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# Common Fixed Point Theorems In Complex Valued Metric Spaces For Weakly Compatible Mappings, E.A. Property And CLR Property

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***Abstract:*** *In this paper, we prove a common fixed point theorem in complex valued metric space for weakly compatible mappings. Also, we prove common fixed point theorems for weakly compatible mappings with E.A. property and CLR property. We will generalized and extended the result of S.M. Kang [7].*
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***Key Words:*** *CLR property, complex valued metric space, E.A. property, weakly compatible mapping.*

## I. Introduction

In 2011, Azam et al. [1] introduced the notion of complex valued metric space which is a generalization of the classical metric space and established some fixed point results for mappings satisfying a rational inequality. Jungck [3] and Vetro [2] introduced the concept of weakly compatible maps. In 2002, Aamri and Moutawakil [4] introduced the notion of E.A. property. In 2011, Sintunavarat and Kumam [8] introduced the notion of CLR property. In 2013, Verma and Pathak [5] defined the 'max' function for partial order relation $\precsim$. A complex number $z \in C$ is an ordered pair of real numbers, whose first coordinate is called Re(z) and second coordinate is called Im(z).

## II. Preliminaries

**Definition 2.1.**[7] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies
(1) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(3) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.
**Example 2.2.** [7] Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by $d(z_1,z_2) = 2i|z_1 - z_2|$ for all $z_1,z_2 \in X$. Then (X, d) is a complex valued metric space.
**Definition 2.3.** [7] Let (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X.
(1) If for every $c \in \mathbb{C}$ with $0 \prec c$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) \prec c$ for all $n \geq N$ then $\{x_n\}$ is said to be convergent to $x \in X$, and we denote this by $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.
(2) If for every $c \in \mathbb{C}$ with $0 \prec c$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_{n+m}) \prec c$ for all $n \geq N$, where $m \in \mathbb{N}$, then $\{x_n\}$ is said to be Cauchy sequence.
(3) If every Cauchy sequence in X is convergent, then (X, d) is said to be a complete complex valued metric space.
**Lemma 2.4.** [7] Let (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.
**Lemma 2.5.** [7] Let (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in N$.
**Definition 2.6.** [7] Let f and g be two self-mappings of a metric space (X, d). Then a pair (f, g) is said to be weakly compatible if they commute at coincidence points.
**Definition 2.7.** [7] Let f and g be two self-mappings of a metric space (X, d). Then a pair (f, g) is said to satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.
**Definition 2.8.** [7] Let f and g be two self-mappings of a metric space (X, d). Then a pair (f, g) is said to satisfy $CLR_f$ property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = fx$ for some $x \in X$.
**Example 2.9.** [7] Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by d(z_1,z_2) = 2i|z_1-z_2| for all $z_1, z_2 \in X$. Then (X, d) is a complex valued metric space. Define S and $T : X \to X$ by $Sz = z + i$ and $Tz = 2z$ for all $z \in X$, respectively. Consider a sequence $\{z_n\} = \{i - \frac{1}{n}\}$ $(n \in \mathbb{N})$ in X. Then $\lim_{n\to\infty} Sz_n = \lim_{n\to\infty}(z_n + i) =$

2i and $\lim_{n\to\infty} Tz_n = \lim_{n\to\infty} 2z_n = 2i$ where $2i \in X$. Thus, S and T satisfy E.A. property. Also, we have $\lim_{n\to\infty} Sz_n = \lim_{n\to\infty} Tz_n = 2i = Si$. where $2i \in X$. Thus, S and T satisfy $CLR_S$ property.

**Definition 2.10.** [7] Define the 'max' function for the partial order relation $\precsim$ by

(1) $\max\{z_1, z_2\} = z_2$ if and only if $z_1 \precsim z_2$.

(2) If $z_1 \precsim \max\{z_2, z_3\}$, then $z_1 \precsim z_2$ or $z_1 \precsim z_3$.

(3) $\max\{z_1, z_2\} = z_2$ if and only if $z_1 \precsim z_2$ or $|z_1| \le |z_2|$.

Using above Definition, we have the following lemma.

**Lemma 2.11.** [7] Let $z_1, z_2, z_3 \dots\dots \in \mathbb{C}$ and the partial order relation $\precsim$ is defined on $\mathbb{C}$. Then following statements are easy to prove.

(i) If $z_1 \precsim \max\{z_2, z_3\}$, then $z_1 \precsim z_2$ if $z_3 \precsim z_2$;

(ii) If $z_1 \precsim \max\{z_2, z_3, z_4\}$, then $z_1 \precsim z_2$ if $\max\{z_3, z_4\} \precsim z_2$;

(iii) If $z_1 \precsim \max\{z_2, z_3, z_4, z_5\}$, then $z_1 \precsim z_2$ if $\max\{z_3, z_4, z_5\} \precsim z_2$, and so on.

## III. Main Result

**Theorem 3.1 :** Let A, B, D, M, S and T be six self mappings of a complex valued metric space (X, d) satisfying:

1. $S(X) \subset BD(X)$ and $T(X) \subset AM(X)$
2. For each $x, y \in X$, there exists $\alpha, \beta, \gamma$ and $\eta$ are non negative real number with $\alpha + \beta + \gamma + \eta < 1$, such that

$$d(Sx, Ty) \precsim \alpha\left[d(BDy, Ty)\frac{1 + d(AMx, Sx)}{1 + d(AMx, BDy)}\right] + \beta[\max\{d(AMx, By), d(AMx, Sx), d(BDy, Ty)\}]$$
$$+\gamma\,[d(Ty, Sx)] + \eta\left[\frac{d(Ty, BDy)d(AMx, Sx)}{d(Ty, AMx) + d(Sx, BDy) + d(Ty, Sx)}\right]$$

3. The pair (AM, S) and (BD, T) are weakly compatible.
4. Suppose that One of A(X), B(X), S(X) and T(X) is complete subspace of X.
5. The pair (AM, S) and (BD, T) are commute.

Then A, B, D, M, S and T have a unique common fixed point.

**Proof:** Let $x_0 \in X$ . Since $S(X) \subset BD(X)$ and $T(X) \subset AM(X)$, define for each $n \ge 0$, the sequence $\{y_n\}$ in X by

$$y_{2n+1} = Sx_{2n} = BDx_{2n+1} \text{ and } y_{2n+2} = Tx_{2n+1} = AMx_{2n+2}$$

**Case I :** Suppose that $y_{2n} = y_{2n+1}$ for some n. Then by (2), we have $y_{2n+2} = y_{2n+1}$, and so, $y_m = y_{2n}$ for every m > 2n. Thus, the sequence $\{y_n\}$ is a Cauchy sequence. The same conclusion holds if $y_{2n+1} = y_{2n+2}$ for some n.

**Case II :** Assume that $y_n \neq y_{n+1}$ for all n. Putting $x = x_{2n}$ and $y = x_{2n-1}$ in (2), we have

$$d(Sx_{2n}, Tx_{2n-1}) \precsim \alpha\left[d(BDx_{2n-1}, Tx_{2n-1})\frac{1 + d(AMx_{2n}, Sx_{2n})}{1 + d(AMx_{2n}, BDx_{2n-1})}\right]$$
$$+\beta[max\{d(AMx_{2n}, BDx_{2n-1}), d(AMx_{2n}, Sx_{2n}), d(BDx_{2n-1}, Tx_{2n-1})\}] +$$
$$\gamma\,[d(Tx_{2n-1}, Sx_{2n})] + \eta\left[\frac{d(Tx_{2n-1}, BDx_{2n-1})d(AMx_{2n}, Sx_{2n})}{d(Tx_{2n-1}, AMx_{2n}) + d(Sx_{2n}, BDx_{2n-1}) + d(Tx_{2n-1}, Sx_{2n})}\right]$$

$$d(y_{2n+1}, y_{2n})$$
$$\precsim \alpha\left[d(y_{2n-1}, y_{2n})\frac{1 + d(y_{2n}, y_{2n+1})}{1 + d(y_{2n}, y_{2n-1})}\right] + \beta[max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\}] +$$
$$\gamma\,[d(y_{2n}, y_{2n+1})] + \eta\left[\frac{d(y_{2n}, y_{2n-1})d(y_{2n}, y_{2n+1})}{d(y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n-1}) + d(y_{2n}, y_{2n+1})}\right]$$
$$d(y_{2n+1}, y_{2n}) \precsim \alpha d(y_{2n}, y_{2n+1}) + \beta d(y_{2n}, y_{2n+1}) + \gamma d(y_{2n}, y_{2n+1}) + \eta d(y_{2n}, y_{2n+1})$$

Thus, we have $|d(y_{2n}, y_{2n+1})| \le (\alpha + \beta + \gamma + \eta)|d(y_{2n}, y_{2n+1})|$

Which is a contradiction to $(\alpha + \beta + \gamma + \eta) < 1$. Conversely we have

$$d(y_{2n+1}, y_{2n}) \precsim d(y_{2n}, y_{2n-1})$$

Thus, we have $d(y_{2n+1}, y_{2n}) \precsim (\alpha + \beta + \gamma + \eta)d(y_{2n}, y_{2n-1})$

On putting $x = x_{2n-2}$ and $y = x_{2n-1}$ in (2), we have

$$d(Sx_{2n-2}, Tx_{2n-1}) \precsim \alpha\left[d(BDx_{2n-1}, Tx_{2n-1})\frac{1 + d(AMx_{2n-2}, Sx_{2n-2})}{1 + d(AMx_{2n-2}, BDx_{2n-1})}\right]$$
$$+\beta[max\{d(AMx_{2n-2}, BDx_{2n-1}), d(AMx_{2n-2}, Sx_{2n-2}), d(BDx_{2n-1}, Tx_{2n-1})\}] +$$
$$\gamma\,[d(Tx_{2n-1}, Sx_{2n-2})] + \eta\left[\frac{d(Tx_{2n-1}, BDx_{2n-1})d(AMx_{2n-2}, Sx_{2n-2})}{d(Tx_{2n-1}, AMx_{2n-2}) + d(Sx_{2n-2}, BDx_{2n-1}) + d(Tx_{2n-1}, Sx_{2n-2})}\right]$$

$$d(y_{2n-1}, y_{2n}) \precsim \alpha\left[d(y_{2n-1}, y_{2n})\frac{1 + d(y_{2n-2}, y_{2n-1})}{1 + d(y_{2n-2}, y_{2n-1})}\right]$$
$$+\beta[max\{d(y_{2n-2}, y_{2n-1}), d(y_{2n-2}, y_{2n-1}), d(y_{2n-1}, y_{2n})\}] + \gamma\,[d(y_{2n}, y_{2n-1})]$$

$$+\eta\left[\frac{d(y_{2n},y_{2n-1})d(y_{2n-2},y_{2n-1})}{d(y_{2n},y_{2n-2})+d(y_{2n-1},y_{2n-1})+d(y_{2n},y_{2n-1})}\right]$$

$d(y_{2n-1},y_{2n}) \precsim \alpha d(y_{2n-1},y_{2n}) + \beta d(y_{2n-1},y_{2n}) + \gamma d(y_{2n-1},y_{2n}) + \eta d(y_{2n-1},y_{2n})$

$(1-\alpha)d(y_{2n-1},y_{2n}) \precsim (\beta+\gamma+\eta)d(y_{2n-1},y_{2n})$

Thus, we have $|d(y_{2n-1},y_{2n})| \leq \left(\frac{\beta+\gamma+\eta}{1-\alpha}\right)|d(y_{2n-1},y_{2n})|$

Which is a contradiction to $(\beta+\gamma+\eta) < 1$. Then we have

$$|d(y_{2n-1},y_{2n})| \leq \left(\frac{\beta+\gamma+\eta}{1-\alpha}\right)|d(y_{2n-2},y_{2n-1})|$$

Define $k = max\left[(\alpha+\beta+\gamma+\eta),\ \left(\frac{\beta+\gamma+\eta}{1-\alpha}\right)\right]$ conversely, it can be concluded that

$d(y_n,y_{n+1}) \precsim kd(y_{n-1},y_n)$

$d(y_n,y_{n+1}) \precsim k^2 d(y_{n-2},y_{n-1}) \precsim .............\precsim k^n d(y_0,y_1)$

Now for all m > n, we have

$$d(y_m,y_n) \precsim d(y_n,y_{n+1}) + d(y_{n+1},y_{n+2}) + \cdots ... + d(y_{m-1},y_m)$$
$$d(y_m,y_n) \precsim k^n d(y_0,y_1) + k^{n+1}d(y_0,y_1) + \cdots + k^{m-1}d(y_0,y_1)$$

Therefore we have,

$$|d(y_m,y_n)| \leq \frac{k^n}{1-k}|d(y_0,y_1)|$$

Hence, we obtain $lim_{n\to\infty}|d(y_m,y_n)| = 0$, hence $\{y_n\}$ is a Cauchy sequence.

**Case III :** Suppose that AM(X) is complete then the sequence $\{y_{2n}\}$ is contained in AM(X) and has a limit in AM(X), say u, that is $lim_{n\to\infty} y_{2n} = u$. Since u ∈ AM(X), there exists v ∈ X such that $AMv = u$.

Now, we shall prove $Sv = u$. Let $Sv \neq u$. From (2) putting $x = v$ and $y = x_{2n-1}$, we have

$$d(Sv,Tx_{2n-1}) \precsim \alpha\left[d(BDx_{2n-1},Tx_{2n-1})\frac{1+d(AMv,Sv)}{1+d(AMv,BDx_{2n-1})}\right]$$
$$+\beta[max\{d(AMv,BDx_{2n-1}),d(AMv,Sv),d(BDx_{2n-1},Tx_{2n-1})\}] + \gamma\,[d(Tx_{2n-1},Sv)] +$$
$$\eta\left[\frac{d(Tx_{2n-1},BDx_{2n-1})d(AMv,Sv)}{d(Tx_{2n-1},AMv)+d(Sv,BDx_{2n-1})+d(Tx_{2n-1},Sv)}\right] \quad (2.1)$$
$$\precsim \alpha\left[d(y_{2n-1},y_{2n})\frac{1+d(u,Sv)}{1+d(u,y_{2n-1})}\right] + \beta[max\{d(u,y_{2n-1}),d(u,Sv),d(y_{2n-1},y_{2n})\}] + \gamma[d(y_{2n},Sv)]$$
$$+\eta\left[\frac{d(y_{2n-1},y_{2n})\,d(Sv,u)}{d(y_{2n},u)+d(Sv,y_{2n-1})+d(y_{2n},Sv)}\right]$$

As the sequence $\{y_{2n-1}\}$ is convergent to u, therefore

$$\lim_{n\to\infty} d(u,y_{2n-1}) = \lim_{n\to\infty} d(y_{2n},y_{2n-1}) = 0$$

Thus letting n→∞, we have

$$\precsim \alpha\left[d(u,u)\frac{1+d(u,Sv)}{1+d(u,u)}\right] + \beta[max\{d(u,u),d(u,Sv),d(u,u)\}] + \gamma[d(u,Sv)]$$
$$+\eta\left[\frac{d(u,u)\,d(Sv,u)}{d(u,u)+d(Sv,u)+d(u,Sv)}\right]$$

$\precsim \beta d(u,Sv) + \gamma d(u,Sv)$

$d(Sv,u) \precsim (\beta+\gamma)d(u,Sv)$

That is, $|d(Sv,u)| \leq (\beta+\gamma)|d(Sv,u)|$

Which is a contradiction to $(\beta+\gamma) < 1$. Hence $u = Sv = AMv$. Now, since $S(X) \subset BD(X)$, $Su = u \in BD(X)$. There exists w ∈ X such that $BDw = u$. By using the same argument as above, one can easily verify that $Tw = u = BDw$, that is, w is the coincidence point of the pair (BD, T).

The same result hold if we assume that BD(X) is complete.

Now if T(X) is complete, then by (2.1), $u \in T(X) \subset AM(X)$. Similarly, if S(X) is complete , then $u \in S(X) \subset$ BD(X).

Now, since the pair (AM, S) and (BD, T) are weakly compatible, so $u = Sv = AMv = Tw = BDw$ and hence they commute at their coincidence point that is, $AMu = AM(Sv) = S(AMv) = Su$ and $BDu = BD(Tw) = T(BDw) = Tu$.

Now, we claim that $Tu = u$. Let $Tu \neq u$. From(2), we have

$$d(Sv,Tu) \precsim \alpha\left[d(BDu,Tu)\frac{1+d(AMv,Sv)}{1+d(AMv,BDu)}\right] + \beta[max\{d(AMv,BDu),d(AMv,Sv),d(BDu,Tu)\}] +$$
$$\gamma\,[d(Tu,Sv)] + \eta\left[\frac{d(Tu,BDu)d(AMv,Sv)}{d(Tu,AMv)+d(Sv,BDu)+d(Tu,Sv)}\right]$$

$$d(u,Tu) \precsim \alpha\left[d(Tu,Tu)\frac{1+d(u,u)}{1+d(u,Tu)}\right]+\beta[max\{d(u,Tu),d(u,u),d(Tu,Tu)\}]+\gamma\,[d(Tu,u)]$$
$$+\eta\left[\frac{d(Tu,Tu)d(u,u)}{d(Tu,u)+d(u,Tu)+d(Tu,u)}\right]$$

Thus, $d(u,Tu) \precsim \beta\, d(u,Tu)+\gamma d(u,Tu)$

That is, $|d(u,Tu)| \le (\beta+\gamma)|d(u,Tu)|$

Which is a contradiction to $(\beta+\gamma) < 1$. Therefore $Tu = u.$ Since $BDu = Tu$, which implies that $BDu = u$. Similarly we can prove that $Su = u.$ Since $u = Su$ , which implies that $AMu = u = Su$.

Now to prove $Mu = u$, using (2), putting $x = Mu$ and $y = u$, we have

$$d(S(Mu),Tu) \precsim \alpha\left[d(BDu,Tu)\frac{1+d\big(AM(Mu),S(Mu)\big)}{1+d(AM(Mu),BDu)}\right]$$
$$+\beta\big[max\big\{d(AM(Mu),BDu),d\big(AM(Mu),S(Mu)\big),d(BDu,Tu)\big\}\big]+\gamma\,[d(Tu,S(Mu)\,)\,]$$
$$+\eta\left[\frac{d(Tu,BDu)\,d(AM(Mu),S(Mu))}{d(Tu,AM(Mu))+d(S(Mu),BDu)+d(Tu,S(Mu))}\right]$$

$$d(Mu,u) \precsim \alpha\left[d(u,u)\frac{1+d(Mu,Mu)}{1+d(Mu,u)}\right]+\beta[max\{d(Mu,u),d(Mu,Mu),d(u,u)\}]+\gamma\,[d(u,Mu)]$$
$$+\eta\left[\frac{d(u,u)\,d(Mu,Mu)}{d(u,Mu)+d(Mu,\ u)+d(u,\ Mu)}\right]$$

$d(Mu,u) \precsim \beta d(Mu,u)+\gamma d(Mu,u)$

That is, $|d(u,Mu)| \le (\beta+\gamma)|d(u,Mu)|$

Which is a contradiction to $(\beta+\gamma) < 1$. Therefore $Mu = u$. Since $AMu = u$ which implies that $Au = u$. Now, to prove $Du = u$, using(2), putting $x = u, y = Du$, we have

$$d(Su,T(Du)) \precsim \alpha\left[d\big(BD(Du),T(Du)\big)\frac{1+d(AMu,Su)}{1+d\big(AMu,BD(Du)\big)}\right]$$
$$+\beta\big[max\big\{d\big(AMu,BD(Du)\big),d(AMu,Su),d\big(BD(Du),T(Du)\big)\big\}\big]+\gamma\,[d(T(Du),Su)]$$
$$+\eta\left[\frac{d(T(Du),BD(Du))\,d(AMu,Su)}{d(T(Du),AMu)+d(Su,BD(Du))+d(T(Du),Su)}\right]$$

$$d(u,Du) \precsim \alpha\left[d(Du,Du)\frac{1+d(u,u)}{1+d(u,Du)}\right]+\beta[max\{d(u,Du),d(u,u),d(Du,Du)\}]+\gamma\,[d(Du,u)]$$
$$+\eta\left[\frac{d(Du,Du)\,d(u,u)}{d(Du,u)+d(u,Du)+d(Du,u)}\right]$$

$d(Du,u) \precsim \beta d(Du,u)+\gamma d(Du,u)$

That is, $|d(u,Du)| \le (\beta+\gamma)|d(u,Du)|$

Which is a contradiction to $(\beta+\gamma) < 1$. Therefore $Du = u$. Since $BDu = u$ which implies that $Bu = u$. Thus combining all the above result, we have $Au = Bu = Du = Mu = Su = Tu = u$. Hence u is a common fixed point of A, B, D, M, S and T.

**Uniqueness:** Let $z$ $(z \neq u)$ be an another common fixed point of A, B, D, M, S and T. Then from (2), we have

$$d(u,z) = d(Su,Tz) \precsim \alpha\left[d(BDz,Tz)\frac{1+d(AMu,Su)}{1+d(AMu,BDz)}\right]$$
$$+\beta[max\{d(AMu,BDz),d(AMu,Su),d(BDz,Tz)\}]+\gamma\,[d(Tz,Su)]$$
$$+\eta\left[\frac{d(Tz,BDz)d(AMu,Su)}{d(Tz,AMu)+d(Su,BDz)+d(Tz,Su)}\right]$$

$$d(u,z) \precsim \alpha\left[d(z,z)\frac{1+d(u,u)}{1+d(u,z)}\right]+\beta[max\{d(u,z),d(u,u),d(z,z)\}]+\gamma[d(z,u)]$$
$$+\eta\left[\frac{d(z,z)d(u,u)}{d(z,u)+d(u,z)+d(z,u)}\right]$$

$d(u,z) \precsim \beta d(u,z)+\gamma d(u,z)$

That is, $|d(u,z)| \le (\beta+\gamma)|d(u,z)|$

Which is a contradiction to $(\beta+\gamma) < 1$. Therefore z$= u$. Hence u is a unique common fixed point of A, B, D, M , S and T.

**Corollary :** Let A, B, S and T be self mappings of a complex valued metric space (X, d) satisfying:

1. $S(X) \subset B(X)$ and $T(X) \subset A(X)$
2. For each x, y ∈ X, there exists α, β, γ and η are non negative real numbers with $\alpha+\beta+\gamma+\eta < 1$, such that

$$d(Sx,Ty) \precsim \alpha\left[d(By,Ty)\frac{1+d(Ax,Sx)}{1+d(Ax,By)}\right] + \beta[max\{d(Ax,By),d(Ax,Sx),d(By,Ty)\}]$$
$$+\gamma\,[d(Ty,Sx)] + \eta\left[\frac{d(Ty,By)d(Ax,Sx)}{d(Ty,Ax)+d(Sx,By)+d(Ty,Sx)}\right]$$

3. The pair (A, S) and (B, T) are weakly compatible.
4. One of A(X), B(X), S(X) and T(X) is complete.

Then A, B, S and T have a unique common fixed point.

**Fixed Point Theorem For Weakly Compatible Mappings With E.A. Property**

**Theorem 3.2 :** Let A, B, D, M, S and T be self mappings of a complex valued metric space (X, d) satisfying:

1. $S(X) \subset BD(X)$ and $T(X) \subset AM(X)$
2. For each x, y ∈ X, there exists α, β, γ and η are non negative real numbers with $\alpha+\beta+\gamma+\eta<1$, such that

$$d(Sx,Ty) \precsim \alpha\left[d(BDy,Ty)\frac{1+d(AMx,Sx)}{1+d(AMx,BDy)}\right] +$$
$$\beta[max\{d(AMx,BDy),d(AMx,Sx),d(BDy,Ty)\}] + \gamma\,[d(Ty,Sx)] +$$
$$\eta\left[\frac{d(Ty,BDy)d(AMx,Sx)}{d(Ty,AMx)+d(Sx,BDy)+d(Ty,Sx)}\right]$$

3. The pair (AM, S) and (BD, T) are weakly compatible.
4. The pair (AM, S) and (BD, T) satisfy the E.A. property.
5. One of the AM(X), BD(X), S(X) and T(X) is closed subspace of X.
6. The pair (AM, S) and (BD, T) are commute.

Then A, B, D, M, S and T have a unique common fixed point.

**Proof:** Suppose that(AM, S) satisfies the E.A. property. Then there exists a sequence $\{x_n\}$ in X such that $AMx_n = Sx_n = z$ for some z ∈ X. Since S(X) ⊂ BD(X), there exists a sequence $\{y_n\}$ in X such that $Sx_n = BDy_n = z$. Hence $lim_{n\to\infty} BDy_n = z$.

We shall show that $lim_{n\to\infty} Ty_n = z$. Let $lim_{n\to\infty} Ty_n = t \neq z$. From (2), putting $x = x_n$ and $y = y_n$, we have

$$d(Sx_n,Ty_n) \precsim \alpha\left[d(BDy_n,Ty_n)\frac{1+d(AMx_n,Sx_n)}{1+d(AMx_n,BDy_n)}\right]$$
$$+\beta[max\{d(AMx_n,BDy_n),d(AMx_n,Sx_n),d(BDy_n,Ty_n)\}] + \gamma[d(Ty_n,Sx_n)]$$
$$+\eta\left[\frac{d(Ty_n,BDy_n)d(AMx_n,Sx_n)}{d(Ty_n,AMx_n)+d(Sx_n,BDy_n)+d(Ty_n,Sx_n)}\right]$$

Letting n→∞, we have

$$d(z,t) \precsim \alpha\left[d(z,t)\frac{1+d(z,z)}{1+d(z,z)}\right] + \beta[max\{d(z,z),d(z,z),d(z,t)\}] + \gamma[d(t,z)]$$
$$+\eta\left[\frac{d(t,z)d(z,z)}{d(t,z)+d(z,z)+d(t,z)}\right]$$

$$d(z,t) \precsim \alpha[d(z,t)] + \beta[d(z,t)] + \gamma[d(t,z)]$$

That is, $|d(z,t)| \le (\alpha+\beta+\gamma)|d(z,t)|$

Which is a contradiction to $(\alpha+\beta+\gamma)<1$. Therefore $t = z$, that is $lim_{n\to\infty} Ty_n = z$.

Suppose that BD(X) is a closed spaces of X. Then there exists u ∈ X such that $z = BDu$. Subsequently, we have

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} AMx_n = \lim_{n\to\infty} BDy_n = z = BDu$$

Now, we prove that $Tu = z$. From (2), putting x = $x_n$ and y = u, we have

$$d(Sx_n,Tu) \precsim \alpha\left[d(BDu,Tu)\frac{1+d(AMx_n,Sx_n)}{1+d(AMx_n,BDu)}\right]$$
$$+\beta[max\{d(AMx_n,BDu),d(AMx_n,Sx_n),d(BDu,Tu)\}] + \gamma[d(Tu,Sx_n)]$$
$$+\eta\left[\frac{d(Tu,BDu)d(AMx_n,Sx_n)}{d(Tu,AMx_n)+d(Sx_n,BDu)+d(Tu,Sx_n)}\right]$$

Letting n→∞, we have

$$d(z,Tu) \precsim \alpha\left[d(z,Tu)\frac{1+d(z,z)}{1+d(z,z)}\right] + \beta[max\{d(z,z),d(z,z),d(z,Tu)\}]$$
$$+\gamma[d(Tu,z)] + \eta\left[\frac{d(Tu,z)d(z,z)}{d(Tu,z)+d(z,z)+d(Tu,z)}\right]$$

$$d(z,Tu) \precsim \alpha[d(z,Tu)] + \beta[d(z,Tu)] + \gamma[d(Tu,z)]$$

That is, $|d(z,Tu)| \le (\alpha+\beta+\gamma)|d(z,Tu)|$

Which is a contradiction to $(\alpha+\beta+\gamma)<1$. Therefore $Tu = z = BDu$. Now, since T(X) ⊂ AM(X), $Tu = z \in AM(X)$. There exists $w \in X$ such that AM$w = z$. By using the same argument as above, one can easily verify that S$w = z = AMw$. Now, since the pair (BD, T) and (AM, S) are weakly compatible, then they commute at their coincidence point that is,
$AMz = AM(Sw) = S(AMw) = Sz$ and $BDz = BD(Tu) = T(BDu) = Tz$.
Now, we claim that $Tz = z$. Let $Tz \neq z$. From(2), putting $x = w$ and $y = z$, we have

$$d(Sw,Tz) \precsim \alpha\left[d(BDz,Tz)\frac{1+d(AMw,Sw)}{1+d(AMw,BDz)}\right]+\beta[max\{d(AMw,BDz),d(AMw,Sw),d(BDz,Tz)\}] +\gamma\,[d(Tz,Sw)]+\eta\left[\frac{d(Tz,BDz)d(AMw,Sw)}{d(Tz,AMw)+d(Sw,BDz)+d(Tz,Sw)}\right]$$

$$d(z,Tz) \precsim \alpha\left[d(Tz,Tz)\frac{1+d(z,z)}{1+d(z,Tz)}\right]+\beta[max\{d(z,Tz),d(z,z),d(Tz,Tz)\}]+\gamma\,[d(Tz,z)] +\eta\left[\frac{d(Tz,Tz)d(z,z)}{d(Tz,z)+d(z,Tz)+d(Tz,z)}\right]$$

Thus, $\quad d(z,Tz) \precsim \beta\, d(z,Tz)+\gamma d(z,Tz)$
That is, $\quad |d(z,Tz)| \le (\beta+\gamma)|d(z,Tz)|$
Which is a contradiction to $(\beta+\gamma) < 1$. Therefore $Tz = z$. Since $BDz = Tz$, which implies that $BDz = z$. Again using (2), putting $x = y = z$, then similarly we can prove that $Sz = z$. Since $z = Sz$ , which implies that $AMz = z = Sz$. Now to prove $Mz = z$, using (2) putting $x = Mz$ and $y = z$, we have

$$d(S(Mz),Tz) \precsim \alpha\left[d(BDz,Tz)\frac{1+d\big(AM(Mz),S(Mz)\big)}{1+d(AM(Mz),BDz)}\right] +\beta\big[max\big\{d(AM(Mz),BDz),d\big(AM(Mz),S(Mz)\big),d(BDz,Tz)\big\}\big]+\gamma\,\big[d\big(Tz,S(Mz)\big)\big]+ \eta\left[\frac{d(Tz,BDz)\,d(AM(Mz),S(Mz))}{d(Tz,AM(Mz))+d(S(Mz),BDz)+d(Tz,S(Mz))}\right]$$

$$d(Mz,z) \precsim \alpha\left[d(z,z)\frac{1+d(Mz,Mz)}{1+d(Mz,z)}\right]+\beta[max\{d(Mz,z),d(Mz,Mz),d(z,z)\}]+\gamma\,[d(z,Mz)] +\eta\left[\frac{d(z,z)\,d(Mz,Mz)}{d(z,Mz)+d(Mz,z)+d(z,Mz)}\right]$$

$d(Mz,z) \precsim \beta d(Mz,z)+\gamma d(Mz,z)$
That is, $\quad |d(z,Mz)| \le (\beta+\gamma)|d(z,Mz)|$
Which is a contradiction to $(\beta+\gamma) < 1$. Therefore $Mz = z$. Since $AMz = z$ which implies that $Az = z$.
Now, to prove $Dz = z$, using(2), putting $x = z, y = Dz$, we have

$$d(Sz,T(Dz)) \precsim \alpha\left[d\big(BD(Dz),T(Dz)\big)\frac{1+d(AMz,Sz)}{1+d\big(AMz,BD(Dz)\big)}\right] +\beta[max\{d(AMz,BD(Dz)),d(AMz,Sz),d(BD(Dz),T(Dz))\}]+\gamma\,[d(T(Dz),Sz)] +\eta\left[\frac{d(T(Dz),BD(Dz))\,d(AMz,Sz)}{d(T(Dz),AMz)+d(Sz,BD(Dz))+d(T(Dz),Sz)}\right]$$

$$d(z,Dz) \precsim \alpha\left[d(Dz,Dz)\frac{1+d(z,z)}{1+d(z,Dz)}\right]+\beta[max\{d(z,Dz),d(z,z),d(Dz,Dz)\}]+\gamma\,[d(Dz,z)] +\eta\left[\frac{d(Dz,Dz)\,d(z,z)}{d(Dz,z)+d(z,Dz)+d(Dz,z)}\right]$$

$d(Dz,z) \precsim \beta d(Dz,z)+\gamma d(Dz,z)$
That is, $\quad |d(z,Dz)| \le (\beta+\gamma)|d(z,Dz)|$
Which is a contradiction to $(\beta+\gamma) < 1$. Therefore $Dz = z$. Since $BDz = z$ which implies that $Bz = z$. Hence u is a common fixed point of A, B, D, M, S and T.

**Uniqueness:** From theorem 3.1, we can easily prove the uniqueness of the theorem. Hence A, B, D, M, S and T have a unique common fixed point in X.

**Fixed Point Theorem For Weakly Compatible Mapping With CLR Property**

**Theorem 3.3 :** Let A, B, D, M, S and T be six self mappings of a complex valued metric space (X, d) satisfying:
1.For each x, y ∈ X, there exists α, β, γ and η are non negative real number with $\alpha+\beta+\gamma+\eta<1$, such that

$$d(Sx,Ty) \precsim \alpha\left[d(BDy,Ty)\frac{1+d(AMx,Sx)}{1+d(AMx,BDy)}\right]+\beta[max\{d(AMx,BDy),d(AMx,Sx),d(BDy,Ty)\}]$$

$$+\gamma\,[d(Ty,Sx)] + \eta\left[\frac{d(Ty,BDy)d(AMx,Sx)}{d(Ty,AMx)+d(Sx,BDy)+d(Ty,Sx)}\right]$$

2. The pair (AM, S) and (BD, T) are weakly compatible.
3. S(X) ⊂ BD(X) and the pair (AM, S) satisfying $CLR_{AM}$ property.
4. T(X) ⊂ AM(X) and the pair (BD, T) satisfy the $CLR_{BD}$ property.
6. The pair (AM, S) and (BD, T) are commute.

Then A, B, D, M, S and T have a unique common fixed point.

**Proof:** Without loss of generality, assume that $S(X) \subset BD(X)$ and the the pair (AM, S) satisfying $CLR_{AM}$ property. Then there exists a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} AMx_n = lim_{n\to\infty} Sx_n = AMx$ for some x ϵ X. Since $S(X) \subset BD(X)$, there exists a sequence $\{y_n\}$ in X such that $Sx_n = BDy_n$. Hence $lim_{n\to\infty} BDy_n = AMx$. We shall show that $lim_{n\to\infty} Ty_n = AMx$. Let $lim_{n\to\infty} Ty_n = z \neq AMx$. From (1), putting $x = x_n$ and $y = y_n$, we have

$$d(Sx_n,Ty_n) \precsim \alpha\left[d(BDy_n,Ty_n)\frac{1+d(AMx_n,Sx_n)}{1+d(AMx_n,BDy_n)}\right]$$
$$+\beta[max\{d(AMx_n,BDy_n),d(AMx_n,Sx_n),d(BDy_n,Ty_n)\}] + \gamma[d(Ty_n,Sx_n)]$$
$$+\eta\left[\frac{d(Ty_n,BDy_n)d(AMx_n,Sx_n)}{d(Ty_n,AMx_n)+d(Sx_n,BDy_n)+d(Ty_n,Sx_n)}\right]$$

Letting n→∞, we have

$$d(AMx,z) \precsim \alpha\left[d(AMx,z)\frac{1+d(AMx,AMx)}{1+d(AMx,z)}\right] + \beta[max\{d(AMx,AMx),d(AMx,AMx),d(AMx,z)\}]$$
$$+\gamma[d(z,AMx)] + \eta\left[\frac{d(z,AMx)d(AMx,z)}{d(z,AMx)+d(AMx,AMx)+d(z,AMx)}\right]$$

$$d(AMx,z) \precsim \alpha[d(AMx,z)] + \beta[d(AMx,z)] + \gamma[d(z,AMx)]$$

That is, $|d(AMx,z)| \leq (\alpha+\beta+\gamma)|d(AMx,z)|$

Which is a contradiction to $(\alpha+\beta+\gamma) < 1$. Therefore $AMx = z$, that is $lim_{n\to\infty} Ty_n = AMx$. Subsequently, we have

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} AMx_n = \lim_{n\to\infty} BDy_n = z = AMx$$

Now, we shall show that $Sx = z$. From (1), putting $y = y_n$, we have

$$d(Sx,Ty_n) \precsim \alpha\left[d(BDy_n,Ty_n)\frac{1+d(AMx,Sx)}{1+d(AMx,BDy_n)}\right]$$
$$+\beta[max\{d(AMx,BDy_n),d(AMx,Sx),d(BDy_n,Ty_n)\}] + \gamma[d(Ty_n,Sx)]$$
$$+\eta\left[\frac{d(Ty_n,BDy_n)d(AMx,Sx)}{d(Ty_n,AMx)+d(Sx,BDy_n)+d(Ty_n,Sx)}\right]$$

Letting n→∞, we have

$$d(Sx,z) \precsim \alpha\left[d(z,z)\frac{1+d(z,Sx)}{1+d(z,z)}\right] + \beta[max\{d(z,z),d(z,Sx),d(z,z)\}] + \gamma[d(z,Sx)] +$$
$$\eta\left[\frac{d(z,z)d(Sx,z)}{d(z,z)+d(Sx,z)+d(z,Sx)}\right]$$

$$d(Sx,z) \precsim \beta[d(Sx,z)] + \gamma[d(z,Sx)]$$

That is, $|d(Sx,z)| \leq (\beta+\gamma)|d(Sx,z)|$

Which is a contradiction to $(\beta+\gamma) < 1$. Therefore $Sx = z = AMx$. Now, since the pair (AM, S) is weakly compatible, then they commute at their coincidence point that is, $AMz = AM(Sx) = S(AMx) = Sz$. Also since S(X) ⊂ BD(X) there exists y ∈ X such that $z = Sx = BDy$. Now, we claim that $Ty = z$. From(1), putting $x = x_n$, we have

$$d(Sx_n,Ty) \precsim \alpha\left[d(BDy,Ty)\frac{1+d(AMx_n,Sx_n)}{1+d(AMx_n,BDy)}\right]$$
$$+\beta[max\{d(AMx_n,BDy),d(AMx_n,Sx_n),d(BDy,Ty)\}] + \gamma[d(Ty,Sx_n)]$$
$$+\eta\left[\frac{d(Ty,BDy)d(AMx_n,Sx_n)}{d(Ty,AMx_n)+d(Sx_n,BDy)+d(Ty,Sx_n)}\right]$$

Letting n→∞, we have

$$d(z,Ty) \precsim \alpha\left[d(z,Ty)\frac{1+d(z,z)}{1+d(z,z)}\right] + \beta[max\{d(z,z),d(z,z),d(z,Ty)\}] + \gamma[d(Ty,z)]$$
$$+\eta\left[\frac{d(Ty,z)d(z,z)}{d(Ty,z)+d(z,z)+d(Ty,z)}\right]$$

$d(z,Ty) \precsim \alpha d(z,Ty) + \beta[d(z,Ty)] + \gamma[d(Ty,z)]$
That is, $|d(z,Ty)| \leq (\alpha+\beta+\gamma)|d(z,Ty)|$
Which is a contradiction to $(\alpha+\beta+\gamma) < 1$. Therefore $Ty = z = BDy$. Now, since pair (BD, T) is weakly compatible, then they commute at their coincidence point that is $BDz = BD(Ty) = T(BDy) = Tz$.
Now we prove that $Sz = Tz$, from(1), putting $x = y = z$, we have

$$d(Sz,Tz) \precsim \alpha\left[d(BDz,Tz)\frac{1+d(AMz,Sz)}{1+d(AMz,BDz)}\right] + \beta[max\{d(AMz,BDz), d(AMz,Sz), d(BDz,Tz)\}] + \gamma\,[d(Tz,Sz)] + \eta\left[\frac{d(Tz,BDz)d(AMz,Sz)}{d(Tz,AMz)+d(Sz,BDz)+d(Tz,Sz)}\right]$$

$$d(Sz,Tz) \precsim \alpha\left[d(Tz,Tz)\frac{1+d(Sz,Sz)}{1+d(Sz,Tz)}\right] + \beta[max\{d(Sz,Tz), d(Sz,Sz), d(Tz,Tz)\}] + \gamma\,[d(Tz,Sz)] + \eta\left[\frac{d(Tz,Tz)d(Sz,Sz)}{d(Tz,Sz)+d(Sz,Tz)+d(Tz,Sz)}\right]$$

$d(Sz,Tz) \precsim \beta[d(Sz,Tz)] + \gamma[d(Sz,Tz)]$
That is, $|d(Sz,Tz)| \leq (\beta+\gamma)|d(Sz,Tz)|$
Which is a contradiction to $(\beta+\gamma) < 1$. Therefore $Sz = Tz$ that is $AMz = Sz = Tz = BDz$.
Now, we prove that $z = Tz$. From (1), put y = z, we have

$$d(Sx,Tz) \precsim \alpha\left[d(BDz,Tz)\frac{1+d(AMx,Sx)}{1+d(AMx,BDz)}\right] + \beta[max\{d(AMx,BDz), d(AMx,Sx), d(BDz,Tz)\}] + \gamma\,[d(Tz,Sx)] + \eta\left[\frac{d(Tz,BDz)d(AMx,Sx)}{d(Tz,AMx)+d(Sx,BDz)+d(Tz,Sx)}\right]$$

$$d(z,Tz) \precsim \alpha\left[d(Tz,Tz)\frac{1+d(z,z)}{1+d(z,Tz)}\right] + \beta[max\{d(z,Tz), d(z,z), d(Tz,Tz)\}] + \gamma\,[d(Tz,z)] + \eta\left[\frac{d(Tz,Tz)d(z,z)}{d(Tz,z)+d(z,Tz)+d(Tz,z)}\right]$$

Thus, $d(z,Tz) \precsim \beta\, d(z,Tz) + \gamma d(z,Tz)$
That is, $|d(z,Tz)| \leq (\beta+\gamma)|d(z,Tz)|$
Which is a contradiction to $(\beta+\gamma) < 1$. Therefore $Tz = Sz = AMz = BDz = z$. similarly by above theorem we can prove that $Mz = Az = z = Dz = Bz$.
**Uniqueness :** From theorem 3.1, we can easily prove the uniqueness of the theorem. Hence A, B, D, M, S and T have a unique common fixed point in X.

## IV. Conclusion

In this paper, we have presented common fixed point theorems in complex valued metric spaces through concept of weak compatibility, E.A. property and CLR property.


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