(s) z(\sigma(s)) + V(s) z(\eta(s))\big) \tag{19}$$

Summing the last inequality from n_1 to $n - 1$, we get

$$z(n) \geq \frac{1}{1+p_0}\sum_{n_1}^{n-1}\frac{1}{r(u)}\sum_{u}^{\infty}\big(Q(s)z(\sigma(s))+V(s)z(\eta(s))\big)$$
$$\geq \frac{1}{1+p_0}\sum_{n_1}^{n-1}\big(Q(s)z(\sigma(s))+V(s)z(\eta(s))\big)\sum_{n_1}^{s}\frac{1}{r(u)}$$

Hence,

$$z(n) \geq \frac{1}{1+p_0}\sum_{n_1}^{n-1}\big(Q(s)z(\sigma(s))+V(s)z(\eta(s))\big) \tag{20}$$

Let us denote the right hand side of (20) by $w(n)$. Since $z(n) \geq w(n)$, we see that $w(n)$ is a positive solution of (E_4). This contradicts our assumption and the proof is complete now.

Corollary 2.6 *Assume that (10) and (16) holds. If, in particular* $\sigma(n) = n+\sigma \leq n+\eta = \eta(n)$ *and also*

$$\liminf_{n\to\infty}\sum_{n+1}^{n+\sigma-1}\big(Q_2(s)+V_2(s)\big) > (1+p_0)\left(\frac{\sigma-1}{\sigma}\right)^{\sigma} \tag{21}$$

and $\sigma(n) \geq \eta(n)$ and also

$$\liminf_{n\to\infty}\sum_{n+1}^{n+\eta-1}\big(Q_2(s)+V_2(s)\big) > (1+p_0)\left(\frac{\eta-1}{\eta}\right)^{\eta} \tag{22}$$

then (E) is oscillatory.

Proof. It follows from Theorem 2.5 that (E) is oscillatory provided that (E_4) has no positive solution. Assume that $w(n)$ is a positive solution of (E_4).

Then $w(n)$ is increasing and if $\sigma(n) \leq \eta(n)$, then $w(\sigma(n) \leq w(\eta(n)))$. Setting the last inequality to (E_4), we see that $w(n)$ is a positive solution of the difference inequality

$$\Delta w(n) - \frac{1}{1+p_0}\big(Q_2(n)+V_2(n)\big)w(\sigma(n)) \geq 0. \tag{E_4^*}$$

But according to Theorem 7.6.1 in [10], the condition (21) guarantees that (E_4^*) has no positive solution. This contradiction finishes proof of the first part of Corollary. The second part can be verified similarly and so the rest of the proof can be omitted.

For our ultimate references, let us denote

$$Q_3(n) = Q(\sigma^{-1}(n))\big(R(n)-R(n_1)\big), \tag{23}$$

$$V_3(n) = V_2(n)\prod_{n_1}^{n-1}\left(\frac{Q_3(s)}{1+p_0}+1\right) \tag{24}$$

where $n \geq n_1$, n_1 is large enough, $Q(n)$ is defined as in (2), while $Q_2(n)$ *and* $V_2(n)$ are defined by (15) and $\sigma^{-1}(n)$ is the inverse function to $\sigma(n)$.

Imposing the assumption that $\sigma(n)$ is the delay and $\eta(n)$ is the advanced argument, we establish the following oscillation criterion for equation (E) with mixed arguments.

Theorem 2.7 *Assume that (10) holds. and*

$$\Delta\sigma(n) > 0,\ \sigma(n) \leq n,\ \eta(n) \geq n. \tag{25}$$

If the first order advanced difference inequality

$$\Delta w(n) - \frac{1}{1+p_0} V_3(n) w(\eta(n)) \geq 0 \qquad (E_5)$$

has no positive solution, then (E) is oscillatory.

Proof. We assume that $x(n)$ is a positive solution of (E). Then proceeding as in the proof of Theorem 2.5, we verify that the corresponding sequence $z(n)$ satisfies (19). On the other hand, using the substitution $\sigma(s) = u$, we see that

$$\sum_{n}^{\infty} Q(s) z(\sigma(s)) = \sum_{\sigma(n)}^{\infty} Q(\sigma^{-1}(u)) z(u)$$

$$\geq \sum_{n}^{\infty} Q(\sigma^{-1}(u)) z(u) \qquad (26)$$

Combining (19) together with (26), one gets

$$r(n)\Delta z(n)(1+p_0) \geq \sum_{n}^{\infty} \left(Q(\sigma^{-1}(s)) z(s) + V(s) z(\eta(s)) \right). \qquad (27)$$

Summing the last inequality from n_1 to $n-1$ with applying the similar process as in the proof of Theorem 2.5, we get

$$z(n) \geq \frac{1}{1+p_0} \sum_{n_1}^{n-1} \left(Q_3(s) z(s) + V_2(s) z(\eta(s)) \right) \qquad (28)$$

Let us denote the right hand side of (28) by $y(n)$.

Since $z(n) \geq y(n)$, we see that $y(n)$ is the positive solution of

$$\Delta y(n) - \frac{1}{1+p_0} Q_3(n) y(n) - \frac{1}{1+p_0} V_2(n) y(\eta(n)) \geq 0 \qquad (E_6)$$

Now, we set

$$y(n) = \prod_{n_1}^{n-1} \left(\frac{Q_3(s)}{1+p_0} + 1 \right) w(n)$$

Then in view of (E_6), it is easy to verify that $w(n)$ is a positive solution of (E_5). This is a contradiction and the proof is complete.

Corollary 2.8. *Assume that (10) and (25) holds. If* $n - \sigma \leq n + \eta$

$$\liminf_{n\to\infty} \sum_{n+1}^{n-\eta-1} V_3(s) \geq (1+p_0) \left(\frac{\eta - 1}{\eta} \right)^{\eta} \qquad (29)$$

then (E) is oscillatory.

Proof. Theorem 7.6.1 in [10] implies that condition (29) guarantees that (E_5) has no positive solution and the assertion now follows from Theorem 2.7.

Now, we turn our attention to the case when $\tau(n)$ is the delay argument. We shall provide the results analogous to Theorem 2.7.

Theorem 2.9. *Assume that (11) holds. If the first order inequality*

$$\Delta w(n) + \frac{1}{1+p_0} Q_1(n) w(\tau^{-1}(n)) + \frac{1}{1+p_0} V_1(n) w(\tau^{-1}(\eta(n))) \leq 0 \qquad (E_7)$$

has no positive solution, then (E) is oscillatory.

Proof. We assume that $x(n)$ is a positive solution of (E). Then $y(n)=r(n)\Delta z(n)>0$ is a decreasing solution of (E_2). We denote $w(n)=y(n)+p_0y(\tau(n))$. What is more (11) implies

$$w(n)\le y(\tau(n))(1+p_0).$$

Substituting this into (E_2), we get that $w(n)$ is a positive solution of (E_7).
A contradiction.

Corollary 2.10. *Assume that (11) holds and* $n-\sigma<n-\tau$, $n+\eta<n-\tau$

$$\sigma(n)<\tau(n),\ \ \eta(n)<\tau(n) \tag{30}$$

If $\sigma(n)\le\eta(n)$ and also

$$\liminf_{n\to\infty}\sum_{n+\tau-\eta}^{n-1}\big(Q_1(s)+V_1(s)\big)>(1+p_0)\left(\frac{\eta-\tau}{\eta-\tau+1}\right)^{\eta-\tau+1}, \tag{31}$$

or $\sigma(n)\ge\eta(n)$ and also

$$\liminf_{n\to\infty}\sum_{n+\tau-\sigma}^{n-1}\big(Q_1(s)+V_1(s)\big)>(1+p_0)\left(\frac{\sigma-\tau}{\sigma-\tau+1}\right)^{\sigma-\tau+1}, \tag{32}$$

then (E) is oscillatory.

Proof. We admit that $w(n)$ is a positive solution of (E_7).

If $\sigma(n)\le\eta(n)$, then $w(\tau^{-1}(\sigma(n)))\ge w(\tau^{-1}(\eta(n)))$ and (E_7) gives that $w(n)$ is a solution of the difference inequality

$$\Delta w(n)+\frac{1}{1+p_0}\big(Q_1(n)+V_1(n)\big)w\big(\tau^{-1}(\eta(n))\big)\le 0 \tag{E_7^*}$$

But according to Theorem 7.6.1 [10] the condition (31) guarantees that (E_7^*) has no positive solution. There fore (E_7) has no positive solution and Theorem 2.9 provides the oscillation of (E). The case $\sigma(n)\ge\eta(n)$ can be treated similarly.
The proof is completed.

For our future reference, let us denote

$$Q_4(n)=Q(n)\big(R(\tau(n))-R(\tau(n_1))\big),\ \ V_4(n)=V(n)\big(R(\tau(n))-R(\tau(n_1))\big) \tag{33}$$

then $n\ge n_1$, n_1 is large enough and Q(n) and V(n) are defined as in (2).

Theorem 2.11. *Assume that (11) holds and*

$$\sigma(n)>\tau(n),\ \ \eta(n)>\tau(n). \tag{34}$$

If the first order advanced difference inequality

$$\Delta w(n)-\frac{1}{1+p_0}Q_4(n)w(\tau^{-1}(\sigma(n)))-\frac{1}{1+p_0}V_4(n)w(\tau^{-1}(\eta(n)))\ge 0$$

has no positive solution, then (E) is oscillatory.

Proof. We assume that $x(n)$ is a positive solution of (E). Then it follows from the proof of Theorem 2.2, that the corresponding sequence $z(n)$ satisfies (17). Since $r(n)\Delta z(n)$ is decreasing and (11) holds, then

$$r(n)\Delta z(n)+p_0r(\tau(n))\Delta z(\tau(n))\le r(\tau(n))\Delta z(\tau(n))(1+p_0). \tag{35}$$

Combining (17) together with (35), we obtain

$$r(\tau(n))\Delta z(\tau(n))(1+p_0) \geq \sum_{n}^{\infty}\big(Q(s)z(\sigma(s))+V(s)z(\eta(s))\big) \tag{36}$$

Multiplying the last inequality by $\dfrac{1}{r(\tau(n))}$ and then summing the result from n_1 *to* $n-1$, we get

$$\begin{aligned} z(\tau(n)) &\geq \frac{1}{1+p_0}\sum_{n_1}^{n-1}\frac{1}{r(\tau(n))}\sum_{u}^{\infty}\big(Q(s)z(\sigma(s))+V(s)z(\eta(s))\big) \\ &\geq \frac{1}{1+p_0}\sum_{n_1}^{n-1}\big(Q(s)z(\sigma(s))+V(s)z(\eta(s))\big)\sum_{n_1}^{s}\frac{1}{r(\tau(n))} \end{aligned}$$

Hence,

$$z(\tau(n)) \geq \frac{1}{1+p_0}\sum_{n_1}^{n-1}\big(Q(s)z(\sigma(s))+V(s)z(\eta(s))\big). \tag{37}$$

Let us denote the right hand side of (37) by $w(n)$. Since $z(n) \geq w(n)$, we see that $w(n)$ is a positive solution of (E_8). This contradicts our assumption and the proof is complete now.

Remark 5. The assumptions imposed in Theorem 2.11 do not require for $\sigma(n)$ and $\eta(n)$ to be advanced arguments. We only need for $\tau^{-1}(\sigma(n))$ and $\tau^{-1}(\eta(n))$ to be advanced arguments. So the conclusions of Theorem 11 hold for all types of equations i.e., advanced, delay, with mixed arguments and even if $n-\sigma(n)$ or $n-\eta(n)$ oscillates.

Corollary 2.12. *Assume that (11) and (34) hold. If, in particular* $\sigma(n) = n-\sigma \leq n-\eta(n)$ *and also*

$$\liminf_{n\to\infty}\sum_{n+1}^{n+\tau-\sigma-1}\big(Q_4(s)+V_4(s)\big) > (1+p_0)\left(\frac{\sigma-\tau-1}{\sigma-\tau}\right)^{\sigma-\tau} \tag{38}$$

or $\sigma(n) \geq \eta(n)$ and also

$$\liminf_{n\to\infty}\sum_{n+1}^{n+\tau-\eta-1}\big(Q_4(s)+V_4(s)\big) > (1+p_0)\left(\frac{\eta-\tau-1}{\eta-\tau}\right)^{\eta-\tau} \tag{39}$$

then (E) is oscillatory.

Proof. We let $w(n)$ is a positive solution of (E_8).
If $\sigma(n) \leq \eta(n)$, then $w(\tau^{-1}(n)(\sigma(n))) \leq w(\tau^{-1}(\eta(n)))$ and (E_8) implies that $w(n)$ satisfies

$$\Delta w(n) - \frac{1}{1+p_0}\big(Q_4(n)+V_4(n)\big)w(\tau^{-1}(\sigma(n))) \geq 0 \tag{E_8^*}$$

But according to Theorem 7.6.1 [10] the condition (38) guarantees that (E_8^*) has no positive solution. This contradiction ensures that (E_8) has no positive solution and taking Theorem 2.11 into account we see that (E) is oscillatory. The case $\sigma(n) \geq \eta(n)$ is left to the reader.

For our incoming references, let us denote

$$Q_5(n) = Q\big(\sigma^{-1}(\tau(n))\big)\big(R(\tau(n)) - R(\tau(n_1))\big) \tag{40}$$

$$V_5(n) = V_4(n) \prod_{n}^{n+\tau-\eta-1} \left(\frac{Q_5(s)}{1+p_0} + 1 \right) \tag{41}$$

where $n \ge n_1$, n_1 is large enough, $Q(n)$ is defined as in (2), while $V_4(n)$ is defined by (33).

Theorem 2.13 *Assume that (11) holds and*

$$\Delta\sigma(n) > 0, \ \sigma(n) \le \tau(n), \ \eta(n) > \tau(n). \tag{42}$$

If the first order advanced difference inequality

$$\Delta w(n) - \frac{1}{1+p_0} V_5(n) w\left(\tau^{-1}(\eta(n))\right) \ge 0 \tag{E_9}$$

has no positive solution, then (E) is oscillatory.

Proof. We assume that $x(n)$ is a positive solution of (E). Then proceeding exactly as in the proof of Theorem 2.11, we verify that the corresponding sequence $z(n)$ satisfies (36). On the other hand, using the substitution $\sigma(s) = \tau(u)$, we see that

$$\sum_{n}^{\infty} Q(s) z(\sigma(s)) = \sum_{\tau^{-1}(\sigma(n))}^{\infty} Q\left(\sigma^{-1}(\tau(u))\right) z(\tau(u))$$

$$\ge \sum_{n}^{\infty} Q\left(\sigma^{-1}(\tau(n))\right) z(\tau(n)) \tag{43}$$

Combining (36) together with (43), one gets

$$r(\tau(u)) \Delta z(\tau(u))(1+p_0) \ge \sum_{n}^{\infty} \left(Q(\sigma^{-1}(\tau(u))) z(\tau(u)) + V(s) z(\eta(s)) \right) \tag{44}$$

Multiplying the last inequality by $\dfrac{1}{r(\tau(n))}$ and then summing the resulting inequality from n_1 to $n-1$, and using the similar process as in the proof of Theorem 2.11, we get

$$z(\tau(n)) \ge \frac{1}{1+p_0} \sum_{n_1}^{n-1} \left(Q_5(s) z(\tau(s)) + V_4(s) z(\eta(s)) \right) \tag{45}$$

Let us denote the right hand side of (45) by $y(n)$. Since $z(\tau(n)) \ge y(n)$, we see that $y(n)$ is a positive solution of

$$\Delta y(n) - \frac{1}{1+p_0} Q_5(n) y(n) - \frac{1}{1+p_0} V_4(n) y\left(z^{-1}(\eta(n))\right) \ge 0 \tag{E_{10}}$$

Now setting

$$y(n) = \prod_{n_1}^{n-1} \left(\frac{Q_5(s)}{1+p_0} + 1 \right) w(n),$$

we see in view of (E_{10}) that $w(n)$ is a positive solution of (E_9). This is a contradiction and the proof is complete.

Corollary 2.14. *Assume that (11) and (42) holds. If*

$$\liminf_{n\to\infty} \sum_{n+1}^{n+\tau-\eta-1} V_5(s) > (1+p_0) \left(\frac{n-\eta-1}{n-\eta} \right)^{n-\eta} \tag{46}$$

then (E) is oscillatory.

Proof. According to Theorem 7.6.1 in [10] the condition (46) guarantees that (E_9) has no positive solution and the assertion now follows from Theorem 2.13.

III. Applications

Example 3.1 *Consider the neutral delay difference equation*

$$\Delta\left(2\Delta[x(n)+\frac{1}{4}x(n+2)]\right)+2^7 5x(n-5)+2^7 x(n-3)=0 \qquad (E_{11})$$

where $p(n)=\frac{1}{4},\ r(n)=2,\ q(n)=2^7 5,\ v(n)=2^7.$

Clearly $\liminf_{n\to\infty}\sum_{n-3}^{n-1}2^7 6>\left(\frac{5}{4}\right)\left(\frac{3}{4}\right)^4$

All conditions of Corollary 2.4 are satisfied. Hence every solution $\{x(n)\}$ of (E_{11}) oscillates. One such solution is $\{x(n)\}=\{(-1)^n 2^n\}$.

Example 3.2 *Consider the neutral advanced difference equation*

$$\Delta[2\Delta[x(n)+4x(n+2)]]+2^4 3^2 x(n+5)+2^2 3^2 x(n+3)=0 \qquad (E_{12})$$

where $p(n)=4,\ r(n)=\frac{1}{3},\ q(n)=2^4 3^2\ \ and\ \ v(n)=2^2 3^2$.

Clearly $\liminf_{n\to\infty}\sum_{n+1}^{n+4}(2^4+2^2)3^2>(1+4)\left(\frac{4}{5}\right)^5$.

All conditions of Corollary 2.6 are satisfied. Every solution of (E_{12}) is oscillatory. One such solution is $\{x(n)\}=\{\frac{(-1)^n}{2^n}\}$.

Example 3.3 *Consider the mixed difference equation*

$$\Delta\left[2\Delta[x(n)+\frac{1}{4}x(n+2)]\right]+10\times 2^5 x(n-3)+2x(n+2)=0 \qquad (E_{13})$$

Clearly $\sum_{n+1}\left(\frac{2^5\times 10}{\frac{5}{4}}+1\right)>\frac{5}{4}\left(\frac{2}{1}\right)^1$.

All the conditions of Corollary 2.8 are satisfied. Hence every solution of equation (E_{13}) is oscillatory. One such solution is $\{x(n)\}=\{(-1)^n 2^n\}$.

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