IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 10, Issue 4 Ver. IV (Jul-Aug. 2014), PP 108-116
www.iosrjournals.org

Chaos in the Dynamics of the Family of Mappings $f_c(x) = x^2 - x + c$

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Abstract: *In this paper, we will study the chaotic behaviour of the family of quadratic mappings $f_c(x) = x^2 - x + c$ through its dynamics. In first few sections, we will take a review of some basic definitions and examples including a dynamical system, orbit, fixed and periodic, etc. Later, we will prove some results that analyse the nature and the stability of the fixed and periodic points of a dynamical system. Using these results, we will study the dynamics of the family of mappings $f_c(x) = x^2 - x + c$ for various values of the real constant c.*
Keywords: *bifurcation, chaos, dynamical system, fixed points, orbits, periodic points, stability*

I. Introduction

Many authors including [1], [2], [3], [4], [5], [6] have defined the notion of a dynamical system, however, we give a formal definition of a dynamical system in general.

A dynamical system is a function $f: R^n \to R^n$, where the set R^n is called as the set of states or the state space. Given a state vector $x \in R^n$, the function f describes the rule by means of which the state vector x changes with time. Different types of dynamical systems with a variety of examples are explained by [2] and [3]. In this paper, we will consider only one dimensional discrete dynamical system $f_c(x) = x^2 - x + c$.

We begin with some formal definitions:

1.1 Iteration of a function

Let $f: S \to S$, $S \subseteq R$, be a given dynamical system. Iteration means the repetition of a particular process again and again. Iteration of a function, which is also called as composition of functions, is simply finding value of the same function over and over where the first output is used as the next input value. For example, let $f(x) = 2x^2 + 1$. The first iteration of f at x is supposed to be $f(x)$ itself. The second iterate of f at x is $(fof)(x) = f(f(x))$. This second iterate is denoted by f^2.

Thus, we have, $f^2(x) = f(f(x)) = f(2x^2 + 1) = 2(2x^2 + 1)^2 + 1 = 8x^4 + 8x^2 + 3$.
The third iteration of f at x is $f^3(x) = f(f(x^2)) = 2(8x^4 + 8x^2 + 3)^2 + 1$.
In general, the k^{th} iterate of f at x is $f^k(x) = f(f^{k-1}(x))$.

1.2 Orbit and seed

Let $f: S \to S$, $S \subseteq R$, be a given dynamical system. Given an initial point $x_0 \in S$, the orbit of x_0 under f is the sequence of iterates x_0, $x_1 = f(x_0)$, $x_2 = f^2(x_0)$, $x_3 = f^3(x_0)$, ..., $x_n = f^n(x_0)$. In this case, the initial point x_0 is called as the seed of the orbit.

For example, let $f(x) = \sqrt{x}$ and $x_0 = 2$.Then $x_1 = f(x_0) = \sqrt{2} \approx 1.4242$, $x_2 = f(x_1) = \sqrt{1.4242} \approx 1.1892$, $x_3 = f(x_2) = \sqrt{1.1892} \approx 1.0905$, $x_4 = f(x_3) = \sqrt{1.0905} \approx 1.0442$,...
Thus, the orbit of 2 is {2, 1.4242, 1.1892, 1.0905,...}

Similarly, the orbit of 3 is {3, 1.7320, 1.3160, 1.1472,...}, the orbit of 0.7 is {0.7, 0.8366, 0.9146, 0.9563,... }.We observe that all these orbits are getting closer to 1.

1.2 Fixed points and periodic points

A point x is said to be a fixed point of a function f if $f(x) = x$. It is clear that if x is a fixed point of f, then $f^n(x) = x$ for all $n \in Z^+$. Also, in this case the orbit of x is the constant sequence $\{x, x, x, ...\}$.

A point x_0 is said to be a periodic point with period n if $f^n(x_0) = x_0$ for some $n \in Z^+$. It is clear that if x_0 is periodic with period n, then it is periodic with period 2n, 3n, 4n,...The smallest n, in this case, is called as the prime period of the orbit. Thus x_0 is a periodic point with period n of f if it is a fixed point of f^n.

From the definition of the fixed point, it is clear that we can find the fixed points of a function $f(x)$ by solving for x the equation $f(x) = x$. For example, the fixed points of $f(x) = x^2 - 5x + 5$ are given by $x^2 - 6x + 5 = 0$, which gives $x = 1$ and $x = 5$. Geometrically, one can obtain the fixed points of $f(x)$ by plotting the graphs of $y = f(x)$ and $y = x$ on the same axes and finding their points of intersection, which are the fixed points of $f(x)$.

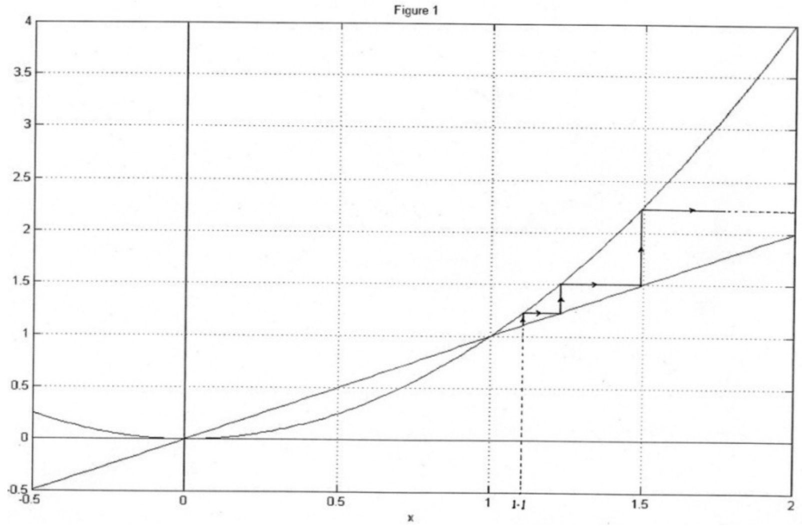
1.3 Graphical analysis of orbits (The orbit diagram)

The trend of an orbit can be studied by graphical analysis. An orbit diagram is a representation of an orbit in a plane which is useful for a one dimensional dynamical system. To draw an orbit diagram, we follow the following steps:

Step 1: Plot the given function $f(x)$ and the line $y = x$ in the plane.
Step 2: If x_0 is the seed whose orbit is to be obtained, draw a vertical line $y = x_0$ and find the point $(x_0, f(x_0))$. This point is simply (x_0, x_1), where $f(x_0) = x_1$.
Step 3: Draw a horizontal line from (x_0, x_1) to the line $y = x$ and obtain the point (x_1, x_1).
Step 4: Return to the **Step 2** with x_1 as the new seed.

Thus the orbit of x_0 looks like a staircase pattern. For example, consider $f(x) = x^2$. For $x_0 = 1.1$, the orbit diagram is as shown in Figure 1.

Figure 1 showing the orbit diagram of $f(x) = x^2$ at $x_0 = 1.1$



1.4 Attracting and repelling fixed points

Let p be a fixed point of a dynamical system $f: S \to S$, $S \subseteq R$.

(1) We say that p is an attracting fixed point or a sink of f if there is some neighbourhood of p such that all points in this neighbourhood are attracted towards p. In other words, p is a sink if there exists an epsilon neighbourhood $N_\epsilon(p) = \{x \in S: |x - p| < \epsilon\}$ such that $\lim_{n\to\infty} f^n(x) = p$ for all $x \in N_\epsilon(p)$.

(2) We say that p is a repelling fixed point or a source of f if there is some neighbourhood $N_\epsilon(p)$ of p such that each x in $N_\epsilon(p)$ except for p maps outside of $N_\epsilon(p)$. In other words, p is a source if there exists an epsilon neighbourhood such that $|f^n(x) - p| > \epsilon$ for infinitely many values of positive integers n.

Consider the function $f(x) = x^2 - ½$. Let us find the fixed points of f and their nature. Solving $f(x) = x$, we get two fixed points $r_1 = \frac{1}{2}(1 + \sqrt{3}) \approx 1.366025403784439$ and $r_2 = \frac{1}{2}(1 - \sqrt{3}) \approx -0.366025403784439$.

Now we verify the nature of the fixed point $r_1 = \frac{1}{2}(1 + \sqrt{3})$. Taking an initial value $x_0 = 1.36$ and finding the first 20 iterating the function f, we obtain the orbit of x_0 as follows:

$x_0 = 1.360000000000000E + 000$,
$x_1 = 1.349600000000000E + 000$,
$x_2 = 1.321420160000001E + 000$,
$x_3 = 1.246151239254428E + 000$,
$x_4 = 1.052892911095348E + 000$,
$x_5 = 6.085834822348364E - 001$,
$x_6 = -1.296261451509206E - 001$,
$x_7 = -4.831970624933125E - 001$,
$x_8 = -2.665205987978339E - 001$,
$x_9 = -4.289667704164440E - 001$,
$x_{10} = -3.159875098784858E - 001$
$x_{11} = -4.001518936007939E - 001$,
$x_{12} = -3.398784620476990E - 001$,
$x_{13} = -3.844826310360908E - 001$,
$x_{14} = -3.521731064315652E - 001$,
$x_{15} = -3.759741031063414E - 001$,
$x_{16} = -3.586434737933821E - 001$,
$x_{17} = -3.713748587054156E - 001$,
$x_{18} = -3.620807143215326E - 001$,
$x_{19} = -3.688975563164088E - 001$,
$x_{20} = -3.639145929437820E - 001$.

Note that $x_0 = 1.360000000000000E + 000$ means the number $x_0 = 1.360000000000000 \times 10^{+000}$ and similarly $x_6 = -1.296261451509206E - 001$ means $x_6 = -1.296261451509206 \times 10^{-001}$.

Continuing this way, the last five iterates among the first 120 iterates of the function f at

$x_0 = 1.36$ are as follows:

$x_{115} = -\,3.660254037844390E - 001,$
$x_{116} = -\,3.660254037844384E - 001,$
$x_{117} = -\,3.660254037844388E - 001,$
$x_{118} = -\,3.660254037844385E - 001,$
$x_{119} = -\,3.660254037844388E - 001,$
$x_{120} = -\,3.660254037844386E - 001.$

We note that the orbit is moving towards the fixed point r_2.
Similarly, taking an initial value $x_0 = 1.37$, we obtain the orbit of x_0 as follows:

$x_0 = 1.370000000000000E + 000,$
$x_1 = 1.376900000000000E + 000,$
$x_2 = 1.395853610000001E + 000,$
$x_3 = 1.448407300550034E + 000,$
$x_4 = 1.597883708286637E + 000,$
$x_5 = 2.053232345207855E + 000,$
$x_6 = 3.715763063407749E + 000,$
$x_7 = 1.330689514338534E + 001,$
$x_8 = 1.765734583570524E + 002,$
$x_9 = 3.117768619616970E + 004,$
$x_{10} = 9.720481160468307E + 008$
$x_{11} = 9.448775399101928E + 017,$
$x_{12} = 8.927935654267380E + 035,$
$x_{13} = 7.970803504673871E + 071,$
$x_{14} = 6.353370851012126E + 143,$
$x_{15} = 4.036532117049055E + 287,$
x_{16} = Infinity,
x_{17} = Infinity,
x_{18} = Infinity,
x_{19} = Infinity,
x_{20} = Infinity.

Thus, for $\epsilon = 0.007$, we observe that $f^{\,n}(x_0) \notin N_\epsilon(p)$ for infinitely many n. Hence r_1 is a repelling fixed point of f. The graph of the function $f(x) = x^2 - ½$ is as shown in Figure 2. The behaviour of the orbits near r_1 can be observed from the orbit diagram as shown in the Figure 2(a).

Figure 2 showing the graph of the function $f(x) = x^2 - ½$

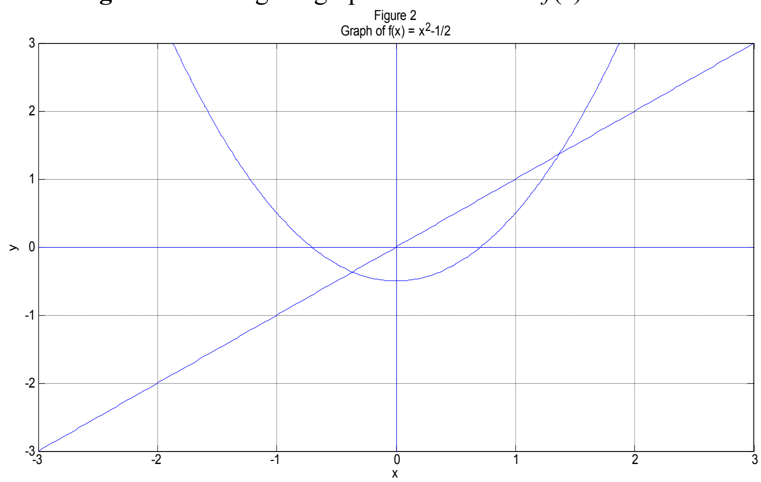
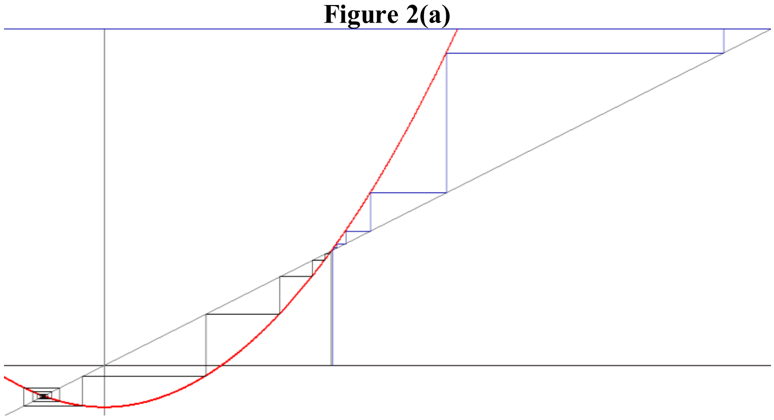


Figure 2(a)



Note that the orbit of $x_0 = 1.36$ is converging towards the fixed point r_2 (lower left corner) and the orbit of $x_0 = 1.37$ is tending towards the infinity (upper right corner), which confirms that r_1 is a repelling fixed point.

Consider now the second fixed point $r_2 = \frac{1}{2}\left(1-\sqrt{3}\right)$. Taking an initial value $x_0 = -\,0.37$ and iterating the function f, we obtain the orbit of x_0 as follows:

x_0 = – 3.700000000000000E – 001,
x_1 = – 3.631000000000000E – 001,
x_2 = – 3.681583900000000E – 001,
x_3 = – 3.644593998726079E – 001,
x_4 = – 3.671693458444985E – 001,
x_5 = – 3.651866714721230E – 001,
x_6 = – 3.666386949791117E – 001,
x_7 = – 3.655760673440139E – 001,
x_8 = – 3.663541389852850E – 001,
x_9 = – 3.657846448483505E – 001,
x_{10} = – 3.662015935931660E – 001,
x_{11} = – 3.658963928498257E – 001,
x_{12} = – 3.661198296994860E – 001,
x_{13} = – 3.659562703008193E – 001,
x_{14} = – 3.660760082275136E – 001,
x_{15} = – 3.659883562002094E – 001.

Continuing in this way, the higher iterates of f are as follows:

x_{90} = – 3.660254037844412E – 001,
x_{91} = – 3.660254037844368E – 001,
x_{92} = – 3.660254037844401E – 001,
x_{93} = – 3.660254037844376E – 001,
x_{94} = – 3.660254037844394E – 001,
x_{95} = – 3.660254037844382E – 001,
x_{96} = – 3.660254037844390E – 001,
x_{97} = – 3.660254037844384E – 001,
x_{98} = – 3.660254037844388E – 001,
x_{99} = – 3.660254037844385E – 001,
x_{100} = – 3.660254037844388E – 001.

We observe that the orbit of $x_0 = -\,0.37$ is converging towards the fixed point r_2.

Similarly, taking an initial value $x_0 = -\,0.35$ and iterating the function f, we obtain the orbit of x_0 as follows:

x_0 = – 3.500000000000000E – 001,
x_1 = – 3.775000000000000E – 001,
x_2 = – 3.574937500000000E – 001,
x_3 = – 3.721982187109375E – 001,
x_4 = – 3.614684859884052E – 001,
x_5 = – 3.693405336372502E – 001,
x_6 = – 3.635875702125513E – 001,
x_7 = – 3.678040787869331E – 001,
x_8 = – 3.647201596276955E – 001,
x_9 = – 3.669792051611484E – 001,
x_{10} = – 3.653262629792918E – 001,
x_{11} = – 3.665367215775853E – 001,
x_{12} = – 3.656508317351557E – 001,
x_{13} = – 3.662994692513889E – 001,
x_{14} = – 3.658246988261507E – 001,
x_{15} = – 3.661722897287561E – 001.

In this way the higher iterates of f are as follows:

x_{90} = – 3.660254037844285E – 001,
x_{91} = – 3.660254037844461E – 001,
x_{92} = – 3.660254037844332E – 001,
x_{93} = – 3.660254037844427E – 001,
x_{94} = – 3.660254037844357E – 001,
x_{95} = – 3.660254037844408E – 001,
x_{96} = – 3.660254037844370E – 001,
x_{97} = – 3.660254037844398E – 001,
x_{98} = – 3.660254037844378E – 001,
x_{99} = – 3.660254037844393E – 001,

$x_{100} = -3.660254037844382\text{E}-001.$

We observe that both the orbits are converging to r_2. The fixed point r_2 is an attracting fixed point.

II. Stability Theorems

In this section, we prove some theorems that analyse the stability of fixed points and periodic points. For these theorems, we need to recall some fundamental theorems from calculus.

2.1 Mean Value Theorem

Let f be a real valued function continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there exists a number c between a and b such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.
(Refer [1], [2])

2.2 Intermediate Value Theorem

Let $F : [a, b] \rightarrow R$ be a function continuous on the closed interval $[a, b]$. If $F(a) \neq F(b)$, then F assumes every value between $F(a)$ and $F(b)$ at least once over the interval $[a, b]$. (Refer [2])

In other words, if r is a number between $F(a)$ and $F(b)$, then there exists at least one c between a and b such that $F(c) = r$. (Refer [2])

Using the Intermediate Value theorem, the following theorem can be easily proved.

2.3 Fixed point theorem

Suppose that $F : [a, b] \rightarrow [a, b]$ is continuous. Then F has at least one fixed point over the interval $[a, b]$. (Refer [2])

2.4 Hyperbolic periodic points

A periodic point p of a mapping f with prime period n is said to be hyperbolic if $|(f^n)'(p)| \neq 1$. (Refer [2])

For example, consider $f(x) = x^2 - x$. This mapping has $x = 2$ as a hyperbolic fixed point whereas $x = 0$ is a non-hyperbolic fixed point.

2.5 Theorem

Let $f : [a, b] \rightarrow R$ be a differentiable function, where f' be continuous and p be a hyperbolic fixed point of f. If $|f'(p)| < 1$, then p is an attracting fixed point of f.

Proof : As $|f'(p)| < 1$, we can find an α with $|f'(p)| < \alpha < 1$. Since f is differentiable on $[a, b]$ and $p \in [a, b]$, an epsilon neighbourhood $N_\epsilon(p)$ can be obtained such that $\frac{|f(x)-f(p)|}{|x-p|} < \alpha$ for all $x \in N_\epsilon(p)$. Thus we have $|f(x) - f(p)| < \alpha|x - p|$. But as p is a fixed point, $|f(x) - p| < \alpha|x - p|$. This inequality implies that $f(x)$ is closer to p than x is by at least a factor $\alpha < 1$. Hence, if $x \in N_\epsilon(p)$, then $f(x) \in N_\epsilon(p)$.

By applying f again, we have $|f^2(x) - f^2(p)| < \alpha|f(x) - f(p)| < \alpha^2|x - p|$.

Hence by mathematical induction,

$$|f^n(p) - p| < \alpha^n |x - p|.$$

As $n \rightarrow \infty$, $\alpha^n \rightarrow 0$. Hence $f^n(x) \rightarrow p$, which proves the result. ⊡

Similar to the theorem 2.5, we have the following:

2.6 Theorem

Let $f : [a, b] \rightarrow R$ be a differentiable function, where f' be continuous and p be a hyperbolic fixed point of f. If $|f'(p)| > 1$, then p is a repelling fixed point of f.

2.7 Neutral fixed point

A fixed point p of a differentiable function f is said to be a neutral fixed point if $|f'(p)| = 1$.

Using theorems 2.5 and 2.6, we can decide the nature of a hyperbolic fixed point, but in case of a neutral fixed point, one needs further information to be processed.

2.8 Attracting and Repelling periodic point

Let p be a periodic point of period n of a function f. Then p is said to be an attracting periodic point or a repelling periodic point according as it is an attracting or a repelling fixed point of the n^{th} iterate f^n.

The following theorem gives a formula to find the derivative of the n^{th} iterate f^n.

By the repeated application of the Chain Rule, this theorem can be easily proved.

2.9 Theorem

Suppose $(x_0, x_1, x_2, ..., x_{n-1})$ is a cycle of period n for a given function F.
Then $(F^n)'(x_0) = F'(x_0).F'(x_1).F'(x_2)...F'(x_{n-1})$. (Refer [2])

For example, let $f(x) = -x^3$. The function f has a periodic point of period 1 at $x = 0$ (i.e. a fixed point) which is an attracting periodic point since $|f'(0)| = 0 < 1$. There is a period-2 point at $x = 1$ whose periodic orbit is $\{1, -1\}$. The periodic 2-cycle $\{1, -1\}$ is repelling since
$|(f^2)'(1)| = |f'(1).f'(-1)| = |-3(1)^2|.|-3(-1)^2| = 9 > 1.$

From the theorems 2.5 and 2.6, we have the following corollary:

2.10 Corollary

Let $f:[a, b] \rightarrow R$ be a differentiable function, where f' be continuous and p be a periodic point of f with period n. Then the periodic orbit of p is attracting or repelling according as $|(f^n)'(p)| < 1$ or $|(f^n)'(p)| > 1$.

The theorems 2.5 and 2.6 do not provide any information in order to study the behaviour of a non-hyperbolic periodic point. In such a situation, the higher ordered derivatives of the function at the periodic point can be effectively used.

Suppose p is an attracting periodic point for which the rate of convergence of an orbit of a seed near p is much slower than normally observed. In this case, we say that the point p is a weakly attracting periodic point. Similarly, if p is repelling with orbits of the seeds near p going away from p much slowly, then we say that p is weakly repelling. The following theorem gives a criterion for the study of the non-hyperbolic fixed points.

2.11 Theorem

Let p be a neutral fixed point of a function f.

(i) If $f''(p) > 0$, then p is weakly attracting from the left and weakly repelling from the right.

(ii) If $f''(p) < 0$, then p is weakly repelling from the left and weakly attracting from the right.

Let p be a neutral fixed point of f with If $f''(p) = 0$.

(iii) If $f'''(p) > 0$ then p is weakly repelling.

(iv) If $f'''(p) < 0$, then p is weakly attracting. (Refer [1],[2],[3])

If this theorem again fails to give any information, then it can be extended to higher order derivatives. Moreover, with slight changes, the theorem can be applied for periodic points also.

III. Dynamics of the Mapping $f_c(x) = x^2 - x + c$

Now we have enough material for the study of the dynamics of the mapping $f_c(x) = x^2 - x + c$. We will find the fixed points of $f_c(x)$ and then analyse their nature.

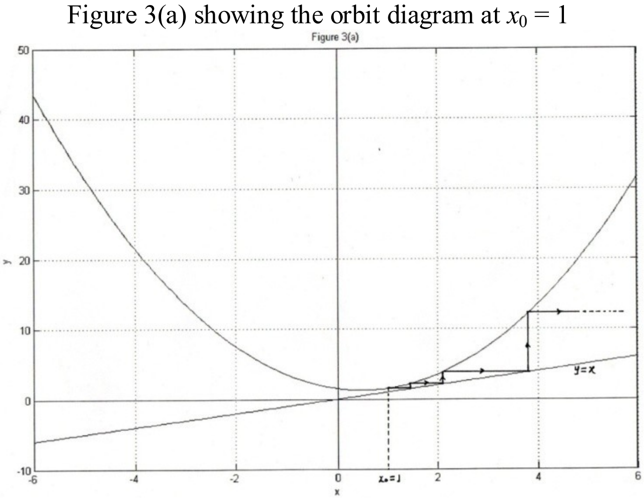
3.1 Fixed points of $f_c(x)$

As mentioned earlier, the fixed points of $f_c(x)$ can be obtained by solving for x the equation $f_c(x) = x$. Thus solving $x^2 - x + c = x$, we get the two roots $r_1 = 1 + \sqrt{1-c}$ and $r_2 = 1 - \sqrt{1-c}$.

We will consider different cases arising from different values of c.

Case 1. $c > 1$: It is clear that the two roots $r_1 = 1 + \sqrt{1-c}$ and $r_2 = 1 - \sqrt{1-c}$ are real if and only if $c \le 1$. For $c > 1$, the line $y = x$ and the function $f_c(x) = x^2 - x + c$ have no point in common. Hence for $c > 1$, there is no fixed point for $f_c(x)$ and all the orbits have a tendency to move towards infinity. For $c = 1.5$, the graph of $f_c(x) = x^2 - x + c$ is as shown in the Figure 3(a). The figure shows the orbit of the initial point $x_0 = 1$ that diverges to infinity.

Figure 3(a) showing the orbit diagram at $x_0 = 1$



Case 2. $c = 1$: For $c = 1$, we have the two roots $r_1 = r_2 = 1$.Thus there is just one fixed point at $x = 1$. What is the nature of the fixed point at $x = 1$? As $f_c'(1) = 1$, the point $x = 1$ is a neutral fixed point and the theorems 2.5 and 2.6 proves to be inconclusive. However, we can use theorem 2.11.
Since $f_c''(1) = 2.(1) = 2 > 0$, we conclude that $x = 1$ is weakly attracting from the left and weakly repelling from the right.
Case 3. $0 < c < 1$: As c becomes just less than 1, we get two fixed points $r_1 = 1 + \sqrt{1-c}$ and $r_2 = 1 - \sqrt{1-c}$. In this case, the line $y = x$ intersects the graph of the function $f_c(x)$ in to two distinct points. For $c < 1$, as $f_c'(r_1) = 2(1 + \sqrt{1-c}) - 1 = 1 + 2\sqrt{1-c} > 1$, by theorem 2.5, the point r_1 happens to be a repelling fixed point. Also, $f_c'(r_2) = 2(1 - \sqrt{1-c}) - 1 = 1 - 2\sqrt{1-c} < 1$. Hence r_2 is an attracting fixed point. However, in order to determine the set of values of c for which r_2 is attracting, we have to do a little more calculations.
Note that $|f_c'(r_2)| < 1 \iff -1 < 1 - 2\sqrt{1-c} < 1$
$\iff 0 < c < 1.$
Therefore, the fixed point $r_2 = 1 - \sqrt{1-c}$ is an attracting fixed point for $0 < c < 1$.

3.2 The occurrence of Saddle-Node Bifurcation

We observe that for $c = 1$, there is just one fixed point at $x = 1$. When c decreases from 1 to a little smaller value, we get two fixed points $r_1 = 1 + \sqrt{1-c}$ and $r_2 = 1 - \sqrt{1-c}$.
As the value of the parameter c passed through 1, the number of the fixed points has changed from 1 to 2. This change in the number and the nature of the fixed points is called as the bifurcation. The particular behaviour of the family $f_c(x) = x^2 - x + c$ as c passes through 1 is known as Saddle-Node Bifurcation or a Tangent Bifurcation. (Refer [2], [3])
Case 4. $c = 0$: When c takes the value 0, the two fixed points are $r_1 = 2$ and $r_2 = 0$. What is the behaviour of the orbits near these fixed points? Since $|f_c'(2)| = 5$, r_1 is a repelling fixed point.
Also, as $|f_c'(0)| = 0$, r_2 is a neutral fixed point. But as $|f_c''(0)| = 2 > 0$, r_2 is weakly attracting from the left and weakly repelling from the right.
Case 5. $-\frac{1}{2} < c < 0$: When c falls down through 0, $\sqrt{1-c} > 1$ so that $|f_c'(r_2)| > 1$. Hence r_2 is a repelling fixed point. But for $c < 0$, r_1 is also repelling. This indicates that there must be some period 2-cycle between these two fixed points.(Refer [2]) To find the periodic points of period 2 of $f_c(x)$, we have to find the fixed points of the mapping $f_c^{\,2}(x)$. Solving $f_c^{\,2}(x) = x$, we get a fourth degree equation
$x^4 - 2x^3 + 2cx^2 - 2cx + c^2 = 0$. Solving this equation, along with the fixed points r_1 and r_2 of $f_c(x)$, we get two periodic points $q_1 = \sqrt{-c}$ and $q_2 = -\sqrt{-c}$. These periodic points are real if and only if $c \le 0$. It can be easily verified that the points $q_1 = \sqrt{-c}$ and $q_2 = -\sqrt{-c}$ are periodic with period 2.Thus we have a period 2-cycle $\{q_1, q_2\}$. Let us find the stability of this 2-cycle. The 2-cycle is hyperbolic if and only if $|(f_c')^2(q_1)| \neq 1$. Hence $|(f_c')^2(q_1)| = 1 \iff -\frac{1}{2} < c < 0$. Thus the periodic 2-cycle is attracting if $-\frac{1}{2} < c < 0$.
Case 6. $c < -\frac{1}{2}$ **:** As c assumes a value less that –1/2, there is period 2-cycle, but loses its stability and an attracting period 4-cycle appears. In this manner, we come across a period doubling bifurcation . For further information, see [1], [2], [3].

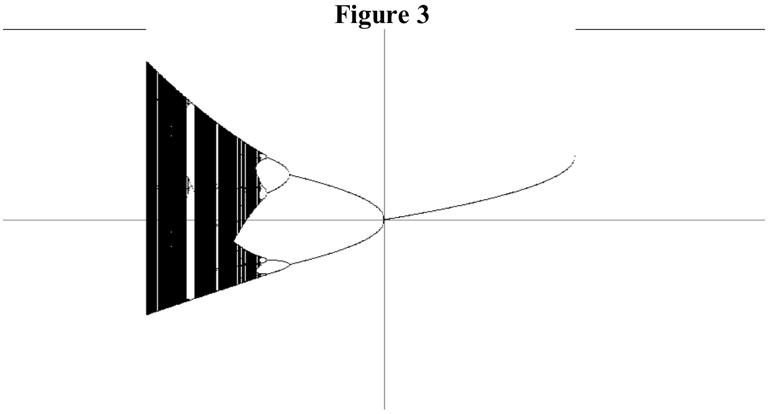
The period 4-cycle can be obtained by solving the equation $f_c^4(x) = x$. The solution gives periodic points of period 1, periodic points of period 2 which are already known to be r_1, r_2, q_1, q_2 and the periodic points of period 4 say s_1, s_2, s_3 and s_4.The values of c for which the periodic 4-cycle is attracting is determined by the equation $|(f_c^4)'(s_1)| = 1$. Call this value as c_1. Thus the period 4-cycle is attracting if $c_1 < c < -\frac{1}{2}$. When c falls down c_1, the period 4-cycle loses its stability and a periodic 8-cycle is born; again a period doubling bifurcation! For lower values of c again, this periodic 8-cycle gives birth to a period 16-cycle and so on. In this way, we witness a period doubling cascade!

3.3 Bifurcation diagram (Refer [1], [2], [3], [4], [5])

This diagram exhibits the most famous transition to chaos through successive period-doubling bifurcations as the parameter c is varied. The bifurcation diagram of a one parameter family $f_\mu(x)$ is a graph for which the horizontal axis represents the values of the parameter μ and the vertical axis represents the higher iterates of the variable x. For each value of the parameter μ, the diagram includes all points of the form $(\mu, f_\mu^n(x))$, for the values of n higher than 100.

In case of the family $f_c(x) = x^2 - x + c$ as all the interesting dynamics occur in the interval $-2 \le c \le 1$, we divide the parameter range [–2, 2] into a number of specified subdivisions. For each parameter value in this subdivision, the orbits are computed and plotted using the initial condition $x_0 = 0$ as shown in the Figure 3.

Figure 3



One can observe the repeated period doublings in this diagram. The first period doubling is observed at $c = 0$ and then as c decreases through 0, we come across successive period doubling bifurcations.

IV. Chaos

The word 'chaos' or sometimes the word 'unpredictable' is used interchangeably in our daily life. Roughly, chaos means a kind of disorder or randomness. We often experience chaos in weather, rising particles of smoke or prices of shares in stock market. This kind of unpredictability can also be observed in some mathematical functions.

Of many definitions of chaos given by different authors, we will consider the one given by Robert L. Devaney. [2]

4.1 Definition

A mapping $f : J \rightarrow J$ is said to be topologically transitive if for any pair of open sets $U, V \subset J$, there exists k > 0 such that $f^k(U) \cap V \neq \varphi$. (See [2])

A topologically transitive map has points that eventually move under iteration from one arbitrary small neighbourhood to any other. This definition implies that in case of a topologically transitive map, for every pair of points x and y and for each $\epsilon > 0$, there exists an orbit that comes within an ϵ-neighbourhood of the two points. If a map has a dense orbit, then it is topologically transitive.

4.2 Definition

A mapping $f : J \rightarrow J$ has sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any neighbourhood N of x, there exists $y \in N$ such that $|f^n(x) - f^n(y)| > \delta$. (Refer [2])

The definition says that f has sensitive dependence on initial conditions if arbitrarily close to any given point x in the domain of the function f, there is a point and an n^{th} iterate that is farther from the n^{th} iterate of x than a distance δ. Thus if a function f possesses sensitive dependence on initial conditions, the higher iterates of an approximate value of x and the computer calculations may be misleading.

As an example, the baker's function B given by

$$B(x) = \begin{cases} 2x \text{ for } 0 \leq x \leq 1/2 \\ 2x - 1 \text{ for } 1/2 < x \leq 1 \end{cases}$$

possesses sensitive dependence on initial conditions since it can be proved that after 10 iterations, the iterates of 1/3 and 0.333 are farther than 1/2 apart.

4.3 Definition

Let V be a set. A mapping $F : V \rightarrow V$ is said to be chaotic on V if

1. F has sensitive dependence on initial conditions.
2. F is topologically transitive.
3. periodic orbits are dense in V. (See[2])

In [2], it has been proved that the quadratic map $F_\mu(x) = \mu x(1- x)$ is chaotic in Λ for $\mu > 2 + \sqrt{5}$. Also, $F_\mu(x)$ is chaotic on the interval $I = [0, 1]$ for $\mu = 4$.

So far, we have analysed the dynamics of the mapping $f_c(x) = x^2 - x + c$ for the values of $c > -½$. As the value of c falls through $-½$, the dynamics of $f_c(x)$ becomes more and more complex. The analysis of $f_c(x)$ done so far points towards the chaotic behaviour and this can be proved in the sense of the above definition of chaos.

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