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Some Coupled Fixed Point Theorems in Dislocated Quasi Metric Spaces

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Abstract: *In this paper, we study some coupled fixed point theorems in a dislocated quasi-metric space which improve and extend some existing results.*
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I. Introduction

The study of partial metric spaces and generalized ultra metric spaces have applications in theoretical computer science had been studied by Matthews [2]. Hitzler and Seda [1] introduced the concept of dislocated metric space as a generalization of metrics where self-distances need not be zero. They also proved a generalized version of Banach contraction mapping principle which was applied to obtain fixed point semantics for logic programs. Recently, many authors studied different properties and fixed point results of dislocated (fuzzy or) metric spaces. Lakshmikantham and ´Ciri´c [7] introduced coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces.

Definition 1.1: Let X be a set and let $d: X \times X \to [0, \infty)$ be a function, called a distance function. Consider the following conditions:
1. For all $x, y \in X$, if $d(x, x) = 0$,
2. For all $x, y \in X$, if $d(x, y) = d(y, x) = 0$, then $x = y$,
3. For all $x, y \in X$, if $d(x, y) = d(y, x)$,
4. For all $x, y, z \in X$, $d(x, y) \le d(x, z) + d(z, y)$.

If d satisfies conditions (1) to (4), then it is called a metric. If it satisfies conditions (1), (2) and (4), it is called a quasi-metric. If it satisfies (2), (3) and (4), we will call it a dislocated metric (or simply d-metric). If it satisfies conditions (2) and (4), it is called a dislocated quasi-metric (or simply dq-metric). In the present paper, we prove some coupled fixed point theorems in dislocated quasi-metric spaces.

II. Fixed Point Results

Definition 2.1: A sequence (x_n) in dq-metric space (X, d) is called Cauchy if for all $\epsilon > 0$, there exists $n_0 \in N$ such that for all $m, n \ge n_0, d(x_m, x_n) < \epsilon, d(x_n, x_m) < \epsilon$. Replacing $d(x_m, x_n) < \epsilon$ and $d(x_n, x_m) < \epsilon$ in this definition by $\max\{d(x_m, x_n), d(x_n, x_m)\} < \epsilon$, the sequence (x_n)in dq-metric space $(X, d)$is called 'bi' Cauchy.

Definition 2.2: A sequence (x_n)in dislocated quasi-converges (for short dq-converges) to x if
$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0$$
In this case x is called dq-limit of(x_n).

Definition 2.3: A dq-metric space (X, d) is called complete if every Cauchy sequences in it is dq-convergent.

Definition 2.4: Let (X, d_1) and (Y, d_2)be a dq-metric spaces and let $f: X \to Y$ be a function. Then f is continuous if for each sequence (x_n) which is d_1q-convergent to x_0 in X, the sequence $(f(x_n))$ is d_2q-convergent to $(f(x_0))$ in Y.

Definition 2.5: let (X, d) be a dq-metric spaces. A map $f: X \to X$ is called contraction if there exists $0 \le \lambda \le 1$ such that $d(f(x), f(y)) \le \lambda d(x, y)$ for all $x, y \in X$.

Lemma 2.6: Every subsequence of dq-convergent sequence to a point $x_0$is dq-convergent to x_0.

Definition 2.7: An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping
$F: X \times X \to X$, if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2.8: An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping
$F: X \times X \to X$ and $g: X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 2.9: Let X be a nonempty set. Then we say that the mappings $F: X \times X \to X$ and $g: X \to X$ commutative if $gF(x, y) = F(gx, gy)$.

Lemma 2.10: Let (X, d) be a dislocated quasi metric space, $g: X \to X$ and $F: X \times X \to X$ be two mappings. Suppose that there exist k_1, k_2 and $k_3$in [0, 1) with $k_1 + k_2 + k_3 < 1$ such that the condition

$$d\big(F(x,y),F(u,v)\big)+d\big(F(y,x),F(v,u)\big)\le\ k_1\big(d(gx,gu)+d(gy,gv)\big)$$
$$+k_2\Big(d\big(gx,F(u,v)\big)+d\big(gy,F(v,u)\big)\Big)+k_3\Big(d\big(gu,F(x,y)\big)+d\big(gv,F(y,x)\big)\Big)$$

Holds for all $x,y,u,v\in X$.

Assume that (x, y) is a coupled coincidence point of the mappings F and g.

If $k_1+k_2+k_3<1$, then $F(x,y)=gx=gy=F(y,x)$.

Proof: Since (x, y) is a coupled coincidence point of the mappings F and g, we have $F(x,y)=gx$ and $F(y,x)=gy$. Assume that $gx\neq gy$.

Then by (1), we get

$$\begin{aligned}d(gx,gy)+d(gy,gx)&=d\big(F(x,y),F(y,x)\big)+d\big(F(y,x),F(x,y)\big)\\&\le k_1\big(d(gx,gy)+d(gy,gx)\big)\\&\quad+k_2\Big(d\big(gx,F(y,x)\big)+d(gy,F(x,y)\Big)\\&\quad+k_3\big(d(gy,gx)+d(gx,gy)\big)\\&\le k_1\big(d(gx,gy)+d(gy,gx)\big)\\&\quad+k_2\big(d(gx,gy)+d(gy,gx)\big)\\&\quad+k_3\big(d(gx,gy)+d(gy,gx)\big)\\&\le(k_1+k_2+k_3)\big(d(gx,gy)+d(gy,gx)\big)\end{aligned}$$

$$d(gx,gy)+d(gy,gx)\le\ (k_1+k_2+k_3)\big(d(gx,gy)+d(gy,gx)\big)$$

Since $k_1+k_2+k_3<1$, we have $d(gx,gy)=d(gy,gx)=0$, we get that $gx=gy$

III. Main result

Let (X,d) be a dislocated quasi metric space, $g\colon X\to X$ and $F\colon X\times X\to X$ be two mappings. Suppose that there exist k_1, k_2 and $k_3$in [0, 1) with $k_1+2k_2+2k_3<1$ such that the condition

$$\begin{aligned}d\big(F(x,y),F(u,v)\big)+d\big(F(y,x),F(v,u)\big)\le\ &k_1\big(d(gx,gu)+d(gy,gv)\big)\\&+k_2\Big(d\big(gx,F(u,v)\big)+d\big(gy,F(v,u)\big)\Big)\\&+k_3\Big(d\big(gu,F(x,y)\big)+d\big(gv,F(y,x)\big)\Big)\end{aligned}$$

Holds for all $x,y,u,v\in X$, also suppose the following hypotheses:

1. $F(X\times X)\subseteq gX$.
2. $g(x)$ is a complete subspace of X with respect to the dislocated-quasi metric space dq.
3. g commutes withF, and then the mapping F and g have a have a unique common coupled fixed point.

Proof: Let $x_0,y_0\in X$.

Since $F(X\times X)\subseteq gX$, we put $gx_1=F(x_0,y_0)$ and $gy_1=F(y_0,x_0)$.

Again, since$F(X\times X)\subseteq gX$, we put $gx_2=F(x_1,y_1)$ and $gy_2=F(y_1,x_1)$.

Continuing the process, we can construct two sequences (gx_n) and (gy_n) in X such that $gx_n=F(x_{n-1},y_{n-1})$, for all $n\in N$, and $gy_n=F(y_{n-1},x_{n-1})$, for all $n\in N$.

Let$n\in N$, then by inequality (2.1), we obtain

$$\begin{aligned}d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1})&=d\big(F(x_{n-1},y_{n-1}),F(x_n,y_n)\big)+d\big(F(y_{n-1},x_{n-1}),F(y_n,x_n)\big)\\&\le k_1\big(d(gx_{n-1},gx_n)+d(gy_{n-1},gy_n)\big)\\&\quad+k_2\Big(d\big(gx_{n-1},F(x_n,y_n)\big)+d\big(gy_{n-1},F(y_n,x_n)\big)\Big)\\&\quad+k_3\Big(d\big(gx_n,F(x_{n-1},y_{n-1})\big)+d\big(gy_n,F(y_{n-1},x_{n-1})\big)\Big)\\&\le k_1\big(d(gx_{n-1},gx_n)+d(gy_{n-1},gy_n)\big)\\&\quad+k_2\big(d(gx_{n-1},gx_{n+1})+d(gy_{n-1},gy_{n+1})\big)\\&\quad+k_3\big(d(gx_n,gx_n)+d(gy_n,gy_n)\big)\\&\le k_1\big(d(gx_{n-1},gx_n)+d(gy_{n-1},gy_n)\big)\\&\quad+k_2\big(d(gx_{n-1},gx_n)+d(gy_{n-1},gy_n)\big)\\&\quad+k_2\big(d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1})\big)\\&\quad+k_3[d(gx_{n-1},gx_n)+d(gy_{n-1},gy_n)]\\&\quad+k_3[d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1})]\end{aligned}$$

From (2.2), we have

$$d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1})\le\left\{\frac{k_1+k_2+k_3}{1-(k_2+k_3)}\right\}\big(d(gx_{n-1},gx_n)+d(gy_{n-1},gy_n)\big).$$

Put $\theta=\{\frac{k_1+k_2+k_3}{1-(k_2+k_3)}\}$. Then $\theta<1$. Repeating (2.3) n-times, we get

$$d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1})\le\theta^n\big(d(gx_0,gx_1)+d(gy_0,gy_1)\big).$$

Let m and n be natural numbers with $m > n$. Then

$$
\begin{aligned}
d(gx_n, gx_m) + d(gy_n, gy_m) &\leq \sum_{i=n}^{m-1}\big(d(gx_i, gx_{i+1}) + d(gy_i, gy_{i+1})\big) \\
&\leq \sum_{i=n}^{m-1} \theta^i\big(d(gx_0, gx_1) + d(gy_0, gy_1)\big) \\
&\leq \left[\frac{\theta^n}{1-\theta}\right]\big(d(gx_0, gx_1) + d(gy_0, gy_1)\big)
\end{aligned}
$$

Let $m, n \to \infty$, we get $\lim_{m,n\to+\infty} d(gx_n, gx_m) = \lim_{m,n\to+\infty} d(gy_n, gy_m) = 0$

By similar arguments as above, we can show that

$\lim_{m,n\to+\infty} d(gx_m, gx_n) = \lim_{m,n\to+\infty} d(gy_m, gy_n) = 0$

Thus the sequences (gx_n) and (gy_n) are Cauchy in (gX, q). Since (gX, q) is complete, there are x and y in X such that $gx_n \to x$ and $gy_n \to y$, and also, f and g are commute, we have

$g(gx_{n+1}) = g\big(F(x_n, y_n)\big) = F((gx_n, gy_n)$ and $g(gy_{n+1}) = g\big(F(y_n, x_n)\big) = F((gy_n, gx_n)$.

We get $gx = g\big(\lim_{n\to\infty} gx_n\big) = \lim_{n\to\infty} d(\,F(x_{n-1}, y_{n-1})\big) = \lim_{n\to\infty} F(gx_{n-1}, gy_{n-1})$

$= F(\lim_{n\to\infty} gx_{n-1},\ \lim_{n\to\infty} gy_{n-1}) = F(x, y)$

Hence $gx = F(x, y)$. Similarly, we may show that $gy = F(y, x)$.

By lemma (2.10), (x, y) is a coupled fixed point of the mappings F and g. So $gx = F(x, y) = F(y, x) = gy$.

Since (gx_{n+1}) is a sub sequence of (gx_n) and (gy_{n+1}) is a sub sequence of (gy_n) and are $dq -$ converges to x and y respectively. Thus

$$
\begin{aligned}
d(gx_{n+1}, gx) + d(gy_{n+1}, gy) &= d(gx_{n+1}, F(x, y) + d(gy_{n+1}, F(y, x)) \\
&= d(F(x_n, y_n), F(x, y) + d(F(y_n, x_n), F(y, x)) \\
&\leq k_1\big(d(gx_n, gx) + d(gy_n, gy)\big) + k_2\big(d(gx_n, gx) + d(gy_n, gy)\big) \\
&\quad + k_3\big(d(gx, gx_n) + d(gy, gy_n\,)\big)
\end{aligned}
$$

$(gx_{n+1}, gx) + d(gy_{n+1}, gy) \leq (k_1 + k_2 + k_3)\big(d(gx_{n+1}, gx) + d(gy_{n+1}, gy)\big)$

Letting $n \to \infty$ in the above inequalities, we get

$d(x, gx) + d(y, gy) \leq (k_1 + k_2 + k_3)\big(d(x, gx) + d(y, gy)\big)$

Since $k_1 + k_2 + k_3 < 1$, we get $d(x, gx) = 0$ and $d(y, gy) = 0$.

By similar arguments as above, we can show that

$d(gx, x) + d(gy, y) \leq (k_1 + k_2 + k_3)\big(d(gx, x) + d(gy, y)\big)$

Since $k_1 + k_2 + k_3 < 1$, we get $d(gx, x) = 0$ and $d(gy, y) = 0$.

From Definition 1.1(2), we have $x = gx$ and $y = gy$.

Thus we get $gx = F(x, x) = x$ and $gy = F(y, y) = gy = y$.

To prove uniqueness, let $z \in X$, such that $z = gz = F(z, z)$.

$$
\begin{aligned}
\text{Then } d\big(F(x, x), F(z, z)\big) + d\big(F(y, y), F(z, z)\big) &= d(gx, gz) + d(gy, gz) \\
&\leq k_1\big(d(gx, gz) + d(gy, gz)\big) \\
&\quad + k_2\Big(d\big(gx, F(z, z)\big) + d\big(gy, F(z, z)\big)\Big) \\
&\quad + k_3\Big(d\big(gz, F(x, y)\big) + d\big(gz, F(y, x)\big)\Big) \\
&\leq k_1\big(d(gx, gz) + d(gy, gz)\big) \\
&\quad + k_2\big(d(gx, gz) + d(gy, gz)\big) \\
&\quad + k_3\big(d(gz, gx) + d(gz, gy)\big) \\
&\leq (k_1 + k_2 + k_3)\big(d(gx, gz) + d(gy, gz)\big)
\end{aligned}
$$

$$d(gx, gz) + d(gy, gz) \leq (k_1 + k_2 + +k_3)\big(d(gx, gz) + d(gy, gz)\big)$$

Which is a contradiction because $(k_1 + k_2 + +k_3) < 1$,

We conclude that $d(gx, gz) = d(gy, gz) = 0$, we get $gx = gz$ and $gy = gz$.

Since$gx = gy$, we get $x = z$ and $x = z$, which proves the uniqueness of common coupled fixed point of F and g.

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