# Stability Analysis of an Infectious Disease Free Equilibrium of Asthma Dynamics Model.

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**Abstract:** *A deterministic model was proposed to study the spread and control of asthma disease and treatment. The model is a system of first order ordinary differential equations. The model equations were analyzed to obtain the characteristics equation and equilibrium states. Stability analysis of the free and endemic equilibrium states was carried out. From our stability analysis, we observed that the free equilibrium state will always be stable if the birth rate β is less than the death rate μ of the population (where $\frac{\beta}{\mu}$, is number of susceptible individuals produced).*

**Key words:** *Disease free equilibrium state, endemic equilibrium, asthma, infectious disease, stability analysis*

## I.     Introduction

Mathematical models are used particularly in the Natural Sciences and Engineering disciplines (such as Physics, Biology, and Electrical Engineering) but also in the Social Sciences (such as Economics, Sociology and Political Science); Physicists, Engineers, Computer Scientists and Economists use mathematical models most extensively.

Eykhoff (1974), defined a mathematical model as 'a representation of the essential aspects of an existing system (or a system to be constructed) which presents knowledge of that system in usable form'.

Mathematical modeling is the process of creating a mathematical representation of some phenomenon in order to gain a better understanding of that phenomenon, has become an important scientific technique over the last two decades and is becoming more and more a powerful tool to solve problems arising from science, engineering, industries and the society in general according to Benyah (2005).

Asthma, disorder of the respiratory system in which the passages that enable air to pass into and out of the lungs periodically narrow, causing coughing, wheezing, and shortness of breath. This narrowing is typically temporary and reversible, but in severe attacks, asthma may result in death. Asthma most commonly refers to bronchial asthma, an inflammation of the airways, but the term is also used to refer to cardiac asthma, which develops when fluid builds up in the lungs as a complication of heart failure. This article focuses on bronchial asthma. Encarta Premium (2009).

Asthma is a disease affecting the airways that carry air to and from your lungs. People who suffer from this chronic condition (long-lasting or recurrent) are said to be asthmatic. Asthma and Allergy Foundation of America.

Asthma is a chronic disease characterized by recurrent attacks of breathlessness and wheezing, which vary in severity and frequency from person to person. During an asthma attack, the lining of the bronchial tubes swells, causing the airways to narrow and reducing the flow of air into and out of the lungs. World Health Organization (WHO 2006).

## II.     Methodology

**Model description:** The S-I-R model is initially starting with three differential equations, the equations representing human population categorized into: S(t)  denoting susceptible human population, I(t) Infected human population and R(t) Recovered/treatment human population, all with respect to time t.

Members of the susceptible human population X(t) can move to the infected population Y(t) via blood relation and contraction with the disease. Infected population Y(t) can as well move to the susceptible population via treatment/ recovery. Recovery/treated human population N(t) comprises members that have been fully vaccinated and are not susceptible to infection.

**The Parameters are Defined as Follows:-**
β = Natural birth rate for the susceptible human population N(t).
μ = Natural death rate for the human population P(t).
c= contracting rate for the susceptible human population as a result of blood

relation with infected human population.

$v$ = The rate of recovery of the infected human population as a result of treatment.

$N(t)$ = The total population at time t.

$x(t)$ = The susceptible class.

$y(t)$ = The infected class.

**The Model Equations**

The dynamics of the disease is given by a system of three differential equations below:-

$$\frac{dx}{dt} = \beta - \frac{xc\beta y}{\mu} - \mu x + vy \qquad (3.1)$$

$$\frac{dy}{dt} = \frac{xc\beta y}{\mu} - \mu y - vy \qquad (3.2)$$

$$\frac{dN}{dt} = \beta - \mu x - \mu y \qquad (3.3)$$

The total population is given by

$$N = x + y$$
$$\therefore N(t) = x(t) + y(t)$$

**Equilibrium State of the Model**

At equilibrium state, let x(t) = x and y(t) = y, the equation (3.1), (3.2) and (3.3) becomes.

$$\beta - \frac{xc\beta y}{\mu} - \mu x + vy = 0 \qquad (3.4)$$

$$\frac{xc\beta y}{\mu} - \mu y - vy = 0 \qquad (3.5)$$

$$\beta - \mu x - \mu y = 0 \qquad (3.6)$$

From equation (3.5) we have

$$\frac{xc\beta y}{\mu} - \mu y - vy = 0$$

$$\left(\frac{xc\beta}{\mu} - \mu - v\right) y = 0$$

Either $\quad y = 0 \quad$ or $\quad \frac{xc\beta}{\mu} - \mu - v = 0$

Therefore $\frac{xc\beta}{\mu} = \mu + v$

Dividing through by $\frac{c\beta}{\mu}$ we get,

$$\frac{xc\beta}{\mu} \div \frac{c\beta}{\mu} = \mu + v \div \frac{c\beta}{\mu}$$

$$\therefore x = v + \mu \times \frac{\mu}{c\beta}$$

$$x = \frac{\mu(v + \mu)}{c\beta} \qquad (3.7)$$

Substituting the value of x into equation (3.6).

$$\beta - \mu x - \mu y = 0 \qquad \text{But} \quad x = \frac{\mu(v+\mu)}{c\beta}$$

$$\beta - \mu\left(\frac{\mu(v + \mu)}{c\beta}\right) - \mu y = 0$$

$$\beta - \frac{\mu^2(\mu + v)}{c\beta} - \mu y = 0$$

$$\beta - \frac{\mu^2(\mu + v)}{c\beta} = \mu y$$

$$\therefore \mu y = \beta - \frac{\mu^2(\mu + v)}{c\beta}$$

$$=> \mu y = \frac{c\beta^2 - \mu^2(\mu + v)}{c\beta}$$

Divided through by $\mu$.

$$\therefore y = \frac{c\beta^2 - \mu^2(\mu + v)}{c\beta} \div \mu$$

$$y = \frac{c\beta^2 - \mu^2(\mu + v)}{c\beta} \times \frac{1}{\mu}$$

Thus, $y = \frac{c\beta^2 - \mu^2(\mu + v)}{\mu c\beta}$ (3.8)

If $y = \frac{c\beta^2 - \mu^2(\mu + v)}{\mu c\beta}$ and $x = \frac{\mu(v + \mu)}{c\beta}$. From equation (3.1) if y = 0 we have

$$\beta - \frac{xc\beta y}{\mu} - \mu x - vy = 0$$

$$\beta - \frac{xc\beta(0)}{\mu} - \mu x - v(0) = 0$$

$$\beta - 0 - \mu x + 0 = 0$$

$$\beta - \mu x = 0$$

$$-\mu x = -\beta$$

$$x = \frac{-\beta}{-\mu}$$

$$\therefore x = \frac{\beta}{\mu}.$$

The equilibrium states are:-
a. The free – equilibrium state is given by.

$$(x, y) = \left[\frac{\beta}{\mu}, \quad 0\right] \text{and}$$

b. The endemic – equilibrium state is given by.

$$x = \frac{\mu(v + \mu)}{c\beta} \text{and} y = \frac{c\beta^2 - \mu^2(\mu + v)}{\mu c\beta}.$$

**The Characteristic Equation**

The Jacobin matrix for the model, from 3.4 and 3.5 is given by the determinant system equation.

$$\begin{vmatrix} -\frac{c\beta y}{\mu} - \mu - \lambda & -\frac{c\beta y}{\mu} + v \\ \frac{c\beta y}{\mu} & \frac{xc\beta y}{\mu} - \mu - v - \lambda \end{vmatrix} = 0 \qquad (3.9)$$

With $\lambda$ as the Eigen – value: which gives the characteristics equation.

$$\left(\frac{-c\beta y}{\mu} - \mu - \lambda\right)\left(\frac{xc\beta}{\mu} - \mu - v - \lambda\right) - \left(\frac{c\beta y}{\mu}\right)\left(\frac{-xc\beta}{\mu} + v\right) = 0$$

$$\left(\frac{-c\beta y}{\mu} - \mu - \lambda\right)\left(\frac{xc\beta}{\mu} - \mu - v - \lambda\right) + \left(\frac{c\beta y}{\mu}\right)\left(\frac{xc\beta}{\mu} - v\right) = 0 \qquad (3.10)$$

### III.     Conclusion

From our stability, we observed that the free equilibrium will always be stable if $\beta < \mu - v$. Stability of the free equilibrium state implies that once asthma disease manifests in any community, the entire population will be wiped out. This means that for the population to be sustained once asthma disease manifest, there should be a way of obtaining a speedy replenishment of the population which could be achieved through friendly immigration policies and increasing number of births per family.

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