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A Common Fixed Point Result for Compatible Mappings of Type (P) in Metrically Convex Metric Spaces

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Abstract: *In the present paper , we improve upon a result on common fixed point theorems for compatible mappings of type (P) in metrically convex metric spaces by relaxing continuity restriction of two out of four mappings.*
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Key Words: *Metrically convex metric spaces , Compatible mappings ,Compatible mappings of type (P)*

I. Introduction:

Compatible mappings of type (P) were introduced in [5] . Later on [4] gave a result on common fixed point theorems for such mappings .We improve one of its results for metrically convex metric spaces .The result is as follows:

THEOREM 1.1: Let (X, d) be a complete metrically convex metric space and K a non empty closed subset of X. Suppose that $S, T: X \to X$ are continuous from X into itself $\partial K \subset S(K) \cap T(K)$, where ∂ denotes boundary of K and $A, B: K \to X$ are continuous mappings with $A(K) \cap K \subset S(K), B(K) \cap K \subset T(K)$. Suppose further that the pairs (A, T) and (B, S) are relatively compatible of type (P) satisfying

$$d(Ax, By) \leq \emptyset(d(Tx, Sy) \qquad \text{..........1.1.1}$$

for all $x, y \in K$,where $\emptyset: [0, \infty) \to [0, \infty)$ is a non-decreasing and upper semi-continuous function such that $\emptyset(t) < t$ and $\sum \emptyset^n(t) < \infty$ for all $t > 0$.If for $x \in K$, $Tx, Sx \in \partial K \Rightarrow Ax, Bx \in K$,then there exists a point $z \in K$ such that $z = Az = Bz = Sz = Tz$.Further if $Av = Bv = Sv = Tv$, then $Tv = Tz$

We relax the restriction of all the four mappings A, B, S, T to be continuous by imposing continuity condition on only two of the four mappings. Also we change the inequality (1.1.1)

II. Preliminaries

DEFINITION 2.1: Let $A, B: (X, d) \to (X, d)$ be mappings .Then A , B are said to be compatible if

$$\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$$

whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some $t \in X$.

DEFINITION 2.2: Let $A, B: (X, d) \to (X, d)$ be mappings .Then A , B are said to be compatible of type (P) if

$$\lim_{n\to\infty} d(AAx_n, BBx_n) = 0$$

whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some $t \in X$.

PROPOSITION 2.3: Let $A, B: (X, d) \to (X, d)$ be mappings .If A , B are compatible of type (P) and $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some $t \in X$,then we have the following

(i) $\lim_{n\to\infty} AAx_n = Bt$ if B is continuous at t.
(ii) $\lim_{n\to\infty} BBx_n = At$ if A is continuous at t.
(iii) $ABt = BAt$ and $At = Bt$ if A and B are continuous at t.

III. MAIN RESULT

THEOREM 3.1: Let (X, d) be a complete metrically convex metric space and K a non empty closed subset of X. Suppose that $M, N: X \to X$ are continuous and $F, G: K \to X$ satisfy the following conditions

1. $\partial K \subset MK \cap NK$, where ∂ denotes boundary of K
2. $FK \cap K \subset NK, GK \cap K \subset MK$
3. $Mx, Nx \in \partial K \Rightarrow Fx, Gx \in K$
4. (F, M) and (G, N) are relatively compatible of type (P)
5. $\emptyset(d(Fx, Gy)) \leq c\max\{\emptyset(d(Mx, Ny), \emptyset(d(Fx, Mx)), \emptyset(d(Gy, Ny)), \emptyset(d(Fx, Ny)) + \emptyset(d(Mx, Gy))\}$

for all $x, y \in X$,where $\emptyset: [0, \infty) \to [0, \infty)$ is an increasing upper semi-continuous function such that $\emptyset(t) = 0 \Rightarrow t = 0$.

PROOF: We construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way:
Let $x \in \partial K$ and $x_0 \in K$ be such that $x = Mx_0$.Then $Fx_0 \in K$ by (3) and hence $Fx_0 \in FK \cap K \subset NK$.

This implies that there exists a point $x_1 \in K$ such that $y_1 = Nx_1 = Fx_0 \in K$. Since $y_1 = Fx_0$ there exists point $y_2 = Gx_1$ such that $d(y_1, y_2) = d(Fx_0, Gx_1)$. Suppose $y_2 \in K$.Then $y_2 \in GK \cap K \subset MK$ which implies that there exists a point $x_2 \in K$ such that $y_2 = Mx_2$. If $y_2 \notin K$,then there exists point $p \in \partial K$ such that

$$d(Nx_1, p) + d(p, y_2) = d(Nx_1, y_2)$$

Since $p \in \partial K \subset MK$ there exists a point $x_2 \in K$ with $p = Mx_2$ such that

$$d(Nx_1, Mx_2) + d(Mx_2, y_2) = d(Nx_1, y_2)$$

Let $y_3 = Fx_2$ be such that $d(y_2, y_3) = d(Gx_1, Fx_2)$.Thus continuing this process by similar arguments we obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that

(i) $y_{2n} = Gx_{2n-1}$, $y_{2n+1} = Fx_{2n}$

(ii) $y_{2n} \in K \Rightarrow y_{2n} = Mx_{2n}$ or $y_{2n} \notin K \Rightarrow Mx_{2n} \in \partial K$ and $d(Nx_{2n-1}, Mx_{2n}) + d(Mx_{2n}, y_{2n}) = d(Nx_{2n-1}, y_{2n})$

(iii) $y_{2n+1} \in K \Rightarrow y_{2n+1} = Nx_{2n+1}$ or $y_{2n+1} \notin K \Rightarrow Nx_{2n+1} \in \partial K$ and $d(Mx_{2n}, Nx_{2n+1}) + d(Nx_{2n+1}, y_{2n+1}) = d(Mx_{2n}, y_{2n+1})$

We denote $P_0 = \{Mx_{2i} \in \{Mx_{2n}\}: Mx_{2i} = y_{2i}\}$

$P_1 = \{Mx_{2i} \in \{Mx_{2n}\}: Mx_{2i} \neq y_{2i}\}$

$Q_0 = \{Nx_{2i+1} \in \{Nx_{2n+1}\}: Nx_{2i+1} = y_{2i+1}\}$

$Q_1 = \{Nx_{2i+1} \in \{Nx_{2n+1}\}: Nx_{2i+1} \neq y_{2i+1}\}$

We observe that $(Mx_{2n}, Nx_{2n+1}) \notin P_1 \times Q_1$ as if $Mx_{2n} \in P_1$,then $y_{2n} \neq Mx_{2n}$ and we infer that $Mx_{2n} \in \partial K \Rightarrow y_{2n+1} = Fx_{2n} \in K$.. Hence $y_{2n+1} = Nx_{2n+1} \in Q_0$. Similarly one can argue that $(Mx_{2n-1}, Nx_{2n}) \notin Q_1 \times P_1$. There arise three cases :

Case I : $(Mx_{2n}, Nx_{2n+1}) \in P_0 \times Q_0$

$$\begin{aligned}
\emptyset(d(Mx_{2n}, Nx_{2n+1})) &= \emptyset(d(y_{2n}, y_{2n+1})) \\
&= \emptyset(d(Gx_{2n-1}, Fx_{2n})) \\
&= \emptyset(d(Fx_{2n}, Gx_{2n-1})) \\
&\leq c \max\{\emptyset(d(Mx_{2n}, Nx_{2n-1})), \emptyset(d(Mx_{2n}, Fx_{2n})), \emptyset(d(Gx_{2n-1}, Nx_{2n-1})), \\
&\qquad \emptyset(d(Mx_{2n}, Gx_{2n-1})) + \emptyset(d(Fx_{2n}, Nx_{2n-1}))\} \\
&\leq c \max\{\emptyset(d(Mx_{2n}, Nx_{2n-1})), \emptyset(d(Mx_{2n}, Nx_{2n+1})), \emptyset(d(Mx_{2n}, Nx_{2n-1})), \\
&\qquad \emptyset(d(Mx_{2n}, Mx_{2n})) + \emptyset(d(Nx_{2n+1}, Nx_{2n-1}))\} \\
&\leq c \max\{\emptyset(d(Mx_{2n}, Nx_{2n-1})), \emptyset(d(Mx_{2n}, Nx_{2n+1})), \emptyset(d(Mx_{2n}, Nx_{2n-1})), \\
&\qquad \emptyset(d(Mx_{2n}, Nx_{2n+1})) + \emptyset(d(Mx_{2n}, Nx_{2n-1}))\} \\
&= \emptyset(d(Mx_{2n}, Nx_{2n+1})) + \emptyset(d(Mx_{2n}, Nx_{2n-1}))
\end{aligned}$$

Thus $\emptyset(d(Mx_{2n}, Nx_{2n+1})) \leq \frac{c}{1-c} \emptyset(d(Mx_{2n}, Nx_{2n-1}))$

Case II : $(Mx_{2n}, Nx_{2n+1}) \in P_0 \times Q_1$

$$\begin{aligned}
\emptyset(d(Mx_{2n}, Nx_{2n+1})) &= \emptyset(d(Mx_{2n}, y_{2n+1})) = \emptyset(d(y_{2n}, y_{2n+1})) \\
&\leq \frac{c}{1-c} \emptyset(d(Mx_{2n}, Nx_{2n-1})) \quad \text{[From Case I]}
\end{aligned}$$

Case III: $(Mx_{2n}, Nx_{2n+1}) \in P_1 \times Q_0$

$$\begin{aligned}
\emptyset(d(Mx_{2n}, Nx_{2n+1})) &= \emptyset(d(Mx_{2n}, y_{2n+1})) \\
&\leq \emptyset(d(Mx_{2n}, y_{2n})) + \emptyset(d(y_{2n}, y_{2n+1})) \\
&\leq \emptyset(d(Mx_{2n}, y_{2n})) + \emptyset(d(Bx_{2n-1}, Ax_{2n})) \\
&\leq \emptyset(d(Nx_{2n-1}, y_{2n})) + c \max\{\emptyset(d(Nx_{2n-1}, y_{2n})), \emptyset(d(Mx_{2n}, Nx_{2n+1})), \\
&\qquad \emptyset(d(Nx_{2n-1}, y_{2n})), \emptyset(d(Mx_{2n}, y_{2n})) + \emptyset(d(Nx_{2n-1}, Nx_{2n+1}))\} \\
&\leq \emptyset(d(Nx_{2n-1}, y_{2n})) + c \max\{\emptyset(d(Nx_{2n-1}, y_{2n})), \emptyset(d(Mx_{2n}, Nx_{2n+1})) \\
&\qquad \emptyset(d(Nx_{2n-1}, y_{2n})), \emptyset(d(Nx_{2n-1}, y_{2n})) + \emptyset(d(Nx_{2n-1}, Mx_{2n})) + \\
&\qquad \emptyset(d(Mx_{2n}, Nx_{2n+1}))\} \\
&\leq \emptyset(d(Nx_{2n-1}, y_{2n})) + c\, \{\emptyset(d(Nx_{2n-1}, y_{2n})) + \emptyset(d(Nx_{2n-1}, Mx_{2n})) + \\
&\qquad \emptyset(d(Mx_{2n}, Nx_{2n+1}))\}
\end{aligned}$$

Therefore, $\emptyset(d(Mx_{2n}, Nx_{2n+1})) \leq \frac{1+2c}{1-c} \emptyset(d(Nx_{2n-1}, y_{2n}))$

$$\leq \frac{1+2c}{(1-c)^2}\emptyset\big(d(Mx_{2n-2}, Nx_{2n-1})\big)$$

Thus in all the three cases

$$\emptyset\big(d(Mx_{2n}, Nx_{2n+1})\big) \leq \max\{\frac{c}{1-c}\emptyset\big(d(Mx_{2n}, Nx_{2n-1})\big), \frac{c(1+2c)}{(1-c)^2}\emptyset\big(d(Mx_{2n-2}, Nx_{2n-1})\big)\}$$
$$= k\max\{\emptyset\big(d(Mx_{2n}, Nx_{2n-1})\big), \emptyset\big(d(Mx_{2n-2}, Nx_{2n-1})\big)\}$$

where $k = \max\{\frac{c}{1-c}, \frac{c(1+2c)}{(1-c)^2}\} < 1$. By induction , for $n \geq 1$, we have

$$\emptyset\big(d(Mx_{2n}, Nx_{2n+1})\big) < k^n\delta \text{ and } \emptyset\big(d(Nx_{2n+1}, Mx_{2n+2})\big) < k^{n+1/2}\delta$$

where $\delta = k^{-1/2}\max\{\emptyset\big(d(Mx_0, Nx_1)\big), \emptyset\big(d(Nx_1, Mx_2)\big)\}$

The sequence $\{Mx_0, Nx_1, Mx_2, Nx_3, \dots Mx_{2n}, Nx_{2n+1}\}$ is Cauchy .Hence there exists at least one subsequence $\{Mx_{2n}\}$ or$\{Nx_{2n+1}\}$ which is contained in P_0 or Q_0 respectively and converges to $q \in K$. Since K is a closed subset of a complete metric space (X, d),therefore

$$q = \lim_{n\to\infty} Nx_{2n+1} = \lim_{n\to\infty} Mx_{2n} \qquad ....(3.1.1)$$

By hypothesis there exists a sequence $\{n_k\}$ in N such that

$$Mx_{2n_k} = Gx_{2n_k-1} \text{ or } Nx_{2n_k+1} = Fx_{2n_k}$$

We observe

$$\emptyset\left(d\big(FFx_{2n_k}, GGx_{2n_k-1}\big)\right) \leq c\max\{\emptyset\left(d\big(MFx_{2n_k}, NGx_{2n_k-1}\big)\right), \emptyset\left(d\big(MFx_{2n_k}, FFx_{2n_k}\big)\right)$$
$$\emptyset\left(d\big(MFx_{2n_k}, GGx_{2n_k-1}\big)\right) + \emptyset\left(d\big(FFx_{2n_k}, NGx_{2n_k-1}\big)\right)\}$$

Letting $k \to \infty$, from (3.1.1) , proposition 2.3(i) and (ii) we have

$$\emptyset\big(d(Mq, Nq)\big) \leq c\max\{\emptyset\big(d(Mq, Nq)\big), \emptyset\big(d(Mq, Mq)\big), \emptyset\big(d(Nq, Nq)\big), \emptyset\big(d(Mq, Nq)\big) + \emptyset\big(d(Mq, Nq)\big)\}$$
$$\leq c\max\{\emptyset\big(d(Mq, Nq)\big), 0, 0, 2\emptyset\big(d(Mq, Nq)\big)\}$$
$$= 2c\emptyset\big(d(Mq, Nq)\big)\}$$

which shows that $\emptyset\big(d(Mq, Nq)\big) = 0$, since $c < \frac{1}{2}$ .Thus showing

$$Mq = Nq \qquad ....(3.1.2)$$

Now ,

$$\emptyset\left(d\big(FFx_{2n_k}, Gq\big)\right) \leq c\max\{\emptyset\left(d\big(MFx_{2n_k}, Nq\big)\right), \emptyset\left(d\big(MFx_{2n_k}, FFx_{2n_k}\big)\right), \emptyset\big(d(Gq, Nq)\big),$$
$$\emptyset\left(d\big(MFx_{2n_k}, Gq\big)\right) + \emptyset\left(d\big(FFx_{2n_k}, Nq\big)\right)\}$$
$$= c\max\{\emptyset\big(d(Mq, Nq)\big), \emptyset\big(d(Mq, Mq)\big), \emptyset\big(d(Gq, Nq)\big), \emptyset\big(d(Mq, Gq)\big) + \emptyset\big(d(Mq, Nq)\big)\}$$

Letting $k \to \infty$, (3.1.1) , (3.1.2) and proposition 2.3 (i)

$\emptyset\big(d(Mq, Gq)\big) \leq c\emptyset\big(d(Nq, Gq)\big) = c\emptyset\big(d(Mq, Gq)\big)$.This gives

$$Mq = Gq \quad \text{since } c < \frac{1}{2}.$$

$$\emptyset\left(d\big(FFx_{2n_k}, Gx_{2n_k-1}\big)\right) \leq c\max\{\emptyset\left(d\big(MFx_{2n_k}, Nx_{2n_k-1}\big)\right), \emptyset\left(d\big(FFx_{2n_k}, MFx_{2n_k}\big)\right),$$
$$\emptyset\left(d\big(Gx_{2n_k-1}, Nx_{2n_k-1}\big)\right), \emptyset\left(d\big(FFx_{2n_k}, Nx_{2n_k-1}\big)\right) +$$
$$\emptyset\left(d\big(MFx_{2n_k}, Nx_{2n_k-1}\big)\right)\}$$

Letting $n \to \infty$, proposition 2.3 (i) and (3.1.1) gives

$$\emptyset\big(d(Mq, q)\big) \leq c\max\{\emptyset\big(d(Mq, q)\big), \emptyset\big(d(Mq, Mq)\big), \emptyset\big(d(q, q)\big), \emptyset\big(d(Mq, q)\big) + \emptyset\big(d(Mq, q)\big)\}$$

Thus $(1 - 2c)\emptyset\big(d(Mq, q)\big) \leq 0$ which gives $Mq = q$ since $c < \frac{1}{2}$.

From (3.1.2) and (3.1.3) we have

$$Mq = Gq = Nq = q \qquad .... (3.1.4)$$

Also

$$\emptyset\big(d(Fq, Gq)\big) \leq c\max\{\emptyset\big(d(Mq, Nq)\big), \emptyset\big(d(Mq, Fq)\big), \emptyset\big(d(Gq, Nq)\big), \emptyset\big(d(Mq, Gq)\big), \emptyset\big(d(Fq, Nq)\big)\}$$

From (3.1.4) we get

$\emptyset\big(d(Fq, q)\big) \leq c\max\{0, \emptyset\big(d(q, Fq)\big), 0, \emptyset\big(d(Fq, q)\big)\}$ showing $Fq = q$

Hence $Mq = Nq = Gq = Fq = q$.To prove the uniqueness of this point , let there be another point t such that $Mt = Nt = Gt = Ft = q$.Then

$$\emptyset\big(d(Fq, Gt)\big) \leq c\max\{\emptyset\big(d(Mq, Nt)\big), \emptyset\big(d(Mq, Fq)\big), \emptyset\big(d(Gt, Nt)\big), \emptyset\big(d(Mq, Gt)\big) + \emptyset\big(d(Fq, Nt)\big)\}$$

which, from above discussion ,yields

$$\emptyset\big(d(q,t)\big) \leq c \max\{\emptyset\big(d(q,t)\big), \emptyset\big(d(q,q)\big), \emptyset\big(d(t,t)\big), \emptyset\big(d(q,t)\big) + \emptyset\big(d(q,t)\big)\}$$

Thus $\emptyset\big(d(q,t)\big) = 0$ giving us $q = t$. Therefore the common fixed point is unique.This proves the result.

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