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On M_n^{**}-Manifold

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Abstract: *In the present paper, after defining an integrated contact metric structure manifold [3] I have defined M_n^{**} and nearly M_n^{**} manifold. It has been shown that M_n^{**} is integrable. Several useful theorems on these manifolds have also been derived.*
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I. Introduction

Let M_n be a differentiable manifold of differentiability class C^∞. Let there exist in M_n a vector valued C^∞ - linear function Φ, a C^∞ - vector field η and a C^∞ -one form ξ such that

(1.1) $$\Phi^2(X) = a^2X - c\xi(X)\eta$$

(1.2) $$(\bar{\eta}) = 0,$$

(1.3) $$G(\bar{X},\bar{Y}) = a^2 G(X,Y) - c\,\xi(X)\xi(Y)$$

Where $\Phi(X) = \bar{X}$, a is a nonzero complex number and c is an integer.

Let us agree to say that Φ gives to M_n a differentiable structure define by algebraic equation (1.1). We shall call (Φ,η,a,c,ξ) as an integrated contact structure.

Remark 1.1: The manifold M_n equipped with an integrated contact structure (Φ,η,a,c,ξ) will be called an integrated contact structure manifold.

Remark 1.2: The C^∞ manifold M_n satisfying (1.1), (1.2) and (1.3) is called an integrated contact metric structure manifold (Φ,η,a,c,G,ξ)

Agreement 1.1: All the equations which follow will hold for arbitrary vector field $X,Y,Z,......$etc.

It is easy to calculate in M_n that

(1.4) $$\xi(\eta) = \frac{a^2}{c}$$

(1.5) $$\Phi(\bar{X}) = 0$$

and

(1.6) $$G(X,\eta) \underline{\underline{def}}\ \xi(X)$$

Remark 1.3: The integrated contact metric structure manifold (Φ,η,a,c,G,ξ) gives an almost norden contact metric manifold [2], Lorentzian Para-contact manifold [1] or an almost Para-contact Riemannian manifold [4] according as $(a^2 = -1,\ c = 1)$,$(a^2 = 1,\ c = -1)$ or $(a^2 = 1, c = 1)$

Agreement 1.2: An integrated contact metric structure manifold will be denoted by M_n.
In the sequel, arbitrary vector fields will be denoted by $X,Y,Z,......$etc.

Definition 1.1: A C^∞-manifold M_n satisfying

(1.7) $$\bar{X} = D_X \eta$$

will be denoted by M_n^*. It is easy to calculate in M_n^*

(1.8) $$(D_X \xi)(Y) = {}'\Phi(X,Y),$$

where

(1.9) $$'\Phi(X,Y) \underset{=\!=}{def} G(\bar{X},Y) = G(X,\bar{Y})$$

(1.10) $$(D_X \xi)(Y) - (D_Y \xi)(X) = 0$$

Definition 1.2: A C^∞-manifold M_n^* satisfying

(1.11) $$(D_X \xi)(\bar{Y}) = -(D_{\bar{X}} \xi)(Y) = -(D_Y \xi)(\bar{X}); \; D_\eta {}'\Phi = 0$$

will be called M_n^{**}-manifold if

(1.12) $$(D_X \Phi)(Y) = -\xi(Y)(D_{\bar{X}}\eta) + (D_Y \xi)(\bar{X})\eta$$

and will be called nearly M_n^{**}-manifold if

(1.13) $$(D_X \Phi)(Y) + (D_Y \Phi)(X) = -\xi(Y) D_{\bar{X}}\eta - \xi(X) D_{\bar{Y}}\eta$$

where D is a Riemannian connection.
The Nijenhuis tensor N with respect to Φ is given by

(1.14) $$N(X,Y) \underset{=\!=}{def} [\bar{X},\bar{Y}] + \overline{\overline{[X,Y]}} - \overline{[\bar{X},Y]} - \overline{[X,\bar{Y}]}$$

which yields

(1.15) $$N(X,Y) = (D_{\bar{X}}\Phi)(Y) - (D_{\bar{Y}}\Phi)(X) - \overline{(D_X \Phi)(Y)} + \overline{(D_Y \Phi)(X)}$$

and

(1.16) $$'N(X,Y,Z) = (D_{\bar{X}} {}'\Phi)(Y,Z) - (D_{\bar{Y}} {}'\Phi)(X,Z) - (D_X {}'\Phi)(Y,Z) + (D_Y {}'\Phi)(X,Z)$$

where

(1.17) $$'N(X,Y,Z) \underset{=\!=}{def} G(N(X,Y),Z)$$

II. On M_n^{**}-Manifold

Theorem 2.1: In M_n^*, we have

(2.1a) $$(D_X {}'\Phi)(Y,Z) = -\xi(Y)\, {}'\Phi(\bar{X},Z) + (D_Y \xi)(\bar{X})\xi(Z)$$

(2.1b) $$(D_X {}'\Phi)(Y,Z) + (D_Y {}'\Phi)(X,Z) = a^2 [\xi(Y)G(X,Z) + \xi(X)G(Y,Z)] + 2c\,\xi(X)\xi(Y)\xi(Z)$$

(2.1c) $$(D_X {}'\Phi)(\bar{Y},Z) + (D_Y {}'\Phi)(Y,\bar{Z}) = [(D_{\bar{Y}}\xi)(\bar{X})\xi(Z) + a^2\,\xi(Y)\, {}'\Phi(X,Z)]$$

Proof: (1.9) yields

(2.2) $$(D_X {}'\Phi)(Y,Z) = G((D_X \Phi)(Y),Z)$$

Operating G on both sides of (1.12) and using (1.3) (1.6) and (2.2), we get

(2.3) $$(D_X {}'\Phi)(Y,Z) = -\xi(Y)G(D_{\bar{X}}\eta,Z) + (D_Y \xi)(\bar{X})\xi(Z)$$

Using (1.7) and (1.9) in the above equation, we get (2.1a). Using (1.9) and (1.1) in (2.1a), we get

(2.4) $$\left(D_X \text{`}\Phi\right)(Y,Z)= a^2\xi(Y)G(X,Z)+c\,\xi(X)\xi(Y)\xi(Z)+\left(D_Y\xi\right)\left(\bar{X}\right)\xi(Z)$$

Interchanging X and Y in above equation, we get

(2.5) $$\left(D_Y \text{`}\Phi\right)(X,Z)= a^2\xi(X)G(Y,Z)+c\,\xi(Y)\xi(X)\xi(Z)+\left(D_X\xi\right)\left(\bar{Y}\right)\xi(Z)$$

adding (2.4) and (2.5) and using (1.11) , we get (2.1b). Barring Y in (2.4) and using (1.5), we get

(2.6) $$\left(D_X \text{`}\Phi\right)\left(\bar{Y},Z\right)=\left(D_{\bar{Y}}\xi\right)\left(\bar{X}\right)\xi(Z)$$

Barring Z in (2.4) and using (1.5) and (1.9), we get

(2.7) $$\left(D_X \text{`}\Phi\right)\left(Y,\bar{Z}\right)=a^2\xi(Y)\text{`}\Phi(X,Z)$$

adding (2.6) and (2.7), we get (2.1c).

Corollary 2.1: In M_n^{**} , we have

(2.8a) $$\left(D_X \text{`}\Phi\right)\left(Y,\bar{Z}\right)=-a^2\xi(Y)\text{`}\Phi(X,Z)$$

(2.8b) $$\left(D_X \text{`}\Phi\right)\left(\bar{Y},\bar{Z}\right)=0$$

(2.8c) $$\left(D_{\bar{\bar{X}}} \text{`}\Phi\right)(Y,Z)+\left(D_Y \text{`}\Phi\right)\left(\bar{X},Z\right)-\left(D_X \text{`}\Phi\right)\left(Y,\bar{Z}\right)=0$$

(2.8d) $$\left(D_{\bar{\bar{X}}} \text{`}\Phi\right)\left(\bar{Y},Z\right)+\left(D_{\bar{Y}} \text{`}\Phi\right)\left(\bar{X},Z\right)=0$$

Proof: Barring Z in (2.1a) and using (1.5), (1.9), (1.1), (1.3), we get (2.8a). Barring Y in (2.8a) and using (1.5), we get (2.8b). Barring X in (2.1b) and using (1.5) and (2.8a), we get (2.8c). Barring X and Y both in (2.1b) and using (1.5), we get (2.8d).

Theorem 2.2: M_n^{**} is integrable.

Proof: Barring X in (1.12), we get

(2.9) $$\left(D_{\bar{X}}\,\Phi\right)(Y)=-\xi(Y)\left(D_{\bar{\bar{X}}}\,\eta\right)+\left(D_Y\,\xi\right)\left(\bar{\bar{X}}\right)\eta$$

Barring both sides of (1.12) and using (1.2), we get

(2.10) $$\overline{\left(D_X\,\Phi\right)(Y)}=-\,\xi(Y)\overline{\left(D_{\bar{X}}\,\eta\right)}$$

Interchanging X and Y in (2.9) and (2.10) separately, we get

(2.11) $$\left(D_{\bar{Y}}\,\Phi\right)(X)=-\xi(X)\left(D_{\bar{\bar{Y}}}\,\eta\right)+\left(D_X\,\xi\right)\left(\bar{\bar{Y}}\right)\eta,$$

and

(2.12) $$\overline{\left(D_Y\,\Phi\right)(X)}=-\,\xi(X)\overline{\left(D_{\bar{Y}}\,\eta\right)}$$

Using (2.9), (2.10), (2.11), (2.12) and (1.7) in (1.15), we get

(2.13) $$N(X,Y)=\left[\left(D_Y\xi\right)\left(\bar{\bar{X}}\right)-\left(D_X\xi\right)\left(\bar{\bar{Y}}\right)\right]\eta$$

(1.1) yields

(2.14) $$\xi\left(\bar{\bar{Y}}\right)=\xi\left(a^2\,Y-c\,\xi(Y)\eta\right)$$

Differentiating corollary (2.14) covariantly along the vector X and using (1.4), we get

(2.15) $$\left(D_X\xi\right)\left(\bar{\bar{Y}}\right)=a^2\left(D_X\xi\right)(Y)$$

Integrating X and Y in the above equation, we get

(2.16) $$\left(D_Y\xi\right)\left(\bar{\bar{X}}\right)=a^2\left(D_Y\xi\right)(X)$$

Using (2.15), (2.16) and (1.10) in (2.13), we get

(2.17) $$N(X,Y)=0$$

which proves the theorem.

Corollary 2.2: In M_n^{**}, we have

(2.18) $$(D_X\Phi)(Y)=-a^2\xi(Y)X+c\xi(X)\xi(Y)\eta+(D_Y\xi)(\bar{X})\eta$$

(2.19) $$c\,\xi((D_X\Phi)(Y))=-a^2(D_Y\xi)(\bar{X})$$

(2.20) $$`N(X,Y,Z)=0$$

Proof: Using (1.7) and (1.1) in (1.12), we get (2.18). Operating ξ on both the sides of (2.18) and using (1.4), we get (2.19). Operating G on both the sides of (2.17) and using (1.17), we get (2.20).

III. Affine Connection

Let B be an affine connection in M_n^{**} defined by

(3.1) $$B_XY \underline{\underline{def}}\ D_XY+H(X,Y)$$

where $H(X,Y)$ is a vector valued bilinear function. If S be the torsion tensor of the connection B, we have

(3.2) $$S(X,Y)=H(X,Y)-H(Y,X)$$

If $H(X,Y)$ is skew-symmetric, we have

(3.3) $$S(X,Y)=2H(X,Y)=-2H(Y,X)$$

Consequently

(3.4) $$`S(X,Y,Z)=2\,`H(X,Y,Z)=-2\,`H(Y,X,Z),$$

where

(3.5a) $$`S(X,Y,Z)\ \underline{\underline{def}}\ G(S(X,Y),Z),$$

and

(3.5b) $$`H(X,Y,Z)\ \underline{\underline{def}}\ G(H(X,Y),Z)$$

Theorem 3.1: On M_n^{**}, we have

(3.6) $$(B_X\Phi)(Y)+\xi(Y)(B_{\bar{X}}\eta)-(B_Y\xi)(\bar{X})\eta=H(X,\bar{Y})-\overline{H(X,Y)}$$
$$+\xi(Y)H(\bar{X},\eta)+\xi(H(Y,\bar{X}))\eta$$

Proof: Using (1.5) in (1.12) and $\Phi(X)=\bar{X}$, we get

(3.7) $$D_X\bar{Y}-\overline{D_XY}=-\xi(Y)(D_{\bar{X}}\eta)-\xi(D_Y\bar{X})\eta$$

Using (3.1) in the above, we get (3.6).

Theorem 3.2: On M_n^{**}, we have

(3.8) $$(B_X\xi)(\bar{Y})=-(B_{\bar{X}}\xi)(Y)=-(B_Y\xi)(\bar{X}),$$

if

(3.9a) $$\xi(H(X,\bar{Y}))=0,$$

and

(3.9b) $H(X,Y)$ is skew-symmetric

Proof: Using (1.5) in (1.11), we have

$$\xi\left(D_X\bar{Y}\right)=-\xi\left(D_Y\bar{X}\right)$$

Using (3.1) in the above equation, we get

(3.10) $$\xi\left(B_X\bar{Y}\right)+\xi\left(B_Y\bar{X}\right)=\xi\left(H\left(X,\bar{Y}\right)\right)+\xi\left(H\left(Y,\bar{X}\right)\right)$$

From (3.9b), we have

(3.11) $$\xi\left(H\left(\bar{X},Y\right)\right)=-\xi\left(H\left(Y,\bar{X}\right)\right)$$

From (1.5), we get

(3.12) $$\xi\left(B_X\bar{Y}\right)=-\left(B_X\xi\right)\left(\bar{Y}\right)$$

From (3.10), (3.11) and (3.12), we get

(3.13) $$\left(B_X\xi\right)\left(\bar{Y}\right)+\left(B_Y\xi\right)\left(\bar{X}\right)=-\xi\left(H\left(X,\bar{Y}\right)\right)+\xi\left(H\left(\bar{X},Y\right)\right)$$

(1.11) yields

(3.14) $$\xi\left(D_X\bar{Y}\right)=\bar{X}\,\xi\left(Y\right)-\xi\left(D_XY\right)$$

Using (3.1) in above, we get

(3.15) $$\left(B_X\xi\right)\left(\bar{Y}\right)+\left(B_{\bar{X}}\xi\right)\left(Y\right)=-\xi\left(H\left(X,\bar{Y}\right)\right)-\xi\left(H\left(\bar{X},Y\right)\right)$$

Thus using (3.9a), (3.9b) in (3.13) and (3.15), we get (3.8).

Theorem 3.3: On M_n^{**}, we have

(3.16) $$\xi\left(B_X\bar{\bar{Y}}\right)+\xi\left(B_{\bar{X}}\bar{Y}\right)=\xi\left(H\left(\bar{X},\bar{Y}\right)\right)+a^2\xi\left(H\left(X,Y\right)\right)-c\,\xi\left(Y\right)\xi\left(H\left(X,\eta\right)\right)$$

Proof: (1.11) yields

$$\xi\left(D_X\bar{Y}\right)=\bar{X}\left(\xi\left(Y\right)\right)-\xi\left(D_{\bar{X}}Y\right)$$

Using (3.1) in the above equation, we get

$$\xi\left(D_X\bar{Y}\right)+\xi\left(B_{\bar{X}}Y\right)=\bar{X}\left(\xi\left(Y\right)\right)+\xi\left(H\left(X,\bar{Y}\right)\right)+\xi\left(H\left(\bar{X},Y\right)\right)$$

Barring Y in the above equation and using (1.1), (1.2), we get (3.16).

Theorem 3.4: In M_n^{**}, we have

(3.17) $$\left(D_X\,{}^{`}\Phi\right)\left(Y,Z\right)+\left(D_Y\,{}^{`}\Phi\right)\left(Z,X\right)+\left(D_Z\,{}^{`}\Phi\right)\left(X,Y\right)=2\Big[\xi\left(X\right)\left(D_X\xi\right)\left(\bar{Y}\right)$$
$$+\xi\left(Y\right)\left(D_X\xi\right)\left(\bar{Z}\right)+\xi\left(Z\right)\left(D_Y\xi\right)\left(\bar{X}\right)\Big]$$

Proof: From (1.7), (1.8) and (1.9), we have

(3.18) $$\left(D_X\xi\right)\left(Y\right)=G\left(D_X\eta,Y\right)$$

Barring X in (3.18), we get

(3.19) $$\left(D_{\bar{X}}\xi\right)\left(Y\right)=G\left(D_{\bar{X}}\eta,Y\right)$$

Using (3.19) in (2.3), we get

(3.20) $$\left(D_X\,{}^{`}\Phi\right)\left(Y,Z\right)=-\xi\left(Y\right)\left(D_{\bar{X}}\xi\right)\left(Z\right)+\left(D_Y\xi\right)\left(\bar{X}\right)\xi\left(Z\right)$$

By the cyclic permutation of X, Y, Z, we also have

(3.21) $$\left(D_Y\,{}^{`}\Phi\right)\left(Z,X\right)=-\xi\left(Z\right)\left(D_{\bar{Y}}\xi\right)\left(X\right)+\left(D_Z\xi\right)\left(\bar{Y}\right)\xi\left(X\right)$$

(3.22) $$\left(D_Z\,{}^{`}\Phi\right)\left(X,Y\right)=-\xi\left(X\right)\left(D_{\bar{Z}}\xi\right)\left(Y\right)+\left(D_X\xi\right)\left(\bar{Z}\right)\xi\left(Y\right)$$

adding (3.20), (3.21) and (3.22) and using (1.11), we get (3.17).

Theorem 3.5: M_n^{**} is necessarily nearly M_n^{**}.

Proof: In M_n^{**}, we have a result (3.21). Interchanging X and Z in (3.21), we get

(3.23) $$(D_Y `\Phi)(X,Z) = -\xi(X)\left((D_{\bar{Y}}\xi)(Z)\right) + (D_X\xi)(\bar{Y})\xi(Z)$$

adding (3.20) and (3.23) and using (1.11), we get

(3.24) $$(D_X `\Phi)(Y,Z) + (D_Y `\Phi)(X,Z) = -\xi(Y)(D_{\bar{X}}\xi)(Z) - \xi(X)(D_{\bar{Y}}\xi)(Z)$$

Using (2.2) and (3.19) in the above equation, we get

$$G\left((D_X\Phi)Y,Z\right) + G\left((D_Y\Phi)X,Z\right) = -\xi(Y)G(D_{\bar{X}}\eta,Z) - \xi(X)G(D_{\bar{Y}}\eta,Z)$$

which yield (1.13). Hence M_n^{**} is necessarily nearly M_n^{**}.

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