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An Internal Construction for Congruence Relations in Lattices

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Abstract: *A method of constructing a smallest congruence relation that is larger than a given equivalence relation on a lattice is explained. A method of constructing a congruence relation in which equivalence classes contain all least upper bounds and all greatest lower bounds for subsets of equivalence classes is explained; and this method constructs a smallest congruence relation with this property which is also larger than a given congruence relation in a lattice.*
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I. Introduction

Let $(L, \leq)$ or $(L, \vee, \wedge)$ be a lattice. Let $\aleph$ be an infinite cardinal number. A subset A of L is $\aleph$-closed in L, if for any subset B of A for which $|B| < \aleph$, the least upper bound of B is in A, whenever it exists in L, and the greatest lower bound of B in A, whenever it exists in L . A subset A of L is said to be closed in L, if it $\aleph$-closed in L, for every $\aleph$. An equivalence relation θ on L is said to be $\aleph$-closed, if every equivalence class induced by θ is $\aleph$-closed in L. The equivalence relation θ on L is said to be closed, if every equivalence class induced by θ is closed. To an equivalence relation on L, let us use an usual notation $x \equiv y \pmod{\theta}$, when x and y are related by θ. Two equivalence relations θ and θ' are ordered by the usual 'refinement' order relation: $\theta \leq \theta'$ if $x \equiv y \pmod{\theta}$ implies $x \equiv y \pmod{\theta'}$). An equivalence relation on L is said to be a congruence relation, if it has the following substitution properties: $x \vee z \equiv y \vee z \pmod{\theta}$ and $x \wedge z \equiv y \wedge z \pmod{\theta}$, whenever $x \equiv y \pmod{\theta}$ and $x, y, z \in L$.

If $(\theta_i)_{i\in I}$ is a collection of $\aleph$-closed (or, simply, closed) congruence relations on L, then the relation θ on L defined by $x \equiv y \pmod{\theta}$ if and only if $x \equiv y \pmod{\theta_i}$, $\forall i \in I$, is also an $\aleph$-closed (or, simply, a closed) congruence relation on L. Thus $\theta = \wedge_{i\in I} \theta_i \in \aleph C$- ConL, the collection of all $\aleph$- closed congruence relations on L, when $\theta_i \in \aleph C$- ConL, $\forall i \in I$. Similarly $\vee_{i\in I} \theta_i$ is in $\aleph C$-ConL, when $\vee_{i\in I}\theta_i$ is considered as $\wedge\{\phi : \phi \in A\}$, when $A=\{\phi \in \aleph C\text{-Con } L: \theta_i \leq \phi, \forall i \in I\}$. If ConL denotes the collection of all congruence relations on L, it is known that ConL is a lattice, and an internal construction for least upper bound of a given sub collection is also known (see the proof of theorem 3.9 in [2]). So, an internal construction of $\vee_{i\in I} \theta_i$ in $\aleph C$- ConL,when $(\theta_i)_{i\in I} \subseteq \aleph C$-ConL, depends on construction of a smallest θ'' in $\aleph C$-ConL such that $\theta'' \geq \theta$, for given $\theta \in$ConL. For this construction, another internal constructon of a smallest congruence relation θ on L such that $\theta \geq \phi$ for a given equivalence relation ϕ on L is developed in this article. It is expected that all types of constructions may be helpful to understand congruence lattices(see: [3]).

II. Construction of equivalence classes through transfinite induction

At every phase, an equivalence relation is to be found from a given equivalence relation, by means of a construction. A finite set of constructions have to be repeated to reach a desirable equivalence relation. So, a common construction procedure is to be defined in this section.

Let $(L,\vee,\wedge)$ be a given lattice. To each i = 1,2,... n, a fixed positive integer, let P_i(say) denote a given common procedure which constructs an equivalence relation $\sim_i$ when it is applied on a given equivalence relation θ_i with a result $\theta_i \leq \sim_i$. Let θ_0 be a given equivalence relation on L. Let θ_i be the equivalence relation that is obtained by following the procedure P_i on θ_{i-1}, for i = 1,2,... n. For $mn < mn + i \leq mn + n$, let θ_{mn+i} denote the equivalence relation that is obtained by the following the procedure P_i on θ_{mn+i-1}, for m = 1,2..... Now $\theta_0 \leq \theta_1 \leq \theta_2 \leq ....$ Let θ_ω be the supremum of the equivalence relations $\theta_0,\theta_1,\theta_2,...$ in the lattice of all equivalence relations on L. Let us construct $\theta_{\omega+1},\theta_{\omega+2},...$ one by one by following the procedures P_1; P_2; ... P_n; P_1; P_2,... P_n,... applied on θ_ω ,$\theta_{\omega+1}$,... respectively. Now define $\theta_{\omega+\omega}$ as the supremum of the equivalence relations θ_0, $\theta_1,...,\theta_\omega,\theta_{\omega+1},...$ in the lattice of equivalence relations on L. Thus, once a limiting ordinal α is reached, θ_α is constructed as the supremum of θ_i, $i < \alpha$; and the procedures P_1, P_2,...P_n, P_1,P_2,...P_n... are applied on θ_α ,$\theta_{\alpha+1}$,... to construct $\theta_{\alpha+1}$, $\theta_{\alpha+2}$,... respectively and successively. This procedure leads to a 'stationary' equivalence relation in the sense that $\theta_\beta=\theta_{\beta+\alpha}$, for every ordinal α. This happens because L is fixed and hence its cardinality is fixed. Let us say that the equivalence relation θ_β is the stationary equivalence relation obtained by following the procedures P_1, P_2,... P_n on θ_0.

III. Equivalence relation to congruence relation

This section provides a construction to obtain a smallest congruence relation θ' from a given equivalence relation θ such that θ≤θ' on a lattice L. This construction is based on the following significant observation.

Lemma 3.1 Let θ be a given equivalence relation on a lattice $(L,\vee,\wedge)$. To each $x\in L$, let [x] denote the equivalence class of θ containing x. Then θ is a congruence relation if and only if the following hold for every $x\in L$: (i) $a\vee b\in [x]$ and $a\wedge b\in [x]$, whenever $a, b\in[x]$; (ii) $(z\vee a)\wedge b\in[x]$ and $(z\wedge a)\vee b\in[x]$, whenever $a,b\in[x]$, and $z\in L$; (iii) $a_1\vee b_1\in[x]$, whenever $a\vee b\in[x]$, $a_1\in[a]$ and $b_1 \in [b]$; (iv) $a_1\wedge b_1\in[x]$, whenever $a\wedge b\in [x]$, $a_1\in[a]$ and $b_1\in[b]$.

Proof: The proof follows from two facts:

(1) θ is a congruence relation if and only if L/θ is a lattice.

(2) A set with two binary operations is a lattice if and only if the binary operations satisfy idempotent law, commutativity law, associativity law, and absorption law (see: Theorem 1 in p.18 in [1]).

Suppose (i),(ii),(iii) and (iv) are true. To each $a,b\in L$, let us define $[a]\vee[b]=[a\vee b]$ and $[a]\wedge[b]=[a\wedge b]$. They are well defined in view of (iii) and (iv). The commutavity and associativity of these operations follow from the corresponding properties of ∨ and ∧ in L. These operations satisfy idempotent law and absorption law, because of (i) and (ii). So L/θ is a lattice so that θ is a congruence relation.

On the other hand, if θ is a congruence relation, then L/θ is a lattice so that (i), (ii),(iii) and (iv) are true.

Construction procedure P_1: Let θ be a given equivalence relation on a lattice (L, ∨,∧). Define $x \sim_1 y$ if there is a finite sequence $x_1,y_1,x_1',y_1',x_2,y_2,x_2'.y_2',...x_n,y_n,x_n', y_n'$,.... in L such that $x_i'\equiv x_i$ (mod θ) and $y_i'\equiv y_i$ (mod θ), $\forall i= 1,2,...n$, and such that $x=x_1\wedge y_1$, $y=x_n'\wedge y_n'$, $x_i'\wedge y_i'=x_{i+1}'\wedge y_{i+1}'$, $\forall i=1,2,...n-1$. Then $\sim_1$ is an equivalence relation on L. Let us say that $\sim_1$ is obtained from θ by following procedure P_1. Observe that if $x_1\equiv x_2$ (mod θ) and $y_1\equiv y_2$ (mod θ), then $x_1\wedge y_1 \sim_1 x_1\wedge y_2$. Note that $\theta\leq \sim_1$.

Construction procedure P_2: Replace $\sim_1$, P_1, ∧ in the previous discussion by $\sim_2$, P_2, ∨, respectively, so that if $x_1\equiv x_2$ (mod θ) and $y_1\equiv y_2$ (mod θ), then $x_1\wedge y_1 \sim_2 x_2\vee y_2$. Note that $\theta\leq \sim_2$.

Construction procedure P3: To each $x\in L$, let [x] denote the equivalence class of a given equivalence relation θ on a given lattice (L,∨,∧). Let us define a relation $\sim_3$ on L by $a\sim_3 b$ if there is a finite sequence $a_0,a_1,a_2,...,a_n$ in L such that : (i) $a\equiv a_0$ (mod θ); (ii) $b\equiv a_n$ (mod θ); and (iii) to each i=0,1,2,....,n-1, there are $b_i,c_i \in[a_i]$ such that $b_i\vee c_i\in[a_{i+1}]$ or $b_i\wedge c_i\in[a_{i+1}]$; or there are $b_{i+1},c_{i+1}\in[a_{i+1}]$ such that $b_{i+1}\vee c_{i+1} \in[a_i]$ or $b_{i+1}\wedge c_{i+1}\in[a_i]$. Then $\sim_3$ is an equivalence relation. Let us say that '$\sim_3$' is obtained from θ by following procedure P_3. Observe that if $b_0,c_0 \in[a_0]$, then $a_0\sim_3 b_0\vee c_0$ and $a_0\sim_3 b_0\wedge c_0$. Note that $\theta\leq \sim_3$.

Construction procedure P_4: Let us fix L and θ, and let us fix the notation [x] as in the previous procedure. Let us define a relation $\sim_4$ on L by $a\sim_4 b$ if there is a finite sequence $a_0,a_1,...,a_n$ in L such that: (i) $a\equiv a_0$(mod θ); (ii) $b\equiv a_n$ (mod θ); and (iii) to each i=1,2,...,n-1 there are $b_i,c_i \in[a_i]$ and $d_i\in L$ such that $(d_i\wedge b_i)\vee c_i\in[a_{i+1}]$ or $(d_i\vee b_i)\wedge c_i\in[a_{i+1}]$; or there are $b_{i+1},c_{i+1}\in[a_{i+1}]$ and $d_{i+1}\in L$ such that $(d_{i+1}\wedge b_{i+1})\vee c_{i+1}\in[a_i]$ or $(d_{i+1}\vee b_{i+1})\wedge c_{i+1}\in[a_i]$. Then '$\sim_4$' is an equivalence relation. Let us say that '$\sim_4$' is obtained from θ by following procedure P_4. Observe that if $b_0,c_0 \in[a_0]$ and $d_0\in L$, then $a_0 \sim_4 (d_0\vee b_0)\wedge c_0$ and $a_0\sim_4 (d_0\wedge b_0)\vee c_0$. Note that $\theta\leq\sim_4$.

Theorem 3.2 Let θ_0 be a given equivalence relation on a lattice (L,∨,∧). Let θ_β be the stationary equivalence relation obtained by following the procedures P_1, P_2, P_3, P_4 on θ_0. Then θ_β is the smallest congruence relation on L such that $\theta_0\leq\theta_\beta$. .**Proof:** To each $x\in L$, let [x] denote the equivalence class of θ_β containing x. The procedure P_1,P_2,P_3, and P_4 reveal that the conditions (i),(ii),(iii) and (iv) of the previous lemma 3.1 are satisfied for the equivalence relation θ_β on L. So, θ_β is a congruence relation on L such that $\theta_{0\leq}\theta_\beta$.

IV. Equivalence relation to closed congruence relation

This section provides a construction to obtain a smallest closed equivalence relation θ' from a given equivalence relation θ such that θ≤θ' on a lattice. This construction may be combined with the construction of the previous section to obtain a smallest closed congruence relation. Let us introduce some notations for the next construction. Let θ be an equivalence relation in a given lattice L. If K_1, K_2 are subsets of two equivalence classes of θ, then let us write $K_1\equiv K_2$(mod θ) if K_1, K_2 are subsets of the same equivalence class of θ, then let us write $x \equiv K (\text{mod } \theta)$ if $x \equiv y (\text{mod } \theta)$, $\forall y \in K$. If K is a subset of L and if a least upper bound of K exists in L, then it will be denoted by $\vee K$. The notation $\wedge K$ refer to a greatest lower bound on K, when it exists.

Construction procedure P_5: Let θ be a given equivalence relation on a lattice $(L,\vee,\wedge)$. Let ℵ be a fixed infinite cardinal number. Let us write H~K for two non empty subsets H,K of equivalence classes of θ, when there is a finite sequence $G_0,G_1,...,G_n (n \geq 1)$ of subsets of equivalence classes of θ such that:

(i) $H \equiv G_0 (\text{mod } \theta)$,
(ii) $K \equiv G_n (\text{mod } \theta)$,
(iii) $|G_i| < \aleph, \forall$ i=1,2,...,n.
(iv) $\vee G_i$ exists and $\vee G_i \equiv G_{i+1} (\text{mod } \theta)$ or
$\wedge G_i$ exists and $\wedge G_i \equiv G_{i+1}$ (mod θ) or
$\vee G_{i+1}$ exists and $\vee G_{i+1} \equiv G_i$ (mod θ) or
$\wedge G_{i+1}$ exists and $\wedge G_{i+1} \equiv G_i$ (mod θ), for i=1,2,...,n-1.

The following can be verified.

(i) If $x \equiv y (\text{mod } \theta)$, then [x]~[y]. For take n=1, G_0=H={x} and G_1=K={y} in the previous description.
(ii) Let H be a subset of an equivalence class of θ such that $|H| < \aleph$.

If $\vee H$ exists, then $H \sim \{\vee H\}$. For, take n=1, G_0=H, and $G_1 = K = \{\vee H\}$in the previous description. Similarly, if $\wedge H$ exists, then $H \sim \{\wedge H\}$.

Let us now define '$\sim_5$' on L by $x \sim_5 y$ if [x]~[y], when $x,y \in L$. This defines an equivalence relation on L such that $\theta \leq \sim_5$. Let us say that '$\sim_5$' is obtained from θ by following the procedure P_5. The properties (i) and (ii) of '$\sim_5$' imply the next theorem 4.1.

Theorem 4.1 Let θ_0 be a given equivalence relation on a lattice $(L,\vee,\wedge)$. Let ℵ be a given infinite cardinal number. Let θ_β be the stationary equivalence relation obtained by following the procedure P_5 on θ_0. Then θ_β is the smallest ℵ-closed equivalence relation on L such that $\theta_0 \leq \theta_\beta$.

The next theorem 4.2 is a combination of Theorem 3.2 and Theorem 4.1.

Theorem 4.2 Let θ_0, $(L,\vee,\wedge)$ and ℵ be as in the previous theorem 4.1. Let θ_β be the stationary equivalence relation obtained by following the procedures P_1, P_2, P_3, P_4, P_5 on θ_0. Then θ_β is the smallest ℵ-closed congruence relation on L such that $\theta_0 \leq \theta_\beta$.

Remark 4.3 If $(\theta_i)_{i \in I}$ is a collection of equivalence relations on a lattice $(L,\vee,\wedge)$, then one can follow an usual procedure (see theorem 4.3 in p.23 in [1]) to construct a smallest equivalence relation θ_0 such that $\theta_i \leq \theta_0$, $\forall i \in I$ If θ_β is the congruence relation constructed in the theorem 4.2, then θ_β is the smallest ℵ-closed congruence relation such that $\theta_i \leq \theta_\beta$, $i \in I$.

Remark 4.4 The word 'ℵ-closed' may be replaced by the word 'closed' in the discussion of this section.

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