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Oscillation results for even-order Quasilinear neutral functional Difference equations

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***Abstract**: In this article, we use the Riccati transformation technique and some inequalities, to establish oscillation theorems for all solutions to even-order quasilinear neutral difference equation*

$$\Delta\left[\Delta^{m-1}\left(x(n)+p(n)x(\tau(n))\right)\right]^{\gamma}+q(n)x^{\gamma}\sigma(n)=0,\ \ n\geq n_0 .$$

Our main results are illustrated with suitable examples.

***Key words and phrases**: Oscillation, even-order, quasilinear, neutral difference equations. 2000 Mathematics Subject Classification 2110: 39A10, 39A12.*

I. Introduction

All over the world, during the last decade or two a lot of of research activity is undertaken on the study of the oscillation of neutral delay difference equation. Such equations appear in a number of important appilcations including problems in population dynamics or in "cobweb" models in Economics. Further, they are frequently used for the study of distributed networks containing lossleds transmission lines see the Hale [11]. Upto now, many studies have been done on the oscillation problem of even order difference equations, and we refer the reader to the papers [2,3,4,5,8,13,14,15,18,20,21,24,25,26,27,29,30,31] and the references cited there in.

In this paper, we consider the oscillatory behaviour of solutions to the even-order neutral difference equation

$$\Delta\left[\Delta^{m-1}\left(x(n)+p(n)x(\tau(n))\right)\right]^{\gamma}+q(n)x^{\gamma}\sigma(n)=0,\ \ n\geq n_0 \tag{1}$$

We will use the following assumptions:

- $m\geq 2$ is even and $\gamma\geq 1$ is the ratio of odd positive integers.
- $\{p(n)\}$ is a positive real sequence such that $0\leq p(n)\leq a$ is not identically zero, where a is a constant
- $\{q(n)\}$ is a positive real sequence such that $q(n)\neq 0$ for $n\geq n_0$.
- $\tau(n)$ and $\sigma(n)$ are positive sequences such that

$\lim_{n\to\infty}\tau(n)=\lim_{n\to\infty}\sigma(n)=\infty$ such that σ^{-1}, $\Delta\sigma^{-1}$ and $\Delta\tau$ exist.

By a solution of (1), we mean a real sequence $\{x(n)\}$ which is defined for $n=1,2,....$ and which satisfies Equation (1) for $n\geq Max\{\tau(n),\sigma(n)\}$. A solution $\{x(n)\}$ of (1) is said to be eventually positive if $x(n)>0$ for all large n, and eventually negative if $x(n)<0$ for all large n.

It is said to be oscillatory if it is neither eventually positive nor eventually negative. We will also say that (1) is oscillatory if all of its solutions are oscillatory.

I.Kubiaczyk and S.H.Sekar [16] studied the second order sublinear delay difference equation

$$\Delta(p_n\Delta x_n)+q_n x_{n-\tau}^{\gamma}=0,\ \ 0<\gamma\leq 1. \tag{E_1}$$

M K Yildiz and O.Ocalan [28], studied the neutral difference equation

$$\Delta^m(y_n-p_n y_{n-k})+q_n y_{n-l}^{\alpha}=0,\qquad n\geq n_1 \tag{E_2}$$

Where $1 > \alpha > 0$ is a quotient of odd positive integers and $\{p_n\}$ satisfies $-1 < p_n < 1$.

In 1998, Wan Tang Li [17] studied the oscillatory behaviour of the following higher order nonlinear difference equation

$$\Delta\left(\gamma_n \Delta^{\delta}(x_n - p_n x_{n-\tau})^{\delta}\right) + f(n, x_{n-\sigma}) = 0 \; n \ge n_0 \qquad (E_3)$$

under the condition $0 \le p_n \le p < 1$.

Guan.X, Yang.J., Liu.S.T and Cheng S.S. [9] studied the nonlinear neutral difference equation

$$\Delta^m\left[x_n - p_n x_{n-\tau}\right] + \sum_{i=1}^{\gamma} Q(n) f(x_n - \sigma_i(n)) = 0 \qquad (E_4)$$

and obtained some results for the oscillation of solutions of (E_4).

Pon.Sundar [22], considered the neutral difference equation of the form

$$\Delta^m\left[x_n - p_n x_{n-\tau}\right] + \sum_{i=1}^{\gamma} Q_i(n) f(x_n - \sigma_i) = 0 \qquad (E_5)$$

and obtained some sufficient conditions for the oscillation of solution of equation (E_5)

In 2010 M.Migda [19] considered the neutral difference equations

$$\Delta^m(x_n + p_n x_{n-\tau}) + f(n, x_n, x_{n-\sigma}) = 0 \qquad (E_6)$$

when $\{p_n\}$ is an oscillatory sequence and obtained some sufficient conditions for the oscillation of all solutions of (E_6).

Y.Bolat and O.Alzabut [6] considered the half-linear delay difference equation

$$\Delta\left[p_n(\Delta^{m-1}(x_n + q_n x_{\tau_n}))^{\alpha}\right] + \gamma_n x_{\sigma_n}^{\beta} = 0 \;\; n \ge n_0 \qquad (E_7)$$

under the condition $\sum_{n_0}^{\infty} \frac{1}{(p_s)^{\frac{1}{\alpha}}} < \infty$ and with using that $\Delta p_n \ge 0$ and derived some oscillation and asymptotic criteria for (E_7).

X.Zhou and J.Yan [32] studied the difference equation

$$\Delta\left[p_{n-1}(\Delta_{y_{n-1}}^{\Delta-1})^{\delta}\right] + s_n y_n^{\delta} = 0 \qquad (E_8)$$

and they obtained some comparison results and necessary and sufficient conditions for the oscillation of Equation (E_8).

S.S.Cheng and T.Patula [7], studied the difference equations

$$\Delta(\Delta y_{k-1})^{p-1} + s_k y_k^{p-1} = 0 \qquad (E_9)$$

when $p > 1$ and proved an existence theorem for equation (E_9).

II. Some Preliminary Lemmas

In the proofs of our main theorems we shall need the following Lemmas

Lemma 2.1 [1] (**Discrete Kneser's Theorem**) Let $\{u(n)\}$ be a sequence of real numbers in $N = \{0,1,2,...\}$. Let $u(n) > 0$ and $\Delta^m u(n)$ be of constant sign with $\Delta^m u(n)$, not being identically zero on any subset $\{n_0, n_0+1, n_0+2,...\}$. Then, there exists an integer l, $0 \le l \le m$ with $m+l$ odd for $\Delta^m u(n) \le 0$ and $m+l$ even for $\Delta^m u(n) \ge 0$ such that $l \le m-1$ implies $(-1)^{l+k}\Delta^k u(n) > 0$ for all $n \in N, l \le k \le m-1,$ and $l \ge 1$ implies $\Delta^k u(n) > 0$, for all $n \in N, 1 \le k \le l-1$.

Lemma 2.2 [1] Let $z(n)$ be as defined in Lemma 2.1 and such that $z(n) > 0$ and $\Delta^m z(n) \le 0$ for all $n \ge n_0 \in N$. Then there exists a sufficiently large integer n_1, such that for all $n \ge n_1 \ge n_0$

$$z(n) \ge \frac{(n-n_1)^{(m-1)}}{(m-1)!}\Delta^{m-1} z(2^{m-l-1} n)$$

where $(n-n_1)^{(m-1)}$ is the usual factorial notation.

Moreover, if $\{z(n)\}$ is increasing, then

$$z(n) \ge \frac{(n)^{(m-1)}}{(m-1)!}\Delta^{m-1} z(n)\frac{1}{(2^{m-1})^{(m-1)}} \quad for\ all\ \ n \ge 2n_1 \tag{2}$$

i.e.

$$z(n) \ge \frac{\lambda (n)^{(m-1)}}{(m-1)!}\Delta^{m-1} z(n) \tag{3}$$

Where $\lambda = \dfrac{1}{(2^{m-1})^{(m-1)}}$ and $0 < \lambda < 1$ for all $n \ge 2n_1$

Lemma 2.3 [23] Assume that (2) holds., Furthermore, Assume that $x(n)$ is an eventually positive solution of (1). Then there exists $n \ge n_0$ such that

$$z(n) > 0, \Delta z(n) > 0, \Delta^{m-1} z(n) > 0, \Delta^m z(n) \le 0 \quad for\ \ all\ \ n \ge n_1$$

Lemma 2.4 [12] Assume that $\gamma \ge 1, x_1, x_2 \in N$, If $x_1 \ge 1$ *and* $x_2 \ge 1$, then

$$x_1^\gamma + x_2^\gamma \ge \frac{1}{2^{\gamma-1}}(x_1 + x_2)^\gamma$$

Lemma 2.5 [10] Consider the oscillatory behavior of solutions of the following linear difference inequality $\Delta u(n) + p(n) y(n-r) \le 0$ where $p(n) \ge 0$ and $\{n-r\}$ is a real sequence of integers such that and $\lim_{n\to\infty}(n-r) = \infty$, If

$$\liminf_{n\to\infty} \sum_{n-r}^{n-1} p(n) > \left(\frac{r}{r+1}\right)^{r+1} \tag{4}$$

Then the inequality has no positive solutions.

We further need the following definition.

Definition 2.6 For any positive integer $n \ge n_0$, define $\tau^{-1}(n) = \{m, m\ is\ an\ integer\ \ge n\}$ and $\tau(m) = n$. The function τ^{-1} defined above is the inverse function of $\tau(n)$. Since $\tau(n)$ is increasing, it is one - to - one. If n is a positive integer greater than or equal to n_0 then $\tau^{-1}(\tau(n)) = n$.

III. Main Results

In this section, we establish some oscillation criteria for (1). For the sake of convenience, we set

$$z(n) = x(n) + p(n) x(\tau(n))$$

$$Q(n) = min\{q(\sigma^{-1}(n)), q(\sigma^{-1}(\tau(n)))\} \text{and } \Delta\rho(n)_+ = max\{0, \Delta\rho(n)\}$$

Theorem 3.1 Assume that $\Delta\sigma^{-1}(n) \geq \sigma_0 > 0$ and $\Delta\tau(n) = \tau_0 > 0$. Further assume that there exists a constant $\lambda,\ 0 < \lambda < 1$, such that

$$\Delta\left[\frac{y(\sigma^{-1}(n))}{\sigma_0} + \frac{a^\gamma y(\sigma^{-1}\tau(n))}{\sigma_0\tau_0}\right] + \frac{1}{2^{\gamma-1}}\left(\frac{\lambda}{(m-1)!}(n)^{m-1}\right)^\gamma Q(n)y(n) \leq 0 \tag{5}$$

has no eventually positive solution. Then (1) is oscillatory.

Proof. Let $x(n)$ be a nonoscillatory solution of equation(1).Without loss of generality, we assume that there exists $n_1 \geq n_0$ such that $x(n) > 0,\ x(\tau(n)) > 0$ and $x(\sigma(n)) > 0$ for all $n \geq n_1$. Then $z(n) > 0$ for all $n \geq n_1$. From (1), we obtain

$$\Delta\left[(\Delta^{m-1}z(n))^\gamma\right] = -q(n)x^\gamma(\sigma(n)) \leq 0,\ n \geq n_0 \tag{6}$$

By lemma 2.3 with m is even, there exists $n_2 \geq n_1$ such that $\Delta^m z(n) \leq 0$ *for* $n \geq n_2$.
Then from Lemma 2.1, there exists $n_3 \geq n_2$ and an odd integer $l \leq m-1$ such that for some large $n_4 \geq n_3$

$$(-1)^{l+k}\Delta^k z(n) > 0,\quad l \leq k \leq m-1 \tag{7}$$

and

$$\Delta^k z(n) > 0 \quad 1 \leq k \leq l-1 \tag{8}$$

Hence in view of (7) and (8), we obtain $\Delta z(n) > 0$ *and* $\Delta^{m-1}z(n) > 0$.
Therefore $\lim_{n\to\infty} z(n) \neq 0$.
Therefore by lemma 2.2, for any $\lambda,\ 0 < \lambda < 1$, there exists N_λ such that, for all $n \geq N_\lambda$

$$z(n) \geq \frac{\lambda}{(m-1)!}(n)^{(m-1)}\Delta^{m-1}z(n) \tag{9}$$

It follows from (1) that

$$\frac{\Delta\left[(\Delta^{m-1}z(\sigma^{-1}(n)))^\gamma\right]}{\Delta(\sigma^{-1}(n))} + q(\sigma^{-1}(n))x^\gamma(n) = 0 \tag{10}$$

Furthermore, from the above inequality and the definition of $z(n)$, we obtain,

$$\frac{\Delta\left[(\Delta^{m-1}z(\sigma^{-1}(n)))^\gamma\right]}{\Delta(\sigma^{-1}(n))} + \frac{a^\gamma\Delta\left[(\Delta^{m-1}z(\sigma^{-1}(\tau(n)))^\gamma\right]}{\Delta(\sigma^{-1}\tau(n))}$$
$$+ q(\sigma^{-1}(n))x^\gamma(n) + a^\gamma q(\sigma^{-1}\tau(n))x^\gamma(\tau(n)) = 0 \tag{11}$$

By(1) and the definition of Q, we obtain

$$\begin{aligned} q(\sigma^{-1}(n))x^\gamma(n) + a^\gamma q(\sigma^{-1}\tau(n))x^\gamma(\tau(n)) &\geq Q(n)\left[x^\gamma(n) + a^\gamma x^\gamma(\tau(n))\right] \\ &\geq \frac{1}{2^{\gamma-1}}Q(n)\left[x(n) + ax(\tau(n))\right]^\gamma \\ &\geq \frac{1}{2^{\gamma-1}}Q(n)z^\gamma(n) \end{aligned} \tag{12}$$

It follows from (11) and (12) that

$$\frac{\Delta\left[(\Delta^{m-1}z(\sigma^{-1}(n)))^\gamma\right]}{\Delta(\sigma^{-1}(n))} + \frac{a^\gamma\Delta\left[(\Delta^{m-1}z(\sigma^{-1}(\tau(n))))^\gamma\right]}{\Delta(\sigma^{-1}\tau(n))} + \frac{1}{2^{\gamma-1}}Q(n)z^\gamma(n) \leq 0 \tag{13}$$

From the inequality, $\Delta(\sigma^{-1}(n)) \geq \sigma_0 > 0$ *and* $\Delta(\tau(n)) \geq \tau_0 > 0$, we obtain

$$\frac{\Delta\left[(\Delta^{m-1}z(\sigma^{-1}(n)))^{\gamma}\right]}{\sigma_0}+\frac{a^{\gamma}\Delta\left[(\Delta^{m-1}z(\sigma^{-1}(\tau(n)))^{\gamma}\right]}{\sigma_0\tau_0}+\frac{1}{2^{\gamma-1}}Q(n)z^{\gamma}(n)\le 0 \tag{14}$$

Set $z(n)=\left(\Delta^{m-1}z(n)\right)^{\gamma}>0$. From (9) and (13), we see that $y(n)$ is an eventually positive solution of

$$\Delta\left[\frac{y(\sigma^{-1}(n))}{\sigma_0}+\frac{a^{\gamma}y(\sigma^{-1}\tau(n))}{\sigma_0\tau_0}\right]+\frac{1}{2^{\gamma-1}}\left(\frac{\lambda}{(m-1)!}(n)^{(m-1)}\right)^{\gamma}Q(n)y(n)\le 0$$

The proof is complete.

Theorem 3.2 Let τ^{-1} exist. Assume that $\tau(n)\le n, \Delta(\sigma^{-1}(n))\ge\sigma_0>0$ and $\Delta(\tau(n))\ge\tau_0>0$. Moreover, assume that there exists a constant λ, $0<\lambda<1$, such that

$$\Delta u(n)+\frac{1}{2^{\gamma-1}\left(\frac{1}{\sigma_0}+\frac{a^{\gamma}}{\sigma_0\tau_0}\right)}\left(\frac{\lambda}{(m-1)!}(n)^{m-1}\right)^{\gamma}Q(n)U(\tau^{-1}(\sigma(n)))\le 0 \tag{15}$$

has no eventually positive solution. Then Equation (1) is oscillatory.

Proof. Let $x(n)$ be a nonoscillatory solution of equation(1).

Without loss of generality, we assume that there exists $n_1\ge n_0$ such that $x(n)>0, x(\tau(n))>0$ and $x(\sigma(n))>0$ for all $n\ge n_1$. Then $z(n)>0$ for all $n\ge n_1$.

Proceeding as in the proof of the Theorem 3.1, we obtain that
$y(n)=\left(\Delta^{m-1}z(n)\right)^{\gamma}>0$ is non-increasing and satisfies the inequality (5), Define

$$u(n)=\frac{y(\sigma^{-1}(n))}{\sigma_0}+\frac{a^{\gamma}}{\sigma_0\tau_0}y(\sigma^{-1}(\tau(n)))$$

Then, from $\tau(n)\le n$ *and* σ^{-1} being increasing, we have

$$u(n)\le\left(\frac{1}{\sigma_0}+\frac{a^{\gamma}}{\sigma_0\tau_0}\right)y(\sigma^{-1}(\tau(n))).$$

Substituting the above formula in (5), we find $u(n)$ is an eventually positive solution of

$$\Delta u(n)+\frac{1}{2^{\gamma-1}\left(\frac{1}{\sigma_0}+\frac{a^{\gamma}}{\sigma_0\tau_0}\right)}\left(\frac{\lambda}{(m-1)!}(n)^{m-1}\right)^{\gamma}Q(n)U(\tau^{-1}(\sigma(n)))\le 0 \tag{16}$$

The proof is complete.

From Theorem 3.2 and Lemma 2.5, we establish the following Corollary.

Corollary 3.3 Let τ^{-1} exist. Assume
$\tau(n)\le n,\quad \Delta(\sigma^{-1}(n))\ge\sigma_0>0\quad and\quad \Delta(\tau(n))\ge\tau_0>0.\ \tau^{-1}(\sigma(n))<n$ and

$$\liminf_{n\to\infty}\lambda^{\gamma}\sum_{\tau^{-1}(\sigma(n))}^{n-1}Q(s)\left((s)^{m-1}\right)^{\gamma}>2^{\gamma-1}\left(\frac{1}{\sigma_0}+\frac{a^{\gamma}}{\sigma_0\tau_0}\right)((m-1)!)^{\gamma}\left(\frac{\sigma-\tau}{\sigma-\tau+1}\right)^{\sigma-\tau+1} \tag{17}$$

Then Equation(1) is oscillatory.

Proof. Taking $\tau(n)=n-\tau$ *and* $\sigma(n)=n-\sigma$, Applying Lemma 2.5 to Equation(16), one can choose a positive constant $0<\lambda<1$ such that

$$\liminf_{n\to\infty} \lambda^{\gamma} \sum_{n+\tau-\sigma}^{n-1} Q(s)\left((s)^{m-1}\right)^{\gamma} > 2^{\gamma-1}\left(\frac{1}{\sigma_0}+\frac{a^{\gamma}}{\sigma_0\tau_0}\right)((m-1)!)^{\gamma}\left(\frac{\tau}{\tau+1}\right)^{\tau+1}$$

This completes the proof.

Theorem 3.4 Assume that $\Delta(\sigma^{-1}(n)) \geq \sigma_0 > 0$ and $\Delta(\tau(n)) \geq \tau_0 > 0, \tau(n) \geq n,$ Furthermore, assume that there exists a constant $\lambda,\ 0 < \lambda < 1$, such that

$$\Delta u(n) + \frac{1}{2^{\gamma-1}\left(\frac{1}{\sigma_0}+\frac{a^{\gamma}}{\sigma_0\tau_0}\right)}\left(\frac{\lambda}{(m-1)!}(n)^{(m-1)}\right)^{\gamma} Q(n)U(\sigma(n)) \leq 0 \tag{18}$$

has no eventually positive solution. Then Equation (1) is oscillatory.

Proof. Let $x(n)$ be a nonoscillatory solution of Equation(1).Without loss of generality, we assume that there exists a $n_1 \geq n_0$ such that $x(n) > 0,\ x(\tau(n)) > 0$ and $x(\sigma(n)) > 0$ for all $n \geq n_1$. Then $z(n) > 0$ for all $n \geq n_1$. Proceeding as in the proof of the Theorem 3.1, we obtain that $y(n) = \left(\Delta^{m-1}z(n)\right)^{\gamma} > 0$ is non-increasing and satisfies the inequality (5), Define

$$u(n) = \frac{y(\sigma^{-1}(n))}{\sigma_0} + \frac{a^{\gamma}}{\sigma_0\tau_0} y(\sigma^{-1}(\tau(n)))$$

Then, from $\tau(n) \leq n$ we have

$$u(n) \leq \left(\frac{1}{\sigma_0}+\frac{a^{\gamma}}{\sigma_0\tau_0}\right) y(\sigma^{-1}(n)).$$

Substituting the above formula in (5), we find $u(n)$ is an eventually positive solution of

$$\Delta u(n) + \frac{1}{2^{\gamma-1}\left(\frac{1}{\sigma_0}+\frac{a^{\gamma}}{\sigma_0\tau_0}\right)}\left(\frac{\lambda}{(m-1)!}(n)^{(m-1)}\right)^{\gamma} Q(n)U(\sigma(n)) \leq 0 \tag{19}$$

The proof is complete.

From Theorem 3.3 and Lemma 2.5, we establish the following Corollary.

Corollary 3.5 Assume that $\Delta(\sigma^{-1}(n)) \geq \sigma_0 > 0$ and $\Delta\tau(n) \geq \tau_0 > 0\tau(n) \geq n,\ \sigma(n) < n$ and

$$\liminf_{n\to\infty} \lambda^{\gamma} \sum_{\sigma(n)}^{n-1} Q(s)(s)^{(m-1)} > 2^{\gamma-1}\left(\frac{1}{\sigma_0}+\frac{a^{\gamma}}{\sigma_0\tau_0}\right)((m-1)!)^{\gamma}\left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1} \tag{20}$$

Then Equation(1) is oscillatory.

Proof. Taking $\tau(n) = n + \tau$ *and* $\sigma(n) = n - \sigma$, Applying Lemma 2.5 to Equation(19), one can choose a positive constant $0 < \lambda < 1$ such that

$$\liminf_{n\to\infty} \lambda^{\gamma} \sum_{n-\sigma}^{n-1} Q(s)(s)^{(m-1)} > 2^{\gamma-1}\left(\frac{1}{\sigma_0}+\frac{a^{\gamma}}{\sigma_0\tau_0}\right)((m-1)!)^{\gamma}\left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1}$$

This completes the proof.

By employing Riccati transformation, we obtain the following oscillation criteria.

Theorem 3.6 Let $\Delta(\sigma^{-1}(n)) \geq \sigma_0 > 0,\ \sigma^{-1}(n) \geq n, \sigma^{-1}(\tau(n)) \geq n\ \ and \Delta(\tau(n)) \geq \tau_0 > 0$. Assume that there exists $\rho(n)$ such that

$$\limsup_{n\to\infty} \sum_{n_0}^{n-1} \left[\frac{1}{2^{\gamma-1}} \rho(s) Q(s) - \frac{\dfrac{1}{\sigma_0} + \dfrac{a^\gamma}{\sigma_0 \tau_0} \left(\Delta\rho(s)_+\right)^{\gamma+1}}{(\gamma+1)^{\gamma+1} \left(Mn^{(m-2)}\right)^\gamma \rho^\gamma(s)} \right] = \infty \tag{21}$$

holds for some constant $M > 0$. Then the Equation(1) is oscilltory.

Proof. Let $x(n)$ be a nonoscillatory solution of equation(1).Without loss of generality, we assume that there exists $n_1 \geq n_0$ such that $x(n) > 0,\ x(\tau(n)) > 0$ and $x(\sigma(n)) > 0$ for all $n \geq n_1$. Then $z(n) > 0$ for all $n \geq n_1$. Proceeding as in the proof of the Theorem 3.1, there exists $n_2 \geq n_1$ such that (7), (8) and (14) hold for all $n \geq n_2$. Using the Riccati Transformation

$$w(n) = \rho(n) \frac{\left(\Delta^{m-1} z(\sigma^{-1}(n))\right)^\gamma}{z^\gamma(n)} \quad n \geq n_2 \tag{22}$$

Then for $w(n) > 0\ \ for\ \ n \geq n_2$. Taking differencing of (22), we obtain

$$\Delta w(n) = \Delta\left[\left(\Delta^{m-1} z(\sigma^{-1}(n))\right)^\gamma\right] \frac{\rho(n)}{z^\gamma(n)} + \left(\Delta^{m-1} z(\sigma^{-1}(n+1))\right)^\gamma \Delta\left[\frac{\rho(n)}{z^\gamma(n)}\right]$$

$$= \Delta\left[\left(\Delta^{m-1} z(\sigma^{-1}(n))\right)^\gamma\right] \frac{\rho(n)}{z^\gamma(n)} + \left(\Delta^{m-1} z(\sigma^{-1}(n+1))\right)^\gamma \left[\frac{\Delta\rho(n) z^\gamma(n) - \rho(n)\Delta z^\gamma(n)}{z^\gamma(n) z^\gamma(n+1)}\right]$$

Now, by using the inequality $x^\gamma - y^\gamma \geq \gamma y^\gamma (x-y)$ for all $x \neq y > 0\ \ and\ \ \gamma > 1$.

$$\Delta w(n) \leq \left(\Delta\rho(n)\right)_+ \frac{\left(\Delta^{m-1} z(\sigma^{-1}(n+1))\right)^\gamma}{z^\gamma(n+1)} + \frac{\rho(n)}{z^{\gamma(n)}} \Delta\left[\left(\Delta^{m-1} z(\sigma^{-1}(n))\right)^\gamma\right]$$

$$- \frac{\left(\Delta^{m-1} z(\sigma^{-1}(n+1))\right)^\gamma}{z^\gamma(n) z^\gamma(n+1)} \rho(n) \gamma z^\gamma(n) \Delta z(n) \tag{23}$$

In view of lemma 2.2, we have

$$\Delta z(n) \geq M(n)^{(m-2)} \Delta^{m-1} z(n) \geq M(n)^{(m-2)} \Delta^{m-1} z(\sigma^{-1}(n+1)) \quad \text{for some } M > 0.$$

Thus from (22) and (23), we obtain

$$\Delta w(n) \leq \left(\Delta\rho(n)\right)_+ \frac{w(n+1)}{\rho(n+1)} + \rho(n) \frac{\Delta\left[\left(\Delta^{m-1} z(\sigma^{-1}(n))\right)^\gamma\right]}{z^\gamma(n)} - \gamma M(n)^{(m-2)} \rho(n) \frac{\left(\Delta^{m-1} z(\sigma^{-1}(n+1))\right)^{\gamma+1}}{z^{\gamma+1}(n+1)}$$

$$\Delta w(n) \leq \left(\Delta\rho(n)\right)_+ \frac{w(n+1)}{\rho(n+1)} + \rho(n) \frac{\Delta\left[\left(\Delta^{m-1} z(\sigma^{-1}(n))\right)^\gamma\right]}{z^\gamma(n)} - \gamma M(n)^{(m-2)} \rho(n) \left(\frac{w(n+1)}{\rho(n+1)}\right)^{\frac{\gamma+1}{\gamma}} \tag{24}$$

Next, We define the Sequence

$$\psi(n) = \frac{\rho(n)\left(\Delta^{m-1} z(\sigma^{-1}(\tau(n)))\right)^\gamma}{z^\gamma(n)},\ \ n \geq n_2 \tag{25}$$

Then $\psi(n) > 0$ for $n \geq n_2$, Differencing (25), we see that

$$\Delta\psi(n) \le (\Delta\rho(n))_+ \frac{\left(\Delta^{m-1} z(\sigma^{-1}(n+1))\right)^\gamma}{z^\gamma(n+1)} + \frac{\rho(n)\Delta\left[\left(\Delta^{m-1} z(\sigma^{-1}(\tau(n)))\right)^\gamma\right]}{z^\gamma(n)}$$
$$-\frac{\rho(n)\left(\Delta^{m-1} z(\sigma^{-1}(\tau(n)))\right)^\gamma}{z^\gamma(n) z^\gamma(n+1)} \gamma z^\gamma(n)\Delta z(n) \qquad (26)$$

In view of lemma 2.2, we have

$$\Delta z(n) \ge M(n)^{(m-2)}\Delta^{m-1} z(n) \ge M(n)^{(m-2)}\Delta^{m-1} z(\sigma^{-1}(\tau(n))) \quad \text{for some } M > 0.$$

Hence by (25) and (26),

$$\Delta\psi(n) \le (\Delta\rho(n))_+ \frac{\psi(n+1)}{\rho(n+1)} + \rho(n)\frac{\Delta\left[\left(\Delta^{m-1} z(\sigma^{-1}(\tau(n)))\right)^\gamma\right]}{z^\gamma(n)}$$
$$-\gamma M(n)^{(m-2)}\rho(n)\left(\frac{\psi(n+1)}{\rho(n+1)}\right)^{\frac{\gamma+1}{\gamma}} \qquad (27)$$

Therefore, from (24) and (27), it follows that

$$\frac{\Delta w(n)}{\sigma_0} + \frac{a^\gamma \Delta\psi(n)}{\sigma_0\tau_0} \le$$
$$\rho(n)\left[\frac{\dfrac{\Delta\left[\left(\Delta^{m-1} z(\sigma^{-1}(n))\right)^\gamma\right]}{\sigma_0} + \dfrac{a^\gamma}{\sigma_0\tau_0}\Delta\left[\left(\Delta^{m-1} z(\sigma^{-1}(\tau(n)))\right)^\gamma\right]}{z^\gamma(n)}\right]$$
$$+\frac{1}{\sigma_0}\left[\frac{(\Delta\rho(n))_+}{\rho(n+1)} w(n+1) - \gamma M(n)^{(m-2)}\rho(n)\left(\frac{w(n+1)}{\rho(n+1)}\right)^{\frac{\gamma+1}{\gamma}}\right]$$
$$+\frac{a^\gamma}{\sigma_0\tau_0}\left[\frac{(\Delta\rho(n))_+}{\rho(n+1)}\psi(n+1) - \gamma M(n)^{(m-2)}\rho(n)\left(\frac{\psi(n+1)}{\rho(n+1)}\right)^{\frac{\gamma+1}{\gamma}}\right] \qquad (28)$$

Thus, from the above inequality and (14),we have

$$\frac{\Delta w(n)}{\sigma_0} + \frac{a^\gamma}{\sigma_0\tau_0}\Delta\psi(n)$$
$$\le -\frac{1}{2^{\gamma-1}}\rho(n)Q(n) + \frac{1}{\sigma_0}\left[\frac{(\Delta\rho(n))_+ w(n+1)}{\rho(n+1)} - \gamma M(n)^{(m-2)}\rho(n)\left(\frac{w(n+1)}{\rho(n+1)}\right)^{\frac{\gamma+1}{\gamma}}\right]$$
$$+\frac{a^\gamma}{\sigma_0\tau_0}\left[\frac{(\Delta\rho(n))_+ \psi(n+1)}{\rho(n+1)} - \gamma M(n)^{(m-2)}\rho(n)\left(\frac{\psi(n+1)}{\rho(n+1)}\right)^{\frac{\gamma+1}{\gamma}}\right] \qquad (29)$$

Set $A = \dfrac{(\Delta\rho(n))_+}{\rho(n+1)}$ $\quad B = \dfrac{\gamma M(n)^{(m-2)}\rho(n)}{(\rho(n+1))^{\frac{\gamma+1}{\gamma}}}$;

$v = w(n+1), \psi(n+1)$ Then by using (29) and the inequality

$$Av - Bv^{\frac{\gamma+1}{\gamma}} \le \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}\frac{A^{\gamma+1}}{B^\gamma} B > 0 \qquad (30)$$

We have

$$\frac{\Delta w(n)}{\sigma_0}+\frac{a^\gamma}{\sigma_0\tau_0}\Delta\psi(n)\le -\frac{1}{2^{\gamma-1}}\rho(n)Q(n)+\frac{\left(\frac{1}{\sigma_0}+\frac{a^\gamma}{\sigma_0\tau_0}\right)\left((\Delta\rho(n))_+\right)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left(M(n)^{(m-2)}\rho(n)\right)^\gamma}$$

Summing the above inequality from n_2 to $n-1$, we have

$$\sum_{n_2}^{n-1}\left[\frac{1}{2^{\gamma-1}}\rho(n)Q(n)-\frac{\left(\frac{1}{\sigma_0}+\frac{a^\gamma}{\sigma_0\tau_0}\right)\left((\Delta\rho(s))_+\right)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left(M(s)^{(m-2)}\rho(s)\right)^\gamma}\right]\le\frac{w(n_2)}{\sigma_0}+\frac{a^\gamma}{\sigma_0\tau_0}\psi(n_0)$$

Which contradicts (21). The proof is complete.

Remark: From (29),Define a Philos-type sequence $H(n,s)$, and obtain some oscillation criteria for Equation (1), the details are left to the reader.

Theorem 3.7 Let $m=2$, $\Delta(\sigma^{-1}(n))\ge\sigma_0>0$, $\sigma^{-1}(n)\ge n$, $\sigma^{-1}(\tau(n))\ge n$ and $\Delta\tau(n)\ge\tau_0>0$. Assume that there exists a positive sequence $\rho(n)$ such that

$$\limsup_{n\to\infty}\sum_{n_0}^{n-1}\left[\frac{1}{2^{\gamma-1}}\rho(s)Q(s)-\frac{\left(\frac{1}{\sigma_0}+\frac{a^\gamma}{\sigma_0\tau_0}\right)\left((\Delta\rho(s))_+\right)^{\gamma+1}}{(\gamma+1)^{(\gamma+1)}\rho^\gamma(s)}\right]=\infty \tag{31}$$

Then the Equation (1) is oscillatory.

Proof. Define $w(n)=\rho(n)\frac{\left(\Delta z(\sigma^{-1}(n))\right)^\gamma}{z^\gamma(n)}$, $\psi(n)=\rho(n)\frac{\left(\Delta z(\sigma^{-1}(\tau(n)))\right)^\gamma}{z^\gamma(n)}$

The remainder of the proof is similar to that of Theorem 3.6

IV. Applications

Example 1. Consider the even-order difference equation

$$\Delta\left[\left(\Delta^{m-1}(x(n)+p(n)x(n-3))\right)^{\frac{5}{3}}\right]+\frac{(-1)^{(n-3)\frac{5}{3}}(1+2^{\frac{5}{3}})3^{\frac{5m}{3}}}{2^{(m+6)\frac{5}{3}}}x^{\frac{5}{3}}(n-6)=0, n>6 \tag{32}$$

Where $\gamma=\frac{5}{3}>1$ is the quotient of odd positive integers.

$p(n)=a=2>0$ Let $\tau(n)=n-3,\ \sigma(n)=n-6$

Then $\tau^{-1}(n)=n+3,\sigma^{-1}(n)=n+6,\ \tau^{-1}(\sigma(n))=n-3,\sigma^{-1}(\tau(n))=n+3 and\ Q(n)=\frac{(-1)^{(n-3)\frac{5}{3}}(1+2^{\frac{5}{3}})3^{\frac{5m}{3}}}{2^{(m+6)\frac{5}{3}}}$

Since

$$\liminf_{n\to\infty}\sum_{n-3}^{n-1}Q(s)(s)^{(m-1)\gamma}=\frac{3^{\frac{5m}{3}}(1+2^{\frac{5}{3}})}{2^{(m+6)\frac{5}{3}}}(n-3+n-2+n-1)^{(m-1)\frac{5}{3}}$$

$$= \frac{3^{\frac{5m}{3}}(1+2^{\frac{5}{3}})}{2^{(m+6)\frac{5}{3}}} 3^{(m-1)\frac{5}{3}} (n-2)^{(m-1)\frac{5}{3}}$$

By applying corollary 3.3, Equation (32) is oscillatory when

$$\frac{3^{\frac{5m}{3}}(1+2^{\frac{5}{3}})}{2^{(m+6)\frac{5}{3}}} 3^{(m-1)\frac{5}{3}} (n-2)^{(m-1)\frac{5}{3}} > 2^{\frac{2}{3}}(1+2^{\frac{5}{3}})\left(\frac{3}{4}\right)^4 \left((m-1)!\right)^{\frac{5}{3}} \quad for \ all \ n \geq 6$$

one such solution is $x(n) = \frac{(-1)^n}{2^n}$

Example 2. Consider the even-order neutral difference equation

$$\Delta\left[\left(\Delta^{m-1}(x(n)+ex(n+3))\right)^{\frac{5}{3}}\right] + \frac{(1+e^{\frac{5}{3}})(e^2-1)^{\frac{5}{3}}(1+e)^{\frac{5}{3}(m-1)}}{e^{(m+5)\frac{5}{3}}} x^{\frac{5}{3}}(n-3) = 0, n \geq 3 \qquad (33)$$

Where $\gamma = \frac{5}{3} > 1$ is the quotient of odd positive integers.

$p(n) = e > 0$. Let $\tau(n) = n+3 > n, \ \sigma(n) = n-3, \ \sigma^{-1}(n) = n+3, \ and \sigma^{-1}(\tau(n)) = n+6$

$$Q(n) = \frac{(1+e^{\frac{5}{3}})(e^2-1)^{\frac{5}{3}}(1+e)^{\frac{5}{3}(m-1)}}{e^{(m+5)\frac{5}{3}}}$$

Since

$$\sum_{n-3}^{n-1} Q(n)(s)^{(m-1)\frac{5}{3}} = \frac{(1+e^{\frac{5}{3}})(e^2-1)^{\frac{5}{3}}(1+e)^{\frac{5}{3}(m-1)}}{e^{(m+5)\frac{5}{3}}} (3n-6)^{(m-1)\frac{5}{3}}$$

$$\to \infty \ as \ n \to \infty$$

By applying corollary 3.5, Equation (33) is oscillatory.

One such solution of Equation (33) is $x(n) = \frac{(-1)^n}{e^n}$

Example 3. Consider the even-order neutral difference equation

$$\Delta\left[\left(\Delta^{m-1}(x(n)+2x(n+2))\right)^{\frac{7}{5}}\right] + \frac{(1+2^{\frac{7}{5}})3^{\frac{7m}{5}}}{2^{\frac{7m}{5}}} x^{\frac{7}{5}}(n+1) = 0, \ n > 1 \qquad (34)$$

Where $\gamma = \frac{7}{5} > 1$ is the quotient of odd positive integers.

$p(n) = 2 > 0$

Let $\tau(n) = n+2, \ \sigma(n) = n+1, \ \sigma^{-1}(n) = n-1, \ and \sigma^{-1}\tau(n) = n+1 \ and \ Q(n) = \frac{(1+2^{\frac{7}{5}})3^{\frac{7m}{5}}}{2^{\frac{7m}{5}}}$

Set $\rho(n) = 1$ Then

$$\sum_{n_0}^{n-1} \frac{1}{2^{\frac{2}{5}}} \frac{(1+2^{\frac{7}{5}})3^{\frac{7m}{5}}}{2^{\frac{7m}{5}}} \to \infty \ as \ n \to \infty$$

Then by Theorem 3.6, Every solution of Equation (34) is oscillatory.

One such solution of equation (34) is $x(n) = \frac{(-1)^n}{2^n}$

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