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α - Generalized & α^* - Separation Axioms for Topological Spaces

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Abstract: *The present paper introduces a new class of separation axioms called α-generalized separation axioms using α-generalized open sets and also includes the study of the connections between these separation axioms and the existing α-separation axioms. Also, here, the concept of* α^* *- closed set has been coined and then* α^* *- separation axioms have been framed w.r.t* α^* *-open sets.*

Key Words: *α-open sets,* α^* *-closed sets, α g-closed sets, α-continuous &* α^* *- continuous / irresolute functions,* α^* *-*T_k*(k = 0,1,2) and αg-* T_k*(k= 0,1,2).*

I. Introduction

In the mathematical paper [3] O.Njastad introduced and defined an α-open/closed set. After the works of O.Njastad on α-open sets, various mathematicians turned their attention to the generalizations of various concepts in topology by considering semi-open, α-open sets.The concept of g-closed [1], s-open[2] and α-open [3] sets has a significant role in the generalization of continuity in topological spaces. The modified form of these sets and generalized continuity were further developed by many mathematicians [4,5].

In 1970, Levine generalized the concept of closed sets to generalized closed sets[1]. After that there is a vast progress occurred in the field of generalized open sets(compliment of respective closed sets) which became the base for separation axioms in the respective context. In this paper, we introduce the generalized forms of α- separation axioms using the concepts of α- generalized open sets called α- generalized – T_k (briefly denoted by αg-T_k)spaces. Also, we define the concepts of α^* - open sets in a topological space in order to frame the another class of separation axioms called α^* -separation axioms. Among other things, the concern basic properties and relative preservation properties of these spaces are projected under α^* -irresolute and α^* - continuous mappings.

II. Preliminaries

Throughout this paper clA and intA respectively closure and the interior of the set A where A is a subset of a topological space (X,τ) on which no separation axioms are assumed unless explicitly stated. The following definitions and results are listed because of their use in the sequel.

Definition 2.1: Let A be a subset of a space X then A is said to be:
(i) A pre-open if A ⊂intclA,
(ii) a semi-open if A ⊂clintA,
(iii) α -open set if A⊂ intclint A,

Definition 2.2: (i) The α -closure of a subset A of X is the intersection of all α -closed sets that contains A and is denoted by αclA.

(ii) The α -interior of a subset A of X is the union of all α -open subsets of X that contained in A and is denoted by α int A.

Definition 2.3: If A be a subset of a space X then A is said to be
(i) an α-generalized closed (i.e. αg-closed) set[6] if αclA ⊂ U whenever A⊂ U and U is α -open set,
(ii) a generalized α -closed (i.e., gα -closed) set[6] if αclA ⊂ U whenever A ⊂U and U is open set in X,
(iii) an α^* -closed set if αclA ⊂ U whenever A ⊂U and U is αg-open set.

Also (iv) α-generalized closure of a subset A of a space X is the intersection of all αg-closed sets containing A and is denoted by αgcl(A) .

(v) A function f : (X, T)→ (Y, σ) is said to be :
(a) α -continuous if $f^{-1}(V)$ is α- open set in (X, τ) for every open set V of(Y, σ).
(b) g-continuous if $f^{-1}(V)$ is g-closed in (X, τ) for every closed set V of (Y, σ).
(c) α- irresolute if $f^{-1}(V)$ is α open in (X, τ) for every α- open set V of (Y, σ).

(d) α -open if f(U) is α -open in (Y, σ) for every α- open set U of (X, τ).

Low separation axioms via α-open sets: α –T_k spaces,(k = 0, 1, 2)

The following are the definitions concerned with α-T_k spaces where (k = 0,1,2).

Definition 2.4: A space (X,τ) is called

(i) α -T_0 iff to each pair of distinct points x, y of X, there exists an α -open set containing one but not the other.

(ii) α –T_1 iff to each pair of distinct points x, y of X, there exists a pair of α -open sets, one containing x but not y, and their other containing y but not x.

(iii) α –T_2 iff to each pair of distinct points x, y of X, there exists a pair of disjoint α- open sets, one containing x and the other containing y.

Theorem 2.5: A space X is α –T_0 iff $\alpha cl\{x\} \neq \alpha cl\{y\}$ for every pair of distinct points x, y of X.

Proof: Let x and y be any two distinct points of α –T_0 space X. We show that $\alpha cl\{x\} \neq \alpha cl\{y\}$: By hypothesis, suppose that U ∈ α O(X) such that x ∈U and y ∉U. Hence y ∈X – U and X – U is α -closed set. Therefore, $\alpha cl\{y\} \subset X - U$. Hence y ∈ $\alpha cl\{y\}$as x ∉ X – U.
Hence, $\alpha cl\{x\} \neq \alpha cl\{y\}$.

Conversely, suppose for any x, y∈ X with x ≠ y, $\alpha cl\{x\} \neq \alpha cl\{y\}$. Without any loss of generality, let z∈X such that z∈ αcl {x} but z∉ αcl {y}. Now, we claim that x ∉$\alpha cl\{y\}$. For if x ∈ $\alpha cl\{y\}$ then{x} ⊂ $\alpha cl\{y\}$ which implies that $\alpha cl\{x\} \subset \alpha cl\{y\}$. This contradicts the fact that z∉ $\alpha cl\{y\}$. Consequently x belongs to the α-open set $[\alpha cl\{y\}]^C$ to which y does not belong .
Hence, the space is an α -T_0 space.

Theorem 2.6: A space (X, τ) is α –T_1 iff the singletons are α-closed sets.

Proof : Let (X, τ) is α –T_1 and x any point of X. Suppose y ∈ $\{x\}^C$. then x ≠y and so there exists an α-open set U_y such that y ∈ U_y but x ∉ U_y . Consequently y∈ $U_y \subset \{x\}^C$ i.e.
$\{x\}^C = \cup\{ U_y / y \in \{x\}^C\}$ which is an α open.

Conversely, suppose {p} is α- closed for every p∈ X. Let x,y∈ X with x ≠y. Now x ≠y implies y ∈ $\{x\}^C$. Hence $\{x\}^C$ is α-open set containing y but not x . Similarly $\{y\}^C$ is α-open set containing x but not y. accordingly X is an α –T_1 space.

Theorem 2.7: A space (X, τ) is α-T_2 iff (X,T^α) is Hausdorff space.

Proof: Let x, y ∈X with x≠ y. Since X is α –T_2, there exist disjoint α-open sets U and V in X such that x ∈U and y∈V , U∩ V = φ. Here, U, V ∈ T^α , so, obviously (X, T^α) ceases to be a T_2-space i.e. a Hausdorff space.

Conversely, whenever (X,T^α) is a T_2-space, there exist a pair of members of T^α , say, P & Q for a pair of distinct points p & q of X such that p∈P & q∈ Q & P∩ Q = φ. But $\alpha o(X,T) = T^\alpha$. Combing all these facts (X,T) is α-T_2 space.

Theorem 2.8: Every open subspace of a α –T_2 space is α-T_2.

Proof: Let U be an open subspace of a α-T_2 space (X, τ). Let x and y be any two distinct points of U. Since X is α-T_2 and U ⊂ X, there exist two disjoint α- open sets G and H in X such that x ∈ G and y ∈H. Let A = U ∩ G and B = U∩ H. Then A and B are α-open sets in U containing x and y . Also, A∩B = φ.
Hence (U, T_U) is α –T_2.

* The contents of the preliminaries have been prepared with the help of the paper produced by M.Caldas et al [7].

III. Invariant property of α-T_k spaces (k = 0,1,2).

We, now, enunciate the invariant property of the α-T_k spaces in the following manner:

Theorem 3.1: If f : X⟶Y be an injective α -irresolute mapping and Y is an α -T_0 then X is α -T_0.

Proof: Let x, y ∈X with x ≠ y. Since f is injective and Y is α -T_0, there exists a α -open set Vx in Y such that f(x) ∈Vx and f(y) ∉ Vx or there exists a α- open set Vy in Y such that f(y) ∈Vy and f(x) ∉ Vy with f(x) ≠ f(y). By α -irresoluteness of f, $f^{-1}(Vx)$ is α -open set in X such that x∈ $f^{-1}(Vx)$ and y∉ $f^{-1}(Vx)$ or $f^{-1}(Vy)$ is α -open set in X such that y ∈ $f^{-1}(Vy)$ and x ∉ $f^{-1}(Vy)$. This shows that X is α –T_0.

Theorem 3.2: If f : X⟶Y be an injective α -irresolute mapping and Y is an α –T_1 then X is α –T_1.

Proof: The argument exists in the similar way as mentioned in theorem 3.1 with suitable changes.

Theorem 3.3: If f : X⟶Y be an injective α -irresolute mapping and Y is an

$\alpha - T_2$, then X is $\alpha - T_2$.
Proof: Similar is the way as in theorem 3.1 for the establishment of the statement of the theorem under proper changes according to the context.

*** New concepts of (i) point α-closure 1-1 mapping and (ii) point α-closed 1- 1 mapping:**
Definition (3.4): A function f : X→Y from one topological space (X,T) to another space (Y,σ) is said to be
(i) point α-closure 1-1 iff x,y ∈ X such that αcl{x}≠ αcl{y} then αcl{f(x)} ≠αcl{f(y)}.
(ii) point α-closed 1-1 iff for every x ∈ X such that {x} is αcl{x} = {x}, then αcl{f(x)} ={f(x)}.

Theorem 3.5: : If f : X→ Y is point –α closure 1-1 and X is α-T_0, then f is 1-1.
Proof: Let x, y ∈ X with x ≠ y. Since X is α-T_0, then αcl{x}≠ αcl{y} by Theorem 2.5. But f is point α-closure 1-1 implies that αcl{f(x)} ≠ αcl{f(y)}. Hence f(x) }≠ f(y). Thus, f is 1-1.
Theorrem 3.6: A point α-closure 1-1 mapping f : X→Y from α-T_0 space X into α-T_0 space Y exists iff f is 1-1.
Proof : The necessity follows from the fact mentioned in theorem 3.1.
For sufficiency, let f : X→Y from α-T_0 space X into α-T_0 space Y be an one-one mapping . Now for every pair of distinct points x & y ∈ X, αcl{x}≠ αcl{y} as X is α-T_0 space. Since, f is 1-1 mapping f(αcl{x}) ≠ f(αcl{y}). i.e., αcl{f(x)}≠ αcl{f(y)}. Consequently, f is point α-closure 1-1 mapping.
Theorrem 3.7: A point α-closed 1-1 mapping f : X→Y from α-T_1 space X into α-T_1 space Y exists iff f is 1-1.
Proof: Necessity : Let f : X→Y be a point α-closed 1-1 mapping , where (X,T) &(Y,σ) are α-T_1 spaces.
Here, for a pair of distict points x,y ∈X, we have as {x},{y} T-α-closed sets & {f(x)},{f(y)} as σ-α- closed spaces. Therefore, αcl{x} = {x} & αcl{f(x)}=f(x). Hence, For a pair of distinct points x,y ∈ X , f(x) ≠ f(y).i.e. f is 1-1.
Sufficiency : Since f is one –one mapping from α-T_1 space X into α-T_1 space Y.
Hence, for x ∈ X, αcl{x} ={x} .Next f(αcl{x}) = f({x}). i.e. αcl(f{x}) = f({x}).
This means that αcl{x} ={x}⇒ αcl(f{x}) = f({x}) for all x ∈ X.

IV. α -Generalized Separation Axioms

Separation axioms using αg-open sets and being weaker than α-separation axiom are, here, framed due to the motivation of the existence & wide application of αg-open sets.
Definition 4.1 : A space X is called α- generalized $-T_0$ (briefly written as αg-T_0) iff to each pair of distinct points x, y of X, there exists a αg-open set containing one but not the other.
.**Definition 4.2:** A space X is called α- generalized $-T_1$ (briefly written as αg-T_1) iff to each pair of distinct points x, y of X, there exists a pair of αg-open sets, one containing x but not y, and the other containing y but not x.
Definition 4.3: A space X is called α- generalized – T_2 space (briefly written as αg-T_2 space) iff to each pair of distinct points x, y of X there exists a pair of disjoint αg-open sets, one containing x and the other containing y.
Clearly, every $\alpha - T_k$ space is αg-T_k spaces(k =0,1,2) since every α -open set is αg-open set.
The following theorems are related to the characterization & invariance nature for αg- T_k spaces:
Theorem 4.4: A space X is αg $-T_0$ iff αcl{x} ≠ αcl{y} for every pair of distinct points x, y of X.
Theorem 4.5: A space (X, τ) is αg $-T_1$ iff the singletons are αg-closed sets.
Theorem 4.6: For a space(X,T) the following are equivalent:
(a) X is αg-T_2.
(b) The diagonal Δ = {(x,x): x ∈ X} is αg –closed in X × X.
Theorem 4.7: If f : X⟶Y be an injective αg -irresolute mapping and Y is an αg $-T_k$ then X is αg $-T_k$ (k= 0,1,2).

Furthermore, we mention the concept of α-$T_{1/2}$ space in the same tune of $T_{1/2}$- space in topology :
Definition 4.8: A space (X, τ) is called an α-$T_{1/2}$ space if every αg –closed set is α-closed.
Definition4.9: In a topological space (X, τ), the following notions are well defined as :
(a) α D(X,τ) = {A: A⊂X and A is αg -closed in (X,τ)}.
(b) $\alpha cl^*(E) = \bigcap \{A : E \subset A(\in \alpha D(X,\tau))\}$
(c) $\alpha O(X,\tau)^* = \{B : \alpha cl^*(B^C) = B^C\}.$

Theorem 4.10 A topological space (X,τ) is a α -$T_{1/2}$ space if and only if $\alpha O(X,\tau) = \alpha O(X,\tau)^*$ holds.
Proof. Necessity:
Let (X,τ) be a topological space which is also α -$T_{1/2}$ space. This means that the α-closed sets and the gα-closed sets coincide by the assumption, $\alpha cl(E) = \alpha cl^*(E)$ holds for every α-closed subset E of (X,τ). Hence, we have $\alpha O(X,\tau) = \alpha O(X,\tau)^*$.
Sufficiency: Let A be a g α -closed set of (X,τ). Then, we have
A = $\alpha cl^*(A)$ & by the accepted criteria $\alpha O(X,\tau) = \alpha O(X,\tau)^*$, we claim that
$A^C \in$ α O(X,τ), which means that A is α -closed. Therefore (X, τ) fulfils the criteria for being α -$T_{1/2}$.
Theorem 4.11. A topological space (X, τ) is a α -$T_{1/2}$ space if and only if, for each x ∈X, {x} is α -open or α -closed.
Proof. Let topological space (X, τ) be a α -$T_{1/2}$ space.
Necessity: Let topological space (X, τ) be a α -$T_{1/2}$ space. Let us Suppose that for some x ∈ X; {x} is not α -closed. Since X is the only α -open set containing $\{x\}^C$, the set $\{x\}^C$ is αg -closed [Definition 2.3] and so it is α -closed in the α -$T_{1/2}$ space (X, τ). Therefore {x} is α -open.
Sufficiency: Since $\alpha O(X,\tau) \subseteq \alpha O(X,\tau)^*$ holds, it is enough to prove that $\alpha O(X,\tau)^* \subseteq \alpha O(X,\tau)$. Let E ⊆ α O(X, τ). Suppose that E ∉ αO(X,τ). Then, $\alpha cl^*(E^C) = E^C$ and $\alpha cl(E^C) \neq E^C$ hold. There exists a point x of X such that x ∈ $\alpha Cl(E^C)$ and x∉ E^C (= $\alpha cl^*(E^C)$). Since x ∉ αCl (E^C) there exists a αg -closed set A such that x∉ A and A ⊃ E^C By the hypothesis, the singleton {x} is α -open or α -closed.
Case 1. {x} is α -open: Since$\{x\}^C$ is a α -closed set with $E^C \subset \{x\}^C$,
we have $\alpha Cl(E^C) \subset \{x\}^C$ i.e. x ∉ αCl (E^C) .This contradicts the fact that x ∈ α Cl(E^C).
E∈ α O(X, τ).
Case 2. {x} is α -closed: Since$\{x\}^C$ is a α -open set containing the αg -closed set A(⊃ E^C), we have$\{x\}^C \supset \alpha Cl(A) \supset \alpha Cl(E^C)$ Therefore x∉ αCl (E^C) This is a contradiction. Therefore E∈ α O(X, τ).
Hence in both cases, we have E∈ α O(X, τ), i.e., $\alpha O(X,\tau)^* \subseteq$ α O(X, τ).
$\therefore \alpha O(X,\tau) = \alpha O(X,\tau)^*$ using theorem (3.1), it follows that (X, τ) is a α -$T_{1/2}$.
Theorem 4.12. A topological space (X, τ) is a α -$T_{1/2}$ space if and only if and only if, every subset of X is the intersection of all α -open sets and all α-closed sets containing it.
Proof. **Necessity:** Let topological space (X, τ) is a α -$T_{1/2}$ space with
$B \subset X$ arbitrary. Then B = $\bigcap \{\{x\}^C; x \notin B\}$ is an intersection of α -open sets and α -closed sets by Theorem 4.11. So the necessity follows.
Sufficiency: For each $x \in X, \{x\}^C$ is the intersection of all α -open sets and
all α -closed sets containing it. Thus $\{x\}^C$ is either α -open or α -closed
and hence X is α - $T_{1/2}$ space .

V. α^* separation axioms:

In this section, we define and study some new separation axioms by defining α^*-open sets which are stronger than α- generalized separation axioms.
Definition 5.1: A subset A of X is called α^*-open set of X if F ⊂ αintA whenever F is αg-closed and F ⊂ A.
Or
A subset A of X is called α^*-closed set of X if αclA⊂ U whenever A⊂ U& U is αg-open set.

Clearly, every open set, α-open set is α^*-open and every α^*-open set is αg-open set.
We, define the following.
Definition 5.2: The union of all α^* -open sets which are contained in A is called the α^* -interior of A and is denoted by α^* -intA.
Definition 5.3: The intersection of all α^* -closed sets which contains A is called the α^* -closure of A and is denoted by α^* -clA.
Definition 5.4: A point x of a space X is called a α^* -limit point of a subset A of X, if for each α^* -open set U containing x, A ∩ (U– {x}) ≠ φ.

Definition 5.4(a): The set of all α^* -limit points of A, denoted by $\alpha^*d(A)$, is called α^*-derived set of A.
Definition 5.5 : A function f : (X, τ)→ (Y, σ) is said to be :
(a) α^* -continuous if $f^{-1}(V)$ is α^* - open set in (X, T) for every open set V of(Y, σ).
(b) α^*- irresolute if $f^{-1}(V)$ is α^* -open in (X, T) for every α^*- open set V of(Y, σ),

Definition 5.6: A space X is called $\alpha^*– T_0$ iff to each pair of distinct points x, y of X, there exists a α^* -open set containing one but not the other.
Definition 5.7: A space X is called α^*-T_1 iff to each pair of distinct points x, y of X, there exists a pair of α^*-open sets, one containing x but not y, and the other containing y but not x.
Definition 5.8: A space X is called $\alpha^* – T_2$ space iff to each pair of distinct points x, y of X there exists a pair of disjoint α^* -open sets, one containing x and the other containing y.
Clearly, every α-T_k space is α^* -T_k and α^* -T_k space is αg-T_k space. Since every α-open set is α^* -open and every α^* -open set is αg-open set where(k = 0,1,2).
The following theorems are related to the characterization & invariance nature for α^*- T_k spaces:
Theorem 5.9: A space X is $\alpha^* –T_0$ iff $\alpha^*cl\{x\} \neq \alpha^*cl\{y\}$ for every pair of distinct points x, y of X.
Theorem 5.10: A space (X, τ) is $\alpha^* –T_1$ iff the singletons are α^* -closed sets.
Theorem 5.11: For a space(X, τ) the following are equivalent:
(a) X is α^* -T_2.
(b) The diagonal $\Delta = \{(x,x): x \in X\}$ is α^* –closed in $X \times X$.
Theorem 5.12: If f : X⟶Y be an injective α^* -irresolute mapping and Y is an $\alpha^* –T_k$ then X is $\alpha^* –T_k$ (k= 0,1,2).
Furthermore, we mention the concept of α^* -$T_{1/2}$ space in the same tune of $T_{1/2}$- space in topology :
Definition 5.12: A space (X, τ) is called an α^* -$T_{1/2}$ space if every α^* –closed set is α-closed.
The following theorem appears as an evidence for the validity of the above definition of an α^* -$T_{1/2}$ space.
Theorem 5.13. A topological space (X, τ) is a α^* **-**$T_{1/2}$ space if and only if, every subset of X is the intersection of all α^*-open sets and all α^*-closed sets containing it.
Proof: The proof is based upon the definitions of the related terms used in the theorem.

VI. Conclusion

Separation axioms in terms of α g-open & α^*-open sets have been formulated and their structural properties have also been discussed and emphasized which opens the future scope of respective normal and regular topological spaces.

References

[1]. N. Levine, Generalized Closed Sets In Topology, Rend.Circ.Mat. Palermo(2) 19(1970),89-96 .
[2]. N.Levine, Semi-Open Sets And Semi- Continuity In Topological Spaces, Amer.Math. Monthly 70(1963),36-41.
[3]. O.Njasted, On Some Classes Of Nearly Open Sets, Pacific J. Math15(1965), 961-970.
[4]. K.Balachandran,P.Sundran, and H.Maxi, On Generalized Continuous Maps In Topological Spaces. Mem. Fac.Sci. Kochi Univ., ser. A. math.,12, 5-13(1991).
[5]. R.Devi, H.Maki,&K. Balachandran, Semi-Generalised Closed Maps And Generalized Semi-Closed Maps, Mem. Fac. Sci. Kochi Univ. Ser. A. math.,14,41-54(1993).
[6]. H. Maki, R.devi and K.Balachandran, Associated topologies of generalized α-closed sets and α-generalized closed sets, Mem, Fac. Sci. Kochi Univ.Ser. A . Math. 15(1994),51-63.
[7]. M.Caldas , D.N.Gorgiou and S.Jafari, Characterizations of low seperartion axioms via α-open sets and α-closure operator, Bol.Soc. Paran. Mat.(3s)v;21 (2003), 1-14.
[8]. A.S. Mashhour, I.A. Hasaanein, & S. N. El –Deep, α-continuous and α-open mappings, Acta Math. Hung.,41(3-4)(1983),213-218.
[9]. K.Balachandran, P. Sundaram, and H. Maki, "On Generalized Continuous Maps in Topological Spaces", Mem.Fac.Sci. Kochi Univ. Ser.A (Math.), 12, (1991), 5-13.
[10]. R.L.Prasad & B.L.Pravakar , Pre-Separation Axioms , Acta Ciencia Indica , Vol XXXIV M, No. 1, 191(2008).
[11]. S.N. Maheshwari, and R. Prasad, "Some New Separation Axioms", Ann.Soc.Sci. Bruxelles, Ser.I., 89 (1975), 395-402.
[12]. D. Andrijevic, "Some Properties of the Topology of α-sets", Mat. Vesnik, 36, (1984),1-10.
[13]. A. Kar, and P. Bhattacharyya, "Some Weak Separation Axioms", Bull.Cal.Math.Soc.,82, 415-422, (1990).
[14]. I.L. Reilly, and M.K. Vamanmurthy, "On α-continuity in Topological Spaces", Acta. Math. Hungar., **45(1-2)**, (1985),27-32.
[15]. Dr. Thakur C.K. Raman & Vidyottama kumari; Characteristic behavior of α-open & β-open sets in topological spaces and their analytical study.", Acta Ciencia Indica Vol XXXIX M.No.4, 495-503(2013).
[16]. Dr. Thakur C.K. Raman & Vidyottama kumari; International Journal of Education and Science Research Review, Volume-1, Issue-2 April- 2014