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A study on gr*-closed sets in Bitopological Spaces

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Abstract: *In this paper we introduce and investigate the notions of gr*-closed set, gr*-open set, gr*-continuous, gr*-neighbourhood, gr*O-connected spaces, pairwise gr*O-compact spaces in Bitopological Spaces.*

Keywords: *(i,j) gr*-closed sets, (i,j) gr*-open sets, $D_r^*(i,j)$- σ_k-continuous, (τ_i,τ_j)-gr*-neighbourhood, pairwise gr*O-connected, pairwise gr* open cover, pairwise gr*O-compact .*

I. Introduction:

A triple (X, τ_1,τ_2) where X is a non-empty set and τ_1 and τ_2 are topologies on X is called a bitopological space and kelly [7] initiated the study of such spaces. In 1985, Fukutake [2] introduced the concept of g-closed sets in bitopological spaces and after that several authors turned their attention towards generalizations of various concepts of topology by considering bitopological spaces and Indirani et.al.[5] introduced gr*-closed sets in topological spaces and investigated its relationship with the other types of closed sets. The purpose of the present paper is to define a new class of closed sets called (i,j) gr*-closed sets and we discuss some basic properties of (i,j) gr*-closed sets in bitopological spaces.

II. Preliminaries:

If A is a subset of X with a topology τ, then the closure of A is denoted by τ-cl(A) of cl(A), the interior of A is denoted by τ-int(A) of int(A) and the complement of A in X is denoted by A^C.

Definition: 2.1. A subset A of a topological space (X, τ) is called

(i) A generalizad closed set [9] (briefly g-closed set) if cl(A)$\subseteq$U whenever A$\subseteq$U ad U is open in X.

(ii) a generalized open set (briefly g-closed set) if A^c is g-closed in X.

(iii) a regular open set [11] if A= int (cl (A))

(iv) a semi-open set [8] if A$\subseteq$cl (int (A))

Definition 2.2 – the intersection of all pre closed sets containing A is called the pre-closure of A and it si denoted by τ-pcl(A) or pcl(A).

Throughout this paper X and Y always represent nonempty bitoplogical spaces (X, τ_1,τ_2) and (Y,σ_1,σ_2) on which no separation axioms are assumed unless explicitly metioned and the integers I,j,k$\in${1,2}. For a subset A of X, τ_i-cl(A) (resp. τ_i-int(A), τ_i-pcl(A)) denote the closure (resp. interior, preclosure) of A with respect to the toplogy τ_i, by GO(X, τ_i) and the family of all τ_j-closed set is denoted by the symbol F_j. By (I,j) we mean the pair of topologies (τ_i,τ_j).

Definition 2.3- A subset A of a topological space (X, τ_1,τ_2) is called

(i) (i,j)-g-closed [2] if τ_j-cl(A)$\subseteq$U whenever A$\subseteq$U and U$\in$ τ_i.

(ii) (i,j)-rg-closed [1] if τ_j cl(A)$\subseteq$ U whenever A $\subseteq$ U and U is regular open in τ_i.

(iii) (i,j)-gpr-closed [4] if τ_j-pcl(A)$\subseteq$ U whenever A $\subseteq$ U and U is regular open in τ_i.

(iv) (i,j)- g -closed [4] if τ_j-cl(A)$\subseteq$ U whenever A $\subseteq$ U and U is semi-open in τ_i.

(v) (i,j)-g^*-closed [12] if τ_j-Cl(A)$\subseteq$U whenever A$\subseteq$U and U $\in$GO(X, τ_i).

The family of all(i,j)-g-closed (resp. (i,j)-rg-closed, (i,j) –gpr-closed, (i,j)-wg-closed and (i,j)-w-closed) subsets of a bitoplogical space(X, τ_1, τ_2) is denoted by D(I,j) (resp. D_r(I,j), ζ (i,j), W(i,j) and C(i,j)).

Definition 2.5 A map f:(X, τ_1, τ_2)$\rightarrow$(Y, τ_1, τ_2) is called:

(1) τ_j-σ_k-continuous [10] if f^1(V)$\in$ τ_j for every V$\in\sigma_k$.

(2) D(i,j)- σ_k-continuous [10] (resp. D^*(i,j)- σ_k-continuous [12], D_r(i,j)-σ_k-continuous [1], ξ (i,j)- σ_k-continuous [4], W(I,j)- σ_k-continuous[3], C(i,j)- σ_k-continuous[4] and D_r^*(i,j)-σ_k-continuous [6]) if the inverse image of every σ_k-closed set is (i,j)-g-closed (resp. (i,j)-g^*-closed,(i,j)-rg-closed, (i,j)-gpr-closed, (i,j)-wg-closed, (i,j)-w-closed and (i,j) gr*-closed) set in (X, τ_1, τ_2).

III. (i,j) gr*-closed sets

Definition 3.1: A subset A of a bitopological space (X, τ_1,τ_2) is said to be an (i,j) gr*-closed set if τ_j-rcl(A) $\subseteq$ U, whenever A $\subseteq$ U and U is τ_i-g-open.
We denote the family of all (i,j) gr*-closed set in (X, τ_1,τ_2) by D_r^*(i,j).
Remark 3.2: By setting $\tau_1 = \tau_2$ in Definition 3.1, a (1,2) gr*-closed sets is a gr*-closed sets.
Proposition 3.3: If A is τ_j-closed subset of (X, τ_1,τ_2) then A is (i,j) gr*-closed.
Proof: Let A be τ_j-closed subset of (X,τ_1,τ_2). To prove that A is (i,j) gr*-closed. Let U be any g-open in (X, τ_i) such that A $\subseteq$ U. Since every r-closed sets are closed, so that A is τ_j-closed, it follows that τ_j-cl(A) $\subseteq$ τ_j-rcl(A) $\subseteq$ U. This implies that τ_j-rcl(A) $\subseteq$ U. Hence A is (i,j) gr*-closed.
The converse of the above proposition is not true as seen from the following example.
Example 3.4: Let X = {a,b,c}, τ_1 = {ϕ,{a},{c},{ab},{ac},X}, τ_2 = {ϕ,{c},{ab},X }. Then the subset {b} and {bc} are (1,2)-gr*-closed set but not τ_2 – closed set in (X,τ_1,τ_2).
Proposition 3.5: In a bitopological space (X,τ_1,τ_2),
(i) Every τ_j-θ-closed is (i,j) gr*-closed.
(ii) Every τ_j-δ-closed is (i,j) gr*-closed.
(iii) Every τ_j-r-closed is (i,j) gr*-closed.
Proof: It follows from
(i) Every θ-closed is closed and by proposition 3.3.
(ii) Every δ-closed is closed and by proposition 3.3.
(iii) Every r-closed is closed and by proposition 3.3.
The converse of the above proposition is not true as seen from the following example.
Example 3.6: Let X = {a,b,c}, τ_1 = {ϕ,{a},{b},{ab},X} and τ_2 = {ϕ,{b},{ac},X }. Then the subset {bc} is (1,2)-gr*-closed set but not τ_2 - θ – closed set in (X,τ_1,τ_2).
Example 3.7: Let X = {a,b,c}, τ_1 = {ϕ,{b},{ac},X} and τ_2 = {ϕ,{a},{b},{ab},X }. Then the subset {ac} is (1,2)-gr*-closed set but not τ_2 - δ – closed set in (X,τ_1,τ_2).
Example 3.8: Let X = {a,b,c}, τ_1 = {ϕ,{a},{ab},{ac},X} and τ_2 = {ϕ,{a},{b},{ab},{ac},X }. Then the subset {c} is (1,2)-gr*-closed set but not τ_2 -r – closed set in (X,τ_1,τ_2).
Proposition 3.9: In a bitopological space (X,τ_1,τ_2),
(i) Every (i,j) gr*-closed is (i,j)-g-closed.
(ii) Every (i,j) gr*-closed is (i,j)-g*-closed.
(iii) Every (i,j) gr*-closed is (i,j)-rg-closed.
(iv) Every (i,j) gr*-closed is (i,j)-gpr-closed.
(v) Every (i,j) gr*-closed is (i,j)- $\hat{g}$ -closed.
Proof: (i) Let A be a (i,j) gr*-closed subset of (X,τ_1,τ_2). Let U $\in$ GO(X,τ_i) be such that A $\subseteq$ U. Then by hypothesis τ_j-rcl(A) $\subseteq$ U. This implies τ_j-cl(A) $\subseteq$ U. Therefore A is (i,j) gr*-closed.
Proof of (ii) to (v) are similar to (i).
The following examples show that the reverse implications of above proposition are not true.
Example 3.10: Let X = {a,b,c}, τ_1 = {ϕ,{a},X} and τ_2 = {ϕ,{a},{c},{ab},{ac},X }. Then the subset {b} is (1,2)-g-closed but not (1,2)-gr*-closed set.
Example 3.11: Let X = {a,b,c}, τ_1 = {ϕ,{a},{ac},X} and τ_2 = {ϕ,{a},{c},{ab},{ac},X }. Then the subset {a} is (1,2)-rg-closed but not (1,2)-gr*-closed set.
Example 3.12: Let X = {a,b,c}, τ_1 = {ϕ,{a},{c},{ab},{ac},X} and τ_2 = {ϕ,{c},X }. Then the subset {ac} is (1,2)-gpr-closed but not (1,2)-gr*-closed set.
Example 3.13: Let X = {a,b,c}, τ_1 = {ϕ,{c},{ac},{bc},X} and τ_2 = {ϕ,{c},X }. Then the subset {a} is (1,2)- $\hat{g}$ -closed but not (1,2)-gr*-closed set.
Proposition 3.14: If A, B $\in$ D_r^*(i,j), then A$\cup$B $\in$ D_r^*(i,j).
Proof: Let U $\in$ GO(X,τ_1) be such that A$\cup$B $\subseteq$ U. Then A $\subseteq$ U and B $\subseteq$ U. Since A, B $\in$ D_r^*(i,j), we have τ_j-rcl(A) $\subseteq$ U and τ_j-rcl(B) $\subseteq$ U. That is τ_j-rcl(A) $\cup$ τ_j-rcl(B) $\subseteq$ U. Also τ_j-rcl(A) $\cup$ τ_j-rcl(B) $\subseteq$ U = τ_j-rcl(A$\cup$B) and so τ_j-rcl(A$\cup$B) $\subseteq$ U. Therefore A$\cup$B $\in$ D_r^*(i,j).
Example 3.15: Let X = {a,b,c}, τ_1 = {ϕ,{a},{ac},X} and τ_2 = {ϕ,{a},{c},{ab},{ac},X }. Then the subset {b} and {c} are (1,2)-gr*-closed set and {b} $\cup${c}={bc} is also (1,2)-gr*-closed set.
Remark 3.16: The intersection of two (1,2)-gr*-closed sets need not be a (1,2)-gr*-closed set.
Example 3.17: Let X = {a,b,c}, τ_1 = {ϕ,{b},{c},{bc},{ac},X } and τ_2 = {ϕ,{a},{bc},X }. Then the subset {ab} and {bc} are (1,2)-gr*-closed set and {ab} $\cap$ {bc}={b} is not in (1,2)-gr*-closed set.
Remark 3.18: D_r^*(i,j) is generally not equal to D_r^*(i,j) as seen from the following example.

Example 3.19: Let $X = \{a,b,c\}$, $\tau_1 = \{\phi,\{a\},\{b\},\{ab\},\{bc\},X\}$ and $\tau_2 = \{\phi,\{a\},\{ab\},\{ac\},X\}$. Then (1,2)-gr*-closed set $= \{\phi,\{ac\},X\}$ and (2,1)-gr*-closed set $= \{\phi,\{a\},\{bc\},X\}$.

Proposition 3.20: If $\tau_1 \subseteq \tau_2$ in (X,τ_1,τ_2), then $D_r^*(1,2) \subseteq D_r^*(2,1)$.

Proof: Let $A \in D_r^*(2,1)$. That is A is a (2,1)-gr*-closed set. To prove that $A \in D_r^*(2,1)$. Let $U \in GO(X, \tau_1)$ be such that $A \subseteq U$. Since $GO(X, \tau_1) \subseteq GO(X, \tau_2)$, we have $U \in GO(X, \tau_2)$. As A is a (2,1)-gr*-closed set, we have τ_1-rcl(A) $\subseteq$ U. Since $\tau_1 \subseteq \tau_2$, we have τ_2-rcl(A) $\subseteq \tau_1$-rcl(A) and it follows that τ_2-rcl(A) $\subseteq$ U. Hence A is (1,2)-gr*-closed set. That is $A \in D_r^*(1,2)$. Therefore $D_r^*(1,2) \subseteq D_r^*(2,1)$.

The converse of the above proposition is not true as seen from the following example.

Example 3.21: Let $X = \{a,b,c\}$, $\tau_1 = \{\phi,\{b\},\{ab\},X\}$ and $\tau_2 = \{\phi,\{b\},X\}$. Then $D_r^*(2,1) \subseteq D_r^*(1,2)$ but τ_1 is not contained in τ_2.

Proposition 3.22: For each element x of (X,τ_1,τ_2), $\{x\}$ is τ_i-g-closed or $\{x\}^c$ is $D_r^*(1,2)$.

Proposition 3.23: If A is (i,j) gr*-closed, then τ_j-cl(A) – A contains no non empty τ_j-g-closed set.

Proof: Let A be a (i,j) gr*-closed set and F be a τ_i-g-closed set such that $F \subseteq \tau_j$-rcl(A) $\subseteq \tau_j$-rcl(A) – A. Since $A \in D_r^*(i,j)$, we have τ_j-cl(A) $\subseteq \tau_j$-rcl(A) $\subseteq F^c$. Thus $F \subseteq \tau_j$-cl(A) $\cap$ $(\tau_j\text{-cl}(A))^c = \phi$.

The converse of the above proposition is not true as seen from the following example.

Example 3.24: Let $X = \{a,b,c\}$, $\tau_1 = \{\phi,\{b\},\{c\},\{bc\},X\}$ and $\tau_2 = \{\phi,\{a\},X\}$. Then the subset $A=\{b\}$,then τ_2-cl(A) – A = $\{c\}$ does not contains any non empty τ_1-g-closed set but A is not (1,2)-gr*-closed set.

Proposition 3.25: If A is (i,j) gr*-closed set of (X,τ_1,τ_2) such that $A \subseteq B \subseteq U \subseteq \tau_j$-rcl(A), then B is also an (i,j)-gr*-closed set of (X,τ_1,τ_2).

Proposition 3.26: Let $A \subseteq Y \subseteq X$ and suppose that A is (i,j)-gr*-closed set in X. Then A is (i,j)-gr*-closed set to Y.

Proposition 3.27: In a bitopological space (X,τ_1,τ_2), $GR^*O(X,\tau_i) \subseteq F_j$ iff every subset of X is a (i,j)-gr*-closed set.

Proof: Suppose that $GR^*O(X,\tau_i) \subseteq F_j$. Let A be a subset of X such that $A \subseteq U$, where $U \in GR^*O(X,\tau_i)$. Then τ_j-rcl(A) $\subseteq \tau_j$-cl(A) $\subseteq \tau_j$-cl(U) = U and hence A is (i,j)-gr*-closed set. Converserly, suppose that every subset of X is (i,j)-gr*-closed. Let $U \in GR^*O(X,\tau_i)$. Since U is (i,j)-gr*-closed, we have τ_j-rcl(U) $\subseteq \tau_j$-cl(U) $\subseteq$ U. Therefore $U \subseteq F_j$ and hence $GR^*O(X,\tau_i) \subseteq F_j$.

IV. (i,j) gr*-open sets

Definition 4.1: A subset A of a bitopological space (X,τ_1,τ_2) is said to be (τ_i,τ_j) gr*-open if its complement is (τ_i,τ_j) gr*- closed in (X,τ_1,τ_2).

Proposition 4.2: In a bitopological space (X,τ_1,τ_2),

(i) Every τ_j-θ-open set is (i,j) gr*-open.

(ii) Every τ_j-δ-open set is (i,j) gr*- open.

(iii) Every τ_j-r-open set is (i,j) gr*- open.

Proposition 4.3: In a bitopological space (X,τ_1,τ_2),

(i) Every (i,j) gr*-open is (i,j)-g- open.

(ii) Every (i,j) gr*- open is (i,j)-g*- open.

(iii) Every (i,j) gr*- open is (i,j)-rg- open.

(iv) Every (i,j) gr*- open is (i,j)-gpr- open.

(v) Every (i,j) gr*- open is (i,j)- $\tilde{g}$ - open.

Proposition 4.4: A subset A of a bitopological space (X,τ_1,τ_2) is (τ_i,τ_j) gr*- open iff $F \subseteq \tau_j$-rint(A) whenever F is τ_i-g-closed and $F \subseteq A$.

Proof: Necessity. Let A be a (τ_i,τ_j) gr*- open set in X and F be a τ_1-g-closed set such that $F \subseteq A$. Then $F^c \subseteq A^c$ where F^c is τ_1-g-open and A^c is (τ_i,τ_j) gr*- closed implies τ_j-rcl(A) $\subseteq (F^c)$, $\tau_j(\text{rint}(A))^c \subseteq F^c$. Hence $F \subseteq \tau_j$-rint(A). Sufficiency. Let $F \subseteq \tau_j$-rint(A) whenever F is τ_1-g-closed and $F \subseteq A$. Then $A^c \subseteq F^c = G$ and G is τ_i-g-open. Then $G^c \subseteq A$ implies $G^c \subseteq \tau_j$-rint(A) or τ_j-rcl$(A^c) = (\tau_j\text{-rint}(A))^c \subseteq G$. Then A^c is (τ_i,τ_j) gr*- closed or A is (τ_i,τ_j) gr*- open.

V. (i,j) gr*-continuous map

Definition 5.1: A map f: $(X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ is called $D_r^*(i,j)$- σ_k-continuous if the inverse image of every σ_k-closed set is (i,j) gr*-closed set.

Proposition 5.2: If f: $(X,\tau_1,\tau_2) \rightarrow (Y,\sigma_1,\sigma_2)$ is τ_j -σ_k-continuous, then it is $D_r^*(i,j)$- σ_k-continuous but not conversely.

Example 5.3: Let X = Y = {a,b,c}, τ_1 = {ϕ,{a},{b},{ab},X}, τ_2 = {ϕ,{b},{ac},X }, σ_1 = {ϕ,{a},X} and σ_2 = {ϕ,{c},X}. Define a map f: $(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$ by f(a) = a, f(b) = c, f(c) = b. Then f is $D_r^*(i,j)$- σ_2-continuous, but not τ_1- σ_2-continuous.

Proposition 5.4: If f: $(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is $D_r^*(i,j)$ - σ_k-continuous then it is

(i) D (i,j) - σ_k-continuous.

(ii) $D^*(i,j)$ - σ_k-continuous.

(iii) D_r (i,j) - σ_k-continuous.

(iv) ζ (i,j) - σ_k-continuous.

(v) C (i,j) - σ_k-continuous.

Proof: (i) Let V be a σ_k-closed set in (Y,σ_1,σ_2). Since f is $D_r^*(i,j)$ - σ_k-continuous, $f^1(V)$ is (i,j)-gr*-closed in (X,τ_1,τ_2). Then by Proposition 3.9, $f^1(V)$ is (i,j)-g-closed. Therefore f is D (i,j) - σ_k-continuous.

proof of (ii) to (v) are similar to (i), using Proposition 3.9.

However the reverse implications of the above proposition are not true in general as seen from the following example.

Example 5.5: Let X = Y = {a,b,c}, τ_1 = {ϕ,{a},{b},{ab},{bc},X}, τ_2 = {ϕ,{b},{ac},X }, σ_1 = {ϕ,{ab},X} and σ_2 = {ϕ,{bc},X}. Then the identity map f on X is D_r (i,j) - σ_k - continuous and ζ (i,j) - σ_k-continuous but not $D_r^*(i,j)$ - σ_2 -continuous.

Example 5.6: Let X = Y = {a,b,c}, τ_1 = {ϕ,{a},X}, τ_2 = {ϕ,{a},{c},{ab},{ac},X }, σ_1 = {ϕ,{b},X} and σ_2 = {ϕ,{ab},X}. Then the identity map f on X is D (i,j) - σ_k – continuous, $D^*(i,j)$ - σ_k-continuous and C (i,j) - σ_k-continuous but not $D_r^*(i,j)$ - σ_2 -continuous.

VI. (τ_i,τ_j)-gr*-Neighbourhood

Definition 6.1. Let (X,τ_i,τ_j) be bitopological space, and let g∈X. A subset N of X is said to be, (τ_i,τ_j)-gr*-neighbourhood (briefly (τ_i,τ_j)- gr*-nhd) if a point g if and only if there exists a (τ_i,τ_j)- gr*-open set G such that g∈G⊆N.

The set of all (τ_i,τ_j)- gr*-nhd of a point g is denoted by (τ_i,τ_j)- gr*-N(g).

Proposition 6.2. Every τ_i-nhd of g∈X is a (τ_i,τ_j)- gr*-nhd of g∈X.

Proof: Since N is τ_i-nhd of g∈X, then there exists τ_i-open set G such that g∈G⊆N. Since every τ_i-open set is (τ_i,τ_j)- gr*-open set, G is (τ_i,τ_j)- gr*-open set. By definition 6.1. N is (τ_i,τ_j)- gr*-nhd of X.

Remark 6.3. The converse of the above Proposition is need not be true as seen from the following example.

Example 6.4. Let X={a,b,c} and τ_i={X, ϕ,{a}}, τ_j={X,ϕ,{a},{a,b}}

D_r^*O (τ_i,τ_j) = {X, ϕ,{a},{b},{c},{a,b},{a,c},{b,c}}, the set {b,c} is (τ_i,τ_j)- gr*-nhd of c, since there exists a (τ_i,τ_j)- gr*-open set G={c} such that c∈{c}⊆{b,c}. However {b,c} is not τ_i-nhd of c , since no τ_i-open set G such that c∈G⊆{b,c}.

Proposition 6.5. If N a subset of a bitopological space (X,τ_i,τ_j) is (τ_i,τ_j)- gr*-open set, then N is (τ_i,τ_j)- gr*-nhd of each of its points.

Proof: Let N be a (τ_i,τ_j)- gr*-open set. By Definition 6.1. N is an (τ_i,τ_j)- gr*-nhd of each of its points.

Proposition 6.6. Let (X,τ_i,τ_j) be bitopological space:

1) ∀ g∈X, (τ_i,τ_j)- gr*-N(g)≠ϕ

2) ∀ N∈(τ_i,τ_j)- gr*-N(g), then g∈N.

3) If N∈(τ_i,τ_j)- gr*-N(g), N⊆M, then M∈(τ_i,τ_j)- gr*-N(g).

4) If N∈(τ_i,τ_j)- gr*-N(g), then there exists M∈(τ_i,τ_j)- gr*-N(g) such that M⊆N and exists M∈(τ_i,τ_j)- gr*-N(h), ∀ h∈M.

Proof. 1) Since X is an (τ_i,τ_j)- gr*-open set, it is (τ_i,τ_j)- gr*-nhd of every g∈X. Hence there exists at least one (τ_i,τ_j)- gr*-nhd G for every g∈X. Therefore (τ_i,τ_j)- gr*-N(g)≠ϕ, ∀ g∈X

2) If N∈(τ_i,τ_j)- gr*-N(g), then N is (τ_i,τ_j)- gr*-nhd G of g. Thus By Definition 6.1 g∈N.

3) If N∈(τ_i,τ_j)- gr*-N(g), then there is an (τ_i,τ_j)- gr*-open set A such that g∈A⊆N, since N⊆M, g∈A⊆M and M is an (τ_i,τ_j)- gr*-nhd of g. Hence M∈(τ_i,τ_j)- gr*-N(g).

4) If N∈(τ_i,τ_j)- gr*-N(g), then there exists is an (τ_i,τ_j)- gr*-open set M such that g∈M⊆N. Since M is an (τ_i,τ_j)- gr*-open set, then it is (τ_i,τ_j)- gr*-nhd of each of its points. Therefore M∈(τ_i,τ_j)- gr*-N(h), ∀ h∈M.

VII. Pairwise gr*O-Connected Space

Definition 7.1. A bitopological space (X,τ_1,τ_2) is pairwise gr*O-connected if X can not be expressed as the union of two non-empty disjoint sets A and B such that [A∩ τ_1- gr*cl(B)] ∪ [τ_2- gr*cl(A)∩B]= ϕ.

Suppose X can be so expressed then X is called pairwise gr*O-disconnected and we write X=A\B and call this pairwise gr*O-separation of X.

Theorem 7.3. The following conditions are equivalent for any bitopological space.

(a) X is pairwise gr*O-connected.
(b) X can not be expressed as the union of two non-empty disjoint sets A and B such that A is τ_1- gr* open and B is τ_2- gr* open.
(c) X contains no non-empty proper subset which is both τ_1- gr* open and B is τ_2- gr* closed.

Proof.(a)⇒(b): Suppose that X is pairwise gr*O-connected. Suppose that X can be expressed as the union of two non-empty disjoint sets A and B such that A is τ_1- gr* open and B is τ_2- gr* open. Then $A\cap B=\phi$. Consequently $A\subseteq B^C$. Then τ_2- gr*cl(A)⊆τ_2- gr*cl(B^C)=B^C. Therefore, τ_2- gr*cl(A)∩B= ϕ. Similarly we can prove A∩τ_1- gr*cl(B) = ϕ. Hence [A∩τ_1- gr*cl(B)]∪[τ_2- gr*cl(A)∩B] = ϕ. This is a contradiction to the fact that X is pairwise gr*O-connected. Therefore, X can not be expressed as the union of two non-empty disjoint sets A and B such that A is τ_1- gr* open and B is τ_2- gr* open.

(b)⇒(c): Suppose that X can not be expressed as the union of two non-empty disjoint sets A and B such that A is τ_1- gr* open and B is τ_2- gr* open. Suppose that X contains a non-empty proper subset A which is both τ_1- gr* open and τ_2- gr* closed. Then $X=A\cup A^C$ where A is τ_1- gr* open, A^C is τ_2- gr*open and A, A^C are disjoint. This is the contradictions to our assumption. Therefore, X contains no non-empty proper subset which is both τ_1- gr* open and τ_2- gr* closed.

(c)⇒(a): Suppose that X contains no non-empty proper subset which is both τ_1- gr* open and τ_2- gr* closed. Suppose that X is pairwise gr*O-connected. Then X can be expressed as the union of two non-empty disjoint sets A and B such that [A∩τ_1- gr*cl(B)]∪[τ_2- gr*cl(A)∩B] = ϕ. Since $A\cap B=\phi$, we have $A=B^C$ and $B=A^C$. Since τ_2- gr*cl(A)∩B= ϕ, we have τ_2- gr*cl(A) ⊆ B^C. Hence τ_2- gr*cl(A)⊆A. Therefore, A is τ_2- gr* closed. Similarly, B is τ_1- gr* closed. Since $A=B^C$, A is τ_1- gr* open. Therefore, there exists a non-empty proper set A which is both τ_1- gr* open and τ_2- gr* closed. This is the contradiction to our assumption. Therefore, X is pairwise gr*O-connected.

Theorem 7.4. If A is pairwise gr*O-connected subset of a bitopological space (X,τ_1,τ_2) which has the pairwise gr*O-separation X=C\D, then A⊆C or A⊆D.

Proof. Suppose that (X,τ_1,τ_2) has the pairwise gr*O-separation X=C\D. Then X=C∪D where C and D are two nonempty disjoint sets such that [C∩τ_1- gr*cl(D)]∪[τ_2- gr*cl(C)∩D] = ϕ. Since C∩D = ϕ, we have $C=D^C$ and $D=C^C$. Now [(C∩A)∩τ_1- gr*cl(D∩A)]∪[τ_2- gr*cl(C∩A)∩(D∩A)] ⊆ [(C∩τ_1- gr*cl(D)]∪[τ_2- gr*cl(C)∩D] = ϕ. Hence A=(C∩A)\(D∩A) is pairwise gr*O-separation of A. Since A is pairwise gr*O-connected, we have either (C∩A) = ϕ or (D∩A) = ϕ. Consequently, $A\subseteq C^C$ or $A\subseteq D^C$. Therefore, A⊆C or A⊆D.

Theorem 7.5. If A is pairwise gr*O-connected and A ⊆ B ⊆ τ_1- gr*cl(A) ∩ τ_2- gr*cl(A) then B is pairwise gr*O-connected.

Proof. Suppose that B is not pairwise gr*O-connected. Then B=C∪D where C and D are two nonempty disjoint sets such that [C∩τ_1- gr*cl(D)]∪[τ_2- gr*cl(C)∩D] = ϕ. Since A is pairwise gr*O-connected, we have A⊆C or A⊆D. Suppose A⊆C. Then D ⊆ D∩B ⊆ D∩τ_2- gr*cl(A) ⊆ D∩τ_2- gr*cl(C) = ϕ. Therefore, ϕ ⊆ D ⊆ ϕ. Consequently, D=ϕ. Similarly, we can prove C=ϕ if A⊆D {by Theorem 7.4}. This is the contradiction to the fact that C and D are nonempty. Therefore, B is pairwise gr*O-connected.

Theorem 7.6. The union of any family of pairwise gr*O-connected sets having a nonempty intersection is pairwise gr*O-connected.

Proof. Let I be an index set and i∈I. Let $A=\cup A_i$ where each A_i is pairwise gr*O-connected with $\cap A_i\neq\phi$. Suppose that A is not pairwise gr*O-connected. Then A = C∪D, where C and D are two nonempty disjoint sets such that [C∩τ_1- gr*cl(D)]∪[τ_2- gr*cl(C)∩D] = ϕ. Since A_i is pairwise gr*O-connected and $A_i\subseteq A$, we have $A_i\subseteq C$ or $A_i\subseteq D$. Therefore, $\cup(A_i)\subseteq C$ or $\cup(A_i)\subseteq D$. Hence, A⊆C or A⊆D. Since $\cap A_i\neq\phi$, we have $x\in\cap A_i$. Therefore, $x\in A_i$ for all i. Consequently, x∈A. Therefore, x∈C or x∈D. Suppose x∈C. Since C∩D=ϕ, we have x∉D. Therefore, A⊈D. Therefore, A⊆C. Therefore, A is not pairwise gr*O-connected. This shows that A is pairwise gr*O-connected.

Theorem 7.7. Let $f:(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$ be a pairwise continous bijective and pairwise r- closed. Then inverse image of a σ_i- gr* closed set is τ_i- gr* closed.

Theorem 7.8. Let $f:(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$ be a pairwise continous bijective and pairwise r- closed function. Then the image of a pairwise gr*O-connected space under f is pairwise gr*O-connected.

Proof. Let $f:(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$ be a pairwise continous surjection and pairwise r- closed. Let X is pairwise gr*O-connected. Suppose that Y is pairwise gr*O-disconnected. Then Y=A∪B where A is σ_1- gr* open and B is σ_2- gr* open in Y. Since f is pairwise continuous and pairwise r-closed, we have $f^1(A)$ is τ_1- gr* open and $f^1(B)$ is τ_2- gr* open in X. Also X = $f^1(A)$ ∪ $f^1(B)$, $f^1(A)$ and $f^1(B)$ are two nonempty disjoint sets. Then X is pairwise gr*O-disconnected. This is the contradiction to the fact that X is pairwise gr*O-connected. Therefore, Y is pairwise gr*O-connected.

VIII. Pairwise gr*O-Compact Space

Definition 8.1. A nonempty collection $\zeta=\{A_i, i\in I$, an index set$\}$ is called a pairwise gr* open cover of a bitopological space (X,τ_1,τ_2) if $X=\cup A_i$ and $\zeta \subseteq \tau_1$- gr*O$(X,\tau_1,\tau_2) \cup \tau_2$- gr*O$(X,\tau_1,\tau_2)$ and ζ contains at least one member of τ_1- gr*O(X,τ_1,τ_2) and one member of τ_2- gr*O(X,τ_1,τ_2).

Defintion 8.2. A bitopological space (X,τ_1,τ_2) is pairwise gr*O-compact if every pairwise gr* open cover of X has a finite subcover.

Definition 8.3. A set A of a bitopological space (X,τ_1,τ_2) is pairwise gr*O-compact relative to X if every pairwise gr* open cover of B by has a finite subcover as a subspace.

Theorem 8.5. Every pairwise gr*O-compact space is pairwise compact.

Proof. Let (X,τ_1,τ_2) be pairwise gr*O-compact. Let $\zeta=\{A_i, i\in I$, an index set$\}$ be a pairwise open cover of X. Then $X=\cup A_i$ and $\zeta \subseteq \tau_1\cup\tau_2$ and ζ contains at least one member of τ_1 and one member of τ_2. Since every τ_i-open set is τ_i- gr* open, we have $X=\cup A_i$ and $\zeta\subseteq\tau_1$- gr*O$(X,\tau_1,\tau_2) \cup \tau_2$- gr*O$(X,\tau_1,\tau_2)$ and ζ contains at least one member of τ_1- gr*O(X,τ_1,τ_2) and one member of τ_2- gr*O(X,τ_1,τ_2). Therefore, ζ is the pairwise gr*-open cover of X. Since X is pairwise gr*O-compact, we have ζ has the finite subcover. Therefore, X is pairwise compact.

But the converse of the above theorem need not be true in general.

Theorem 8.6. Let $f:(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$ be a pairwise continous, bijective and pairwise r- closed. Then the image of a pairwise gr*O-compact space under f is pairwise gr*O-compact.

Proof. Let $f:(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$ be a pairwise continous surjection and pairwise r-closed. Let X be pairwise gr*O-compact. Let $\zeta=\{A_i, i\in I$, an index set$\}$ be a pairwise gr*-open cover of Y. Then $Y=\cup A_i$ and $\zeta \subseteq \sigma_1$- gr*O$(Y,\sigma_1,\sigma_2) \cup \sigma_2$- gr*O$(Y,\sigma_1,\sigma_2)$ and ζ contains at least one member of σ_1- gr*O(Y,σ_1,σ_2) and one member of σ_2- gr*O(Y,σ_1,σ_2).Therefore, $X = f^1[\cup(A_i)] = \cup f^1(A_i)$ and $f^1(\zeta) \subseteq \tau_1$- gr*O$(X,\tau_1,\tau_2) \cup \tau_2$- gr*O$(X,\tau_1,\tau_2)$ and $f^1(\zeta)$ contains at least one member of τ_1- gr*O(X,τ_1,τ_2) and one member of τ_2- gr*O(X,τ_1,τ_2). Therefore, $f^1(\zeta)$ is the pairwise gr*-open cover of X. Since X is pairwise gr*O-compact, we have $X=\cup f^1(A_i)$, i = 1….. n. $\Rightarrow$ $Y=f(X) = \cup(A_i)$, i=1…. n. Hence, ζ has the finite subcover. Therefore, Y is pairwise gr*O-compact.

Theorem 8.7. If Y is τ_1- gr* closed subset of a pairwise gr*O-compact space (X,τ_1,τ_2), then Y is τ_2- gr*O-compact.

Proof. Let (X,τ_1,τ_2) be a pairwise gr*O-compact space. Let $\zeta = \{A_i, i\in I$, an index set$\}$ be a τ_2- gr* open cover of Y. Since Y is τ_1- gr* closed subset, Y^C is τ_1- gr* open. Also $\zeta \cup Y^C = Y^C \cup \{A_i, i\in I$, an index set$\}$ be a pairwise gr* open cover of X. Since X is pairwise gr*O-compact, $X = Y^C \cup A_1 \cup \ldots\ldots \cup A_n$. Hence $Y= A_1 \cup \ldots\ldots \cup A_n$. Therefore, Y is τ_2- gr*O-compact.

Since every τ_1-closed set is τ_1- gr* closed, we have the following.

Therorem 8.8. If Y is τ_1-closed subset of a pairwise gr*O-compact space (X,τ_1,τ_2), then Y is τ_2- gr*O-compact.

Theorem 8.9. If (X,τ_1) and (X,τ_2) are Hausdorff and (X,τ_1,τ_2) is pairwise gr*O-compact, then $\tau_1= \tau_2$.

Proof. Let (X,τ_1) and (X,τ_2) are Hausdorff and (X,τ_1,τ_2) is pairwise gr*O-compact. Since every pairwise gr*O-compact space is pairwise compact, we have (X,τ_1) and (X,τ_2) are Hausdorff and (X,τ_1,τ_2) is pairwise compact. Let F be τ_1-closed in X. Then F^C is τ_1-open in X. Let $\zeta = \{A_i, i\in I$, an index set$\}$ be the τ_2-open cover for X. Therefore, $\zeta \cup F^C$ is the pairwise open cover for X. Since X is pairwise compact, $X = F^C \cup A_1 \cup \ldots\ldots \cup A_n$. Hence $F = A_1 \cup \ldots\ldots \cup A_n$. Hence F is τ_2-compact. Since (X,τ_2) is Hausdorff, we have F is τ_2-closed. Similarly, every τ_2-closed set is τ_1-closed. Therefore $\tau_1= \tau_2$.

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