# Numerical Solutions of Eleventh Order Boundary Value Problems Using Piecewise Polynomials

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***Abstract:*** *The aim of this paper is to apply Galerkin weighted residual method for solving eleventh-order BVPs. In this method, we exploit Bernstein and Legendre polynomials as basis functions which are modified into to a new set of basis functions to satisfy the corresponding homogeneous form of boundary conditions where the essential types of boundary conditions are mentioned. The method is formulated as a rigorous matrix form. Examples of both linear and nonlinear BVPs are presented to illustrate the reliability and efficiency of the proposed method. It is observed that the present method is a more effective tool and yields better results.*
***Keywords:*** *Galerkin Method, Eleventh-order Linear and Nonlinear BVPs, Bernstein and Legendre Polynomials.*

## I.     Introduction

Agarwal [1] discussed the theorems stating the conditions for the existence and uniqueness of solutions of high order boundary value problems (BVP) without numerical methods. In the literature of BVPs we observe that the higher order differential equations arise in some branches of applied mathematics, engineering and many other fields of advanced physical sciences [2-3]. Particularly, eighth, tenth and more even higher order BVP arise in hydromagnetic stability analysis [4]. Thus, some researchers have solved high even order BVP numerically; e.g. Siddiqi and Twizell [5] solved eighth order BVPs with spline functions while Wazwaz [6] considered tenth order problems by the modified decomposition method. But few researchers have paid their attention to solve eleventh order BVP. Such as Siddiqi *et al* [7] used Variational iteration technique to obtain numerical approximations for eleventh-order BVPs by converting the original problem into a system of integral equations. Very recently Amjad Hussain *et al* [8] derived the numerical solutions of eleventh-order BVPs using differential transformation method.

In the present paper, the solution of eleventh-order BVP is derived by the Galerkin method [2] with standard (Bernstein and Legendre) polynomials as basis functions. The problem has the   following form:

$$a_{11}\frac{d^{11}u}{dx^{11}}\,a_{10}\frac{d^{10}u}{dx^{10}} + a_9\frac{d^9u}{dx^9} + a_8\frac{d^8u}{dx^8} + a_7\frac{d^7u}{dx^7} + a_6\frac{d^6u}{dx^6} + a_5\frac{d^5u}{dx^5} + a_4\frac{d^4u}{dx^4} + a_3\frac{d^3u}{dx^3} + a_2\frac{d^2u}{dx^2}$$

$$+ a_1\frac{du}{dx} + a_0 u = r\,,\ a < x < b \tag{1a}$$

Subject to the boundary conditions:

$$u(a) = A_0, u(b) = B_0, u'(a) = A_1, u'(b) = B_1, u''(a) = A_2, u''(b) = B_2, u'''(a) = A_3, u'''(b) = B_3,$$

$$u^{(iv)}(a) = A_4, u^{(iv)}(b) = B_4, u^{(v)}(a) = A_5 \tag{1b}$$

where $A_i$, $i = 0,1,2,3,4,5$ and $B_j$, $j = 0,1,2,3,4$ are finite real constants and $a_i$, $i = 0,1,\cdots 11$ and $r$ are all continuous functions defined on the interval [a, b]. The BVP (1) is solved with the boundary conditions of eqn. (1b).

However, in section 2 of this paper, we give a short description on Bernstein and Legendre polynomials. In section 3, the formulation for solving linear eleventh-order BVP by the Galerkin weighted residual method with Bernstein and Legendre polynomials are presented. Numerical examples and results for both linear and nonlinear BVPs are considered to verify the proposed formulation and the solutions are compared with the existing methods in the literature in section 4. Finally the conclusions of the paper are given in the last section.

## II.     Piecewise Polynomials

### 2.1 Bernstein polynomials
The general form of the Bernstein polynomials of *n*th degree over the interval $[a,b]$ is defined by

$$B_{i,n}(x) = \binom{n}{i} \frac{(x-a)^i (b-x)^{n-i}}{(b-a)^n}, \qquad a \le x \le b \qquad\qquad i = 0,1,2,\ldots,n.$$

For example, the first 11 Bernstein polynomials of degree 10 over the interval $[0,\ 1]$ are given bellow:

$$B_0(x) = (1-x)^{10} \qquad\qquad B_4(x) = 210(1-x)^6 x^4 \qquad\qquad B_8(x) = 45(1-x)^2 x^8$$

$$B_1(x) = 10(1-x)^9 x \qquad\qquad B_5(x) = 252(1-x)^5 x^5 \qquad\qquad B_9(x) = 10(1-x)x^9$$

$$B_2(x) = 45(1-x)^8 x^2 \qquad\qquad B_6(x) = 210(1-x)^4 x^6 \qquad\qquad B_{10}(x) = x^{10}$$

$$B_3(x) = 120(1-x)^7 x^3 \qquad\qquad B_7(x) = 120(1-x)^3 x^7$$

Note that each of these $n+1$ polynomials having degree $n$ satisfies the following properties:

(i) $B_{i,n}(x) = 0$ if $i < 0$ or $i > n$.

(ii) $\sum_{i=0}^{n} B_{i,n}(x) = 1$

(iii) $B_{i,n}(a) = B_{i,n}(b) = 0,\ \ i = 1,2,\ldots,n-1$

For these properties, Bernstein polynomials are used in the trail functions satisfying the corresponding homogeneous form of the *essential* boundary conditions in the Galerkin method to solve a BVP.

## 2.2 Legendre Polynomials

The general form of the Legendre polynomials of degree $n$ is defined by

$$P_n(x) = \frac{(-1)^n}{2^n (n!)} \frac{d^n}{dx^n}[(1-x^2)^n], \qquad n \ge 1.$$

Now we modify above Legendre polynomials as

$$p_n(x) = \left[ \frac{1}{n!} \frac{d^n}{dx^n} \left(x^2 - x\right)^n - (-1)^n \right](x-1), \ \ n \ge 1$$

We write first few modified Legendre polynomials over the interval [0, 1]:

$$p_1(x) = 2x(x-1),\ \ p_2(x) = 6x(x-1)^2,\ \ p_3(x) = 2x(x-1)(10x^2 - 15x + 6)$$

$$p_4(x) = 20x - 110x^2 + 230x^3 - 210x^4 + 70x^5$$

$$p_5(x) = -30x + 240x^2 - 770x^3 + 1190x^4 - 882x^5 + 252x^6$$

$$p_6(x) = 42x - 462x^2 + 2100x^3 - 4830x^4 + 5922x^5 - 3696x^6 + 924x^7$$

$$p_7(x) = -56x + 812x^2 - 4956x^3 + 15750x^4 - 28182x^5 + 28644x^6 - 15444x^7 + 3432x^8$$

$$p_8(x) = 72x - 1332x^2 + 10500x^3 - 43890x^4 + 106722x^5 - 156156x^6 + 135564x^7 - 64350x^8$$
$$+ 12870x^9$$

$$p_9(x) = -90x + 2070x^2 - 20460x^3 + 108570x^4 - 342342x^5 + 672672x^6 - 832260x^7$$
$$+ 630630x^8 - 267410x^9 + 48620x^{10}$$

$$p_{10}(x) = 110x - 3080x^2 + 37290x^3 - 244530x^4 + 966966x^5 - 2438436x^6 + 4015440x^7$$
$$- 4302870x^8 + 2892890x^9 - 1108536x^{10} + 184756x^{11}$$

Since the modified Legendre polynomials have special properties at $x = 0$ and $x = 1$: $p_n(0) = 0$ and $p_n(1) = 0,\ n \ge 1$ respectively, so that they can be used as set of basis function to satisfy the corresponding homogeneous form of the *Dirichlet* boundary conditions to derive the matrix formulation of fourth order BVP over the interval [0, 1].

## III.        Matrix Formulation of Eleventh-order BVP

In this section we first derived the matrix formulation for eleventh-order linear BVP and then we extend our idea for solving nonlinear BVP. To solve the boundary value problem (1) by the Galerkin method we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x) \tag{3}$$

Here $\theta_0(x)$ is specified by the essential boundary conditions and $N_{i,n}(a) = N_{i,n}(b) = 0$ for each $i = 1, 2, \ldots n$.
Using eqn. (3) into eqn. (1), the weighted residual equations are

$$\int_a^b \left[ a_{11} \frac{d^{11}u}{dx^{11}} a_{10} \frac{d^{10}u}{dx^{10}} + a_9 \frac{d^9u}{dx^9} + a_8 \frac{d^8u}{dx^8} + a_7 \frac{d^7u}{dx^7} + a_6 \frac{d^6u}{dx^6} + a_5 \frac{d^5u}{dx^5} + a_4 \frac{d^4u}{dx^4} + a_3 \frac{d^3u}{dx^3} + a_2 \frac{d^2u}{dx^2} \right.$$

$$\left. + a_1 \frac{du}{dx} + a_0 u - r \right] N_{j,n}(x)dx = 0 \qquad (4)$$

Integrating by parts the terms up to second derivative on the left hand side of (4), we get

$$\int_a^b a_{11} \frac{d^{11}\tilde{u}}{dx^{11}} N_{j,n}(x)dx = -\left[ \frac{d}{dx}[a_{11}N_{j,n}(x)]\frac{d^9\tilde{u}}{dx^9} \right]_a^b + \left[ \frac{d^2}{dx^2}[a_{11}N_{j,n}(x)]\frac{d^8\tilde{u}}{dx^8} \right]_a^b - \left[ \frac{d^3}{dx^3}[a_{11}N_{j,n}(x)]\frac{d^7\tilde{u}}{dx^7} \right]_a^b$$

$$+ \left[ \frac{d^4}{dx^4}[a_{11}N_{j,n}(x)]\frac{d^6\tilde{u}}{dx^6} \right]_a^b - \left[ \frac{d^5}{dx^5}[a_{11}N_{j,n}(x)]\frac{d^5\tilde{u}}{dx^5} \right]_a^b + \left[ \frac{d^6}{dx^6}[c_{11}N_{j,n}(x)]\frac{d^4\tilde{u}}{dx^4} \right]_a^b$$

$$- \left[ \frac{d^7}{dx^7}[a_{11}N_{j,n}(x)]\frac{d^3\tilde{u}}{dx^3} \right]_a^b + \left[ \frac{d^8}{dx^8}[a_{11}N_{j,n}(x)]\frac{d^2\tilde{u}}{dx^2} \right]_a^b - \left[ \frac{d^9}{dx^9}[a_{11}N_{j,n}(x)]\frac{d\tilde{u}}{dx} \right]_a^b$$

$$+ \int_a^b \frac{d^{10}}{dx^{10}}[a_{11}N_{j,n}(x)]\frac{d\tilde{u}}{dx} dx \qquad (5)$$

$$\int_a^b a_{10} \frac{d^{10}\tilde{u}}{dx^{10}} N_{j,n}(x)dx = -\left[ \frac{d}{dx}[a_{10}N_{j,n}(x)]\frac{d^8\tilde{u}}{dx^8} \right]_a^b + \left[ \frac{d^2}{dx^2}[a_{10}N_{j,n}(x)]\frac{d^7\tilde{u}}{dx^7} \right]_a^b - \left[ \frac{d^3}{dx^3}[a_{10}N_{j,n}(x)]\frac{d^6\tilde{u}}{dx^6} \right]_a^b$$

$$+ \left[ \frac{d^4}{dx^4}[a_{10}N_{j,n}(x)]\frac{d^5\tilde{u}}{dx^5} \right]_a^b - \left[ \frac{d^5}{dx^5}[a_{10}N_{j,n}(x)]\frac{d^4\tilde{u}}{dx^4} \right]_a^b + \left[ \frac{d^6}{dx^6}[a_{10}N_{j,n}(x)]\frac{d^3\tilde{u}}{dx^3} \right]_a^b$$

$$- \left[ \frac{d^7}{dx^7}[a_{10}N_{j,n}(x)]\frac{d^2\tilde{u}}{dx^2} \right]_a^b + \left[ \frac{d^8}{dx^8}[a_{10}N_{j,n}(x)]\frac{d\tilde{u}}{dx} \right]_a^b - \int_a^b \frac{d^9}{dx^9}[a_{10}N_{j,n}(x)]\frac{d\tilde{u}}{dx} dx \quad (6)$$

$$\int_a^b a_9 \frac{d^9\tilde{u}}{dx^9} N_{j,n}(x)dx = -\left[ \frac{d}{dx}[a_9N_{j,n}(x)]\frac{d^7\tilde{u}}{dx^7} \right]_a^b + \left[ \frac{d^2}{dx^2}[a_9N_{j,n}(x)]\frac{d^6\tilde{u}}{dx^6} \right]_a^b - \left[ \frac{d^3}{dx^3}[a_9N_{j,n}(x)]\frac{d^5\tilde{u}}{dx^5} \right]_a^b$$

$$+ \left[ \frac{d^4}{dx^4}[a_9N_{j,n}(x)]\frac{d^4\tilde{u}}{dx^4} \right]_a^b - \left[ \frac{d^5}{dx^5}[a_9N_{j,n}(x)]\frac{d^3\tilde{u}}{dx^3} \right]_a^b + \left[ \frac{d^6}{dx^6}[a_9N_{j,n}(x)]\frac{d^2\tilde{u}}{dx^2} \right]_a^b$$

$$- \left[ \frac{d^7}{dx^7}[a_9N_{j,n}(x)]\frac{d\tilde{u}}{dx} \right]_a^b + \int_a^b \frac{d^8}{dx^8}[a_9N_{j,n}(x)]\frac{d\tilde{u}}{dx} dx \qquad (7)$$

$$\int_a^b a_8 \frac{d^8\tilde{u}}{dx^8} N_{j,n}(x)dx = -\left[ \frac{d}{dx}[a_8N_{j,n}(x)]\frac{d^6\tilde{u}}{dx^6} \right]_a^b + \left[ \frac{d^2}{dx^2}[a_8N_{j,n}(x)]\frac{d^5\tilde{u}}{dx^5} \right]_a^b - \left[ \frac{d^3}{dx^3}[a_8N_{j,n}(x)]\frac{d^4\tilde{u}}{dx^4} \right]_a^b$$

$$+ \left[ \frac{d^4}{dx^4}[a_8N_{j,n}(x)]\frac{d^3\tilde{u}}{dx^3} \right]_a^b - \left[ \frac{d^5}{dx^5}[a_8N_{j,n}(x)]\frac{d^2\tilde{u}}{dx^2} \right]_a^b + \left[ \frac{d^6}{dx^6}[a_8N_{j,n}(x)]\frac{d\tilde{u}}{dx} \right]_a^b$$

$$- \int_a^b \frac{d^7}{dx^7}[a_8N_{j,n}(x)]\frac{d\tilde{u}}{dx} dx \qquad (8)$$

$$\int_a^b a_7 \frac{d^7\tilde{u}}{dx^7} N_{j,n}(x)dx = -\left[ \frac{d}{dx}[a_7N_{j,n}(x)]\frac{d^5\tilde{u}}{dx^5} \right]_a^b + \left[ \frac{d^2}{dx^2}[a_7N_{j,n}(x)]\frac{d^4\tilde{u}}{dx^4} \right]_a^b - \left[ \frac{d^3}{dx^3}[a_7N_{j,n}(x)]\frac{d^3\tilde{u}}{dx^3} \right]_a^b$$

$$+ \left[ \frac{d^4}{dx^4}[a_7N_{j,n}(x)]\frac{d^2\tilde{u}}{dx^2} \right]_a^b - \left[ \frac{d^5}{dx^5}[a_7N_{j,n}(x)]\frac{d\tilde{u}}{dx} \right]_a^b + \int_a^b \frac{d^6}{dx^6}[a_7N_{j,n}(x)]\frac{d\tilde{u}}{dx} dx \,(9)$$

$$\int_a^b a_6 \frac{d^6\tilde{u}}{dx^6} N_{j,n}(x)dx = -\left[\frac{d}{dx}\left[a_6 N_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_a^b + \left[\frac{d^2}{dx^2}\left[a_6 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_a^b - \left[\frac{d^3}{dx^3}\left[a_6 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_a^b$$

$$+\left[\frac{d^4}{dx^4}\left[a_6 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_a^b - \int_a^b \frac{d^5}{dx^5}\left[a_6 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \tag{10}$$

$$\int_a^b a_5 \frac{d^5\tilde{u}}{dx^5} N_{j,n}(x)dx = -\left[\frac{d}{dx}\left[a_5 N_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_a^b + \left[\frac{d^2}{dx^2}\left[a_5 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_a^b - \left[\frac{d^3}{dx^3}\left[a_5 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_a^b$$

$$+\int_a^b \frac{d^4}{dx^4}\left[a_5 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \tag{11}$$

$$\int_a^b a_4 \frac{d^4\tilde{u}}{dx^4} N_{j,n}(x)dx = -\left[\frac{d}{dx}\left[a_4 N_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_a^b + \left[\frac{d^2}{dx^2}\left[a_4 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_a^b - \int_a^b \frac{d^3}{dx^3}\left[a_4 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \tag{12}$$

$$\int_a^b a_3 \frac{d^3\tilde{u}}{dx^3} N_{j,n}(x)dx == -\left[\frac{d}{dx}\left[a_3 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_a^b + \int_a^b \frac{d^2}{dx^2}\left[a_3 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \tag{13}$$

$$\int_a^b a_2 \frac{d^2\tilde{u}}{dx^2} N_{j,n}(x)dx = -\int_a^b \frac{d}{dx}\left[a_2 N_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \tag{14}$$

Putting eqns. (5) – (14) into eqn. (4) and using approximation for $\tilde{u}(x)$ given in eqn. (3) and after applying the boundary conditions given in eqn. (1b) and rearranging the terms for the resulting equations we get a system of equations in matrix form as

$$\sum_{i=1}^n D_{i,j}\,\alpha_i = F_j,\, j=1,2,\ldots,n \tag{15a}$$

where

$$D_{i,j} = \int_a^b \left\{\left[\frac{d^{10}}{dx^{10}}\left[a_{11}N_{j,n}(x)\right] - \frac{d^9}{dx^9}\left[a_{10}N_{j,n}(x)\right] + \frac{d^8}{dx^8}\left[a_9 N_{j,n}(x)\right] - \frac{d^7}{dx^7}\left[a_8 N_{j,n}(x)\right] + \frac{d^6}{dx^6}\left[a_7 N_{j,n}(x)\right]\right.\right.$$

$$+\frac{d^2}{dx^2}\left[a_3 N_{j,n}(x)\right] - \frac{d}{dx}\left[a_2 N_{j,n}(x)\right] - \frac{d^5}{dx^5}\left[a_6 N_{j,n}(x)\right] + \frac{d^4}{dx^4}\left[a_5 N_{j,n}(x)\right] - \frac{d^3}{dx^3}\left[a_4 N_{j,n}(x)\right]$$

$$+ a_1 N_{j,n}(x)\left.\right]\frac{d}{dx}\left[N_{i,n}(x)\right] + a_0 N_{i,n}(x)N_{j,n}(x)\left.\right\}dx - \left[\frac{d}{dx}\left[a_{11}N_{j,n}(x)\right]\frac{d^9}{dx^9}\left[N_{i,n}(x)\right]\right]_{x=b}$$

$$+ \left[\frac{d}{dx}\left[a_{11}N_{j,n}(x)\right]\frac{d^9}{dx^9}\left[N_{i,n}(x)\right]\right]_{x=a} + \left[\frac{d^2}{dx^2}\left[a_{11}N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=b}$$

$$- \left[\frac{d^2}{dx^2}\left[a_{11}N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=a} - \left[\frac{d^3}{dx^3}\left[a_{11}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=b}$$

$$+ \left[\frac{d^3}{dx^3}\left[a_{11}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=a} + \left[\frac{d^4}{dx^4}\left[a_{11}N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=b}$$

$$- \left[\frac{d^4}{dx^4}\left[a_{11}N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=a} - \left[\frac{d^5}{dx^5}\left[a_{11}N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=b}$$

$$- \left[\frac{d}{dx}\left[a_{10}N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=b} + \left[\frac{d}{dx}\left[a_{10}N_{j,n}(x)\right]\frac{d^8}{dx^8}\left[N_{i,n}(x)\right]\right]_{x=a}$$

$$+ \left[\frac{d^2}{dx^2}\left[a_{10}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=b} - \left[\frac{d^2}{dx^2}\left[a_{10}N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=a}$$

$$-\left[\frac{d^3}{dx^3}\left[a_{10}N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=b}+\left[\frac{d^3}{dx^3}\left[a_{10}N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=a}$$

$$+\left[\frac{d^4}{dx^4}\left[a_{10}N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=b}-\left[\frac{d}{dx}\left[a_9N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=b}$$

$$+\left[\frac{d}{dx}\left[a_9N_{j,n}(x)\right]\frac{d^7}{dx^7}\left[N_{i,n}(x)\right]\right]_{x=b}+\left[\frac{d^2}{dx^2}\left[a_9N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=b}$$

$$-\left[\frac{d^2}{dx^2}\left[a_9N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=a}-\left[\frac{d^3}{dx^3}\left[a_9N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=b}$$

$$-\left[\frac{d}{dx}\left[a_8N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=b}+\left[\frac{d}{dx}\left[a_8N_{j,n}(x)\right]\frac{d^6}{dx^6}\left[N_{i,n}(x)\right]\right]_{x=a}$$

$$+\left[\frac{d^2}{dx^2}\left[a_8N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=b}-\left[\frac{d}{dx}\left[a_7N_{j,n}(x)\right]\frac{d^5}{dx^5}\left[N_{i,n}(x)\right]\right]_{x=b}\qquad(15b)$$

$$F_j=\int_a^b\left\{sN_{j,n}(x)+\left[-\frac{d^{10}}{dx^{10}}\left[a_{11}N_{j,n}(x)\right]+\frac{d^9}{dx^9}\left[a_{10}N_{j,n}(x)\right]-\frac{d^8}{dx^8}\left[a_9N_{j,n}(x)\right]+\frac{d^7}{dx^7}\left[a_8N_{j,n}(x)\right]\right.\right.$$

$$-\frac{d^6}{dx^6}\left[a_7N_{j,n}(x)\right]+\frac{d^5}{dx^5}\left[a_6N_{j,n}(x)\right]-\frac{d^4}{dx^4}\left[a_5N_{j,n}(x)\right]+\frac{d^3}{dx^3}\left[a_4N_{j,n}(x)\right]+\frac{d^2}{dx^2}\left[a_3N_{j,n}(x)\right]$$

$$\left.+\frac{d}{dx}\left[a_2N_{j,n}(x)\right]-a_1N_{j,n}(x)\right]\frac{d\theta_0}{dx}-a_0\theta_0N_{j,n}(x)\right\}dx+\left[\frac{d}{dx}\left[a_{11}N_{j,n}(x)\right]\frac{d^9\theta_0}{dx^9}\right]_{x=b}$$

$$-\left[\frac{d}{dx}\left[a_{11}N_{j,n}(x)\right]\frac{d^9\theta_0}{dx^9}\right]_{x=0}-\left[\frac{d^2}{dx^2}\left[a_{11}N_{j,n}(x)\right]\frac{d^8\theta_0}{dx^8}\right]_{x=b}+\left[\frac{d^2}{dx^2}\left[a_{11}N_{j,n}(x)\right]\frac{d^8\theta_0}{dx^8}\right]_{x=a}$$

$$+\left[\frac{d^3}{dx^3}\left[a_{11}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=b}-\left[\frac{d^3}{dx^3}\left[a_{11}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=a}-\left[\frac{d^4}{dx^4}\left[a_{11}N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=b}$$

$$+\left[\frac{d^4}{dx^4}\left[a_{11}N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=a}+\left[\frac{d^5}{dx^5}\left[a_{11}N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=b}+\left[\frac{d}{dx}\left[a_{10}N_{j,n}(x)\right]\frac{d^8\theta_0}{dx^8}\right]_{x=b}$$

$$-\left[\frac{d}{dx}\left[a_{10}N_{j,n}(x)\right]\frac{d^8\theta_0}{dx^8}\right]_{x=a}-\left[\frac{d^2}{dx^2}\left[a_{10}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=b}+\left[\frac{d^2}{dx^2}\left[a_{10}N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=a}$$

$$+\left[\frac{d^3}{dx^3}\left[a_{10}N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=b}-\left[\frac{d^3}{dx^3}\left[a_{10}N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=a}-\left[\frac{d^4}{dx^4}\left[a_{10}N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=b}$$

$$+\left[\frac{d}{dx}\left[a_9N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=b}-\left[\frac{d}{dx}\left[a_9N_{j,n}(x)\right]\frac{d^7\theta_0}{dx^7}\right]_{x=a}-\left[\frac{d^2}{dx^2}\left[a_9N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=b}$$

$$+\left[\frac{d^2}{dx^2}\left[a_9N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=a}+\left[\frac{d^3}{dx^3}\left[a_9N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=b}+\left[\frac{d}{dx}\left[a_8N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=b}$$

$$-\left[\frac{d}{dx}\left[a_8N_{j,n}(x)\right]\frac{d^6\theta_0}{dx^6}\right]_{x=a}-\left[\frac{d^2}{dx^2}\left[a_8N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=b}+\left[\frac{d}{dx}\left[a_7N_{j,n}(x)\right]\frac{d^5\theta_0}{dx^5}\right]_{x=b}$$

$$-\left[\frac{d^5}{dx^5}\left[a_{11}N_{j,n}(x)\right]\right]_{x=a}\times A_5-\left[\frac{d^6}{dx^6}\left[a_{11}N_{j,n}(x)\right]\right]_{x=b}\times B_4+\left[\frac{d^6}{dx^6}\left[a_{11}N_{j,n}(x)\right]\right]_{x=a}\times A_4$$

$$+\left[\frac{d^7}{dx^7}\left[a_{11}N_{j,n}(x)\right]\right]_{x=b}\times B_3-\left[\frac{d^7}{dx^7}\left[a_{11}N_{j,n}(x)\right]\right]_{x=a}\times A_3-\left[\frac{d^8}{dx^8}\left[a_{11}N_{j,n}(x)\right]\right]_{x=b}\times B_2$$

$$+\left[\frac{d^8}{dx^8}\left[a_{11}N_{j,n}(x)\right]\right]_{x=a}\times A_2+\left[\frac{d^9}{dx^9}\left[a_{11}N_{j,n}(x)\right]\right]_{x=b}\times B_1-\left[\frac{d^9}{dx^9}\left[a_{11}N_{j,n}(x)\right]\right]_{x=a}\times A_1$$

$$+\left[\frac{d^4}{dx^4}\left[a_{10}N_{j,n}(x)\right]\right]_{x=a}\times A_5+\left[\frac{d^5}{dx^5}\left[a_{10}N_{j,n}(x)\right]\right]_{x=b}\times B_4-\left[\frac{d^5}{dx^5}\left[a_{10}N_{j,n}(x)\right]\right]_{x=a}\times A_4$$

$$-\left[\frac{d^6}{dx^6}\left[a_{10}N_{j,n}(x)\right]\right]_{x=b}\times B_3+\left[\frac{d^6}{dx^6}\left[a_{10}N_{j,n}(x)\right]\right]_{x=a}\times A_3+\left[\frac{d^7}{dx^7}\left[a_{10}N_{j,n}(x)\right]\right]_{x=b}\times B_2$$

$$-\left[\frac{d^7}{dx^7}\left[a_{10}N_{j,n}(x)\right]\right]_{x=a}\times A_2-\left[\frac{d^8}{dx^8}\left[a_{10}N_{j,n}(x)\right]\right]_{x=b}\times B_1+\left[\frac{d^8}{dx^8}\left[a_{10}N_{j,n}(x)\right]\right]_{x=a}\times A_1$$

$$-\left[\frac{d^3}{dx^3}\left[a_9N_{j,n}(x)\right]\right]_{x=a}\times A_5-\left[\frac{d^4}{dx^4}\left[a_9N_{j,n}(x)\right]\right]_{x=b}\times B_4+\left[\frac{d^4}{dx^4}\left[a_9N_{j,n}(x)\right]\right]_{x=a}\times A_4$$

$$+\left[\frac{d^5}{dx^5}\left[a_9N_{j,n}(x)\right]\right]_{x=b}\times B_3-\left[\frac{d^5}{dx^5}\left[a_9N_{j,n}(x)\right]\right]_{x=a}\times A_3-\left[\frac{d^6}{dx^6}\left[a_9N_{j,n}(x)\right]\right]_{x=b}\times B_2$$

$$+\left[\frac{d^6}{dx^6}\left[a_9N_{j,n}(x)\right]\right]_{x=a}\times A_2+\left[\frac{d^7}{dx^7}\left[a_9N_{j,n}(x)\right]\right]_{x=b}\times B_1-\left[\frac{d^7}{dx^7}\left[a_9N_{j,n}(x)\right]\right]_{x=a}\times A_1$$

$$-\left[\frac{d^2}{dx^2}\left[a_8N_{j,n}(x)\right]\right]_{x=a}\times A_5+\left[\frac{d^3}{dx^3}\left[a_8N_{j,n}(x)\right]\right]_{x=b}\times B_4-\left[\frac{d^3}{dx^3}\left[a_8N_{j,n}(x)\right]\right]_{x=a}\times A_4$$

$$-\left[\frac{d^4}{dx^4}\left[a_8N_{j,n}(x)\right]\right]_{x=b}\times B_3+\left[\frac{d^4}{dx^4}\left[a_8N_{j,n}(x)\right]\right]_{x=a}\times A_3+\left[\frac{d^5}{dx^5}\left[a_8N_{j,n}(x)\right]\right]_{x=b}\times B_2$$

$$-\left[\frac{d^5}{dx^5}\left[a_8N_{j,n}(x)\right]\right]_{x=a}\times A_2-\left[\frac{d^6}{dx^6}\left[a_8N_{j,n}(x)\right]\right]_{x=b}\times B_1+\left[\frac{d^6}{dx^6}\left[a_8N_{j,n}(x)\right]\right]_{x=a}\times A_1$$

$$-\left[\frac{d}{dx}\left[a_7N_{j,n}(x)\right]\right]_{x=a}\times A_5-\left[\frac{d^2}{dx^2}\left[a_7N_{j,n}(x)\right]\right]_{x=b}\times B_4+\left[\frac{d^2}{dx^2}\left[a_7N_{j,n}(x)\right]\right]_{x=a}\times A_4$$

$$+\left[\frac{d^3}{dx^3}\left[a_7N_{j,n}(x)\right]\right]_{x=b}\times B_3-\left[\frac{d^3}{dx^3}\left[a_7N_{j,n}(x)\right]\right]_{x=a}\times A_3-\left[\frac{d^4}{dx^4}\left[a_7N_{j,n}(x)\right]\right]_{x=b}\times B_2$$

$$+\left[\frac{d^4}{dx^4}\left[a_7N_{j,n}(x)\right]\right]_{x=a}\times A_2+\left[\frac{d^5}{dx^5}\left[a_7N_{j,n}(x)\right]\right]_{x=b}\times B_1-\left[\frac{d^5}{dx^5}\left[a_7N_{j,n}(x)\right]\right]_{x=a}\times A_1$$

$$+\left[\frac{d}{dx}\left[a_6N_{j,n}(x)\right]\right]_{x=b}\times B_4-\left[\frac{d}{dx}\left[a_6N_{j,n}(x)\right]\right]_{x=a}\times A_4-\left[\frac{d^2}{dx^2}\left[a_6N_{j,n}(x)\right]\right]_{x=b}\times B_3$$

$$+\left[\frac{d^2}{dx^2}\left[a_6N_{j,n}(x)\right]\right]_{x=a}\times A_3+\left[\frac{d^3}{dx^3}\left[a_6N_{j,n}(x)\right]\right]_{x=b}\times B_2-\left[\frac{d^3}{dx^3}\left[a_6N_{j,n}(x)\right]\right]_{x=a}\times A_2$$

$$-\left[\frac{d^4}{dx^4}\left[a_6N_{j,n}(x)\right]\right]_{x=b}\times B_1+\left[\frac{d^4}{dx^4}\left[a_6N_{j,n}(x)\right]\right]_{x=a}\times A_1+\left[\frac{d}{dx}\left[a_5N_{j,n}(x)\right]\right]_{x=b}\times B_3$$

$$-\left[\frac{d}{dx}\left[a_5N_{j,n}(x)\right]\right]_{x=a}\times A_3-\left[\frac{d^2}{dx^2}\left[a_5N_{j,n}(x)\right]\right]_{x=b}\times B_2+\left[\frac{d^2}{dx^2}\left[a_5N_{j,n}(x)\right]\right]_{x=a}\times A_2$$

$$+\left[\frac{d^3}{dx^3}\left[a_5N_{j,n}(x)\right]\right]_{x=b}\times B_1-\left[\frac{d^3}{dx^3}\left[a_5N_{j,n}(x)\right]\right]_{x=a}\times A_1+\left[\frac{d}{dx}\left[a_4N_{j,n}(x)\right]\right]_{x=b}\times B_2$$

$$-\left[\frac{d}{dx}\left[a_4N_{j,n}(x)\right]\right]_{x=a}\times A_2-\left[\frac{d^2}{dx^2}\left[a_4N_{j,n}(x)\right]\right]_{x=b}\times B_1+\left[\frac{d^2}{dx^2}\left[a_4N_{j,n}(x)\right]\right]_{x=a}\times A_1$$

$$+\left[\frac{d}{dx}\left[a_3 N_{j,n}(x)\right]\right]_{x=b} \times B_1 - \left[\frac{d}{dx}\left[a_3 N_{j,n}(x)\right]\right]_{x=a} \times A_1 \qquad (15c)$$

Solving the system (15a), we find the values of the parameters $\alpha_i$ and then substituting these parameters into eqn. (3), we get the approximate solution of the BVP (1). For nonlinear eleventh-order BVP, we first compute the initial values on neglecting the nonlinear terms and using the system (15). Then using the Newton's iterative method we find the numerical approximations for desired nonlinear BVP. This formulation is described through the numerical examples in the next section.

## IV. Numerical Examples And Results

To test the applicability of the proposed method, we consider two linear and one nonlinear problem which are available in the existing literature. For all examples, the solutions obtained by the proposed method are compared with the exact solutions. All the calculations are performed by **MATLAB 10**. The convergence of linear BVP is calculated by

$$E = \left|\tilde{u}_{n+1}(x) - \tilde{u}_n(x)\right| < 10^{-13}$$

where $\tilde{u}_n(x)$ denotes the approximate solution using $n$-th polynomials while the convergence of nonlinear BVP is calculated by the absolute error of two consecutive iterations such that

$$\left|\tilde{u}_n^{N+1} - \tilde{u}_n^N\right| < 10^{-12}$$

where $N$ is the Newton's iteration number.

**Example 1**: Consider the linear differential equation of eleventh-order [7, 8]

$$\frac{d^{11}u}{dx^{11}} - u = -22(5+x)e^x, \quad 0 \le x \le 1 \qquad (16a)$$

subject to boundary conditions:

$$u(0)=1,\ u(1)=0,\ u'(0)=1, u'(1)=-2e, u''(0)=-1, u''(1)=-6e, u'''(0)=-5, u'''(1)=-12e, u^{(iv)}(0)=-11,$$
$$u^{(iv)}(1)=-20e, u^{(v)}(0)=-19 \qquad (16b)$$

The analytic solution of the above problem is $u(x)=(1-x^2)e^x$. The maximum absolute errors by the present method are summarized in Table 1. On the other hand, it is observed that the accuracy is found nearly the order $10^{-12}$ in [7] by Siddiqi *et al* and nearly the order $10^{-13}$ in [8] by Amjad Hussain *et al*. Now the exact and approximate solutions are depicted in Fig. 1of example 1 for $n=14$.

Table 1: Maximum Absolute Errors for the Example 1

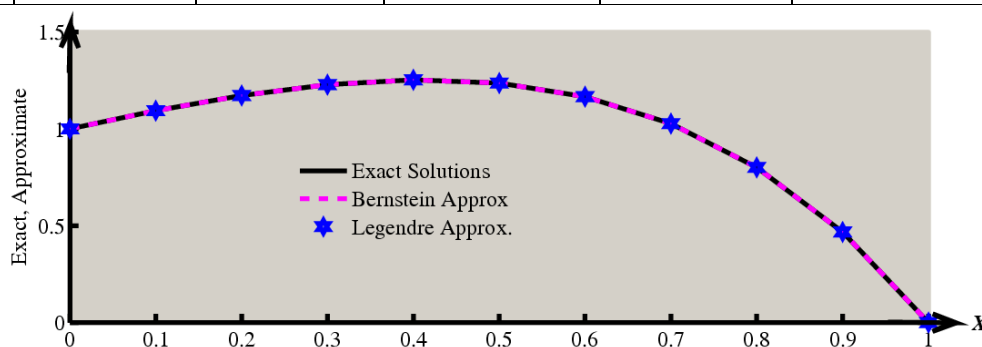| $x$ | Exact Results | 14, Bernstein Polynomials | | 14, Legendre Polynomials | |
|---|---|---|---|---|---|
| | | Approximate | Abs. Error | Approximate | Abs. Error |
| 0.0 | 1.0000000000 | 1.0000000000 | 0.0000000E+000 | 1.0000000000 | 0.0000000E+000 |
| 0.1 | 1.0941192089 | 1.0941192089 | 1.3322676E-015 | 1.0941192089 | 2.4424907E-015 |
| 0.2 | 1.1725466478 | 1.1725466478 | 4.4408921E-016 | 1.1725466478 | 1.9984014E-015 |
| 0.3 | 1.2283715149 | 1.2283715149 | 1.1102230E-015 | 1.2283715149 | 3.3306691E-015 |
| 0.4 | 1.2531327460 | 1.2531327460 | 6.6613381E-016 | 1.2531327460 | 5.1070259E-015 |
| 0.5 | 1.2365409530 | 1.2365409530 | 4.4408921E-016 | 1.2365409530 | 8.8817842E-016 |
| 0.6 | 1.1661560322 | 1.1661560322 | 8.8817842E-016 | 1.1661560322 | 4.8849813E-015 |
| 0.7 | 1.0270138808 | 1.0270138808 | 2.2204460E-016 | 1.0270138808 | 6.2172489E-015 |
| 0.8 | 0.8011947343 | 0.8011947343 | 9.9920072E-016 | 0.8011947343 | 2.1094237E-015 |
| 0.9 | 0.4673245911 | 0.4673245911 | 5.5511151E-017 | 0.4673245911 | 3.1641356E-015 |
| 1.0 | 0.0000000000 | 0.0000000000 | 0.0000000E+000 | 0.0000000000 | 0.0000000E+000 |



Fig. 1: Graphical representation of exact and approximate solutions of example 1.

**Example 2:** Consider the linear BVP [7, 8]

$$\frac{d^{11}u}{dx^{11}} + u = 22(5\sin x + x\cos x) + (1-x^2)(\sin x + \cos x), \quad 0 \le x \le 1 \tag{17a}$$

subject to boundary conditions:

$$u(0) = 1, u(1) = 0, u'(0) = 0, u'(1) = -2\cos 1, u''(0) = -3, u''(1) = 4\sin 1 - 2\cos 1, u'''(0) = 0, u'''(1) = 6\sin 1 + 6\cos 1,$$

$$u^{(iv)}(0) = 13, u^{(iv)}(1) = -8\sin 1 + 12\cos 1, u^{(v)}(0) = 0 . \tag{17b}$$

The analytic solution of the above problem is $u(x) = (1-x^2)\cos x$. The maximum absolute errors, shown in Table 2, are listed to compare with existing results. The maximum absolute errors were found by Siddiqi *et al* [7] is $5.148 \times 10^{-12}$ and by Amjad Hussain *et al* [8] is $9.560 \times 10^{-13}$. We have shown the exact and approximate solutions in Fig. 2 of example 2 for $n = 14$.

Table 2: Maximum Absolute Errors for the Example 2

| $x$ | Exact Results | 14, Bernstein Polynomials | | 14, Legendre Polynomials | |
|---|---|---|---|---|---|
| | | Approximate | Abs. Error | Approximate | Abs. Error |
| 0.0 | 1.0000000000 | 1.0000000000 | 0.0000000E+000 | 1.0000000000 | 0.0000000E+000 |
| 0.1 | 0.9850541236 | 0.9850541236 | 5.5511151E-016 | 0.9850541236 | 5.5511151E-016 |
| 0.2 | 0.9408639147 | 0.9408639147 | 4.4408921E-016 | 0.9408639147 | 5.5511151E-016 |
| 0.3 | 0.8693562051 | 0.8693562051 | 5.5511151E-016 | 0.8693562051 | 8.8817842E-016 |
| 0.4 | 0.7736912350 | 0.7736912350 | 1.1102230E-016 | 0.7736912350 | 1.3322676E-015 |
| 0.5 | 0.6581869214 | 0.6581869214 | 4.4408921E-016 | 0.6581869214 | 0.0000000E+000 |
| 0.6 | 0.5282147935 | 0.5282147935 | 5.5511151E-016 | 0.5282147935 | 1.3322676E-015 |
| 0.7 | 0.3900695155 | 0.3900695155 | 5.5511151E-017 | 0.3900695155 | 1.2767565E-015 |
| 0.8 | 0.2508144154 | 0.2508144154 | 2.7755576E-016 | 0.2508144154 | 1.6653345E-016 |
| 0.9 | 0.1181058940 | 0.1181058940 | 8.3266727E-017 | 0.1181058940 | 4.7184479E-016 |
| 1.0 | 0.0000000000 | 0.0000000000 | 0.0000000E+000 | 0.0000000000 | 0.0000000E+000 |

**Example 3:** Consider the following **nonlinear** differential equation [7]

$$\frac{d^{11}u}{dx^{11}} = 11(\cos x - \sin x) - x(\cos x + \sin x) - u^2 + x^2(1 - 2\sin x\cos x), 0 \le x \le 1 \tag{18a}$$

subject to boundary conditions:

$$u(0) = 0, u(1) = \sin 1 - \cos 1, u'(0) = -1, u'(1) = 2\sin 1, u''(0) = 2, u''(1) = \sin 1 + 3\cos 1, u'''(0) = 3,$$

$$u'''(1) = -4\sin 1 + 2\cos 1, u^{(iv)}(0) = -4, u^{(iv)}(1) = -3\sin 1 - 5\cos 1, u^{(v)}(0) = -5 \tag{18b}$$

The exact solution of this BVP is $u(x) = x(\sin x - \cos x)$. Consider the approximate solution of $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i N_{i,n}(x), \ n \ge 1 \tag{19}$$

Here $\theta_0(x) = 1 + x(\sin 1 - \cos 1 - 1)$ is specified by the essential boundary conditions in eqn. (18b). Also $N_{i,n}(0) = N_{i,n}(1) = 0$ for each $i = 1, 2, \ldots, n$. Putting eqn. (19) into eqn. (18a), the Galerkin weighted residual equations are

$$\int_0^1 \left[ \frac{d^{11}\tilde{u}}{dx^{11}} + \tilde{u}^2 - 11(\cos x - \sin x) + x(\cos x + \sin x) - x^2(1 - 2\sin x\cos x) \right] N_{k,n}(x)dx = 0 \tag{20}$$

Integrating $1^{st}$ term of (20) by parts, we obtain

$$\int_0^1 \frac{d^{11}\tilde{u}}{dx^{11}} N_{k,n}(x) = -\left[ \frac{dN_{k,n}(x)}{dx} \frac{d^9\tilde{u}}{dx^9} \right]_0^1 + \left[ \frac{d^2N_{k,n}(x)}{dx^2} \frac{d^8\tilde{u}}{dx^8} \right]_0^1 - \left[ \frac{d^3N_{k,n}(x)}{dx^3} \frac{d^7\tilde{u}}{dx^7} \right]_0^1 + \left[ \frac{d^4N_{k,n}(x)}{dx^4} \frac{d^6\tilde{u}}{dx^6} \right]_0^1$$

$$- \left[ \frac{d^5N_{k,n}(x)}{dx^5} \frac{d^5\tilde{u}}{dx^5} \right]_0^1 + \left[ \frac{d^6N_{k,n}(x)}{dx^6} \frac{d^4\tilde{u}}{dx^4} \right]_0^1 - \left[ \frac{d^7N_{k,n}(x)}{dx^7} \frac{d^3\tilde{u}}{dx^3} \right]_0^1 + \left[ \frac{d^8N_{k,n}(x)}{dx^8} \frac{d^2\tilde{u}}{dx^2} \right]_0^1$$

$$- \left[ \frac{d^9N_{k,n}(x)}{dx^9} \frac{d\tilde{u}}{dx} \right]_0^1 + \int_0^1 \frac{d^{10}N_{k,n}(x)}{dx^{10}} \frac{d\tilde{u}}{dx} dx \tag{21}$$

Using eqn. (21) into eqn. (20) and using approximation for $\tilde{u}(x)$ given in eqn. (19) and after applying the conditions given in eqn. (18b) and rearranging the terms for the resulting equations we obtain

$$\sum_{i=1}^{n}\left[\int_0^1\left[\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{dN_{i,n}(x)}{dx}+2\theta_0 N_{i,n}(x)N_{k,n}(x)+\sum_{j=1}^{n}\alpha_j\int_0^1(N_{i,n}(x)N_{j,n}(x)N_{k,n}(x))\right]\right]dx-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^9N_{i,n}(x)}{dx^9}\right]_{x=1}$$

$$+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^9N_{i,n}(x)}{dx^9}\right]_{x=0}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^8N_{i,n}(x)}{dx^8}\right]_{x=1}-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^8N_{i,n}(x)}{dx^8}\right]_{x=0}-\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^7N_{i,n}(x)}{dx^7}\right]_{x=1}$$

$$+\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^7N_{i,n}(x)}{dx^7}\right]_{x=0}+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^6N_{i,n}(x)}{dx^6}\right]_{x=1}-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^6N_{i,n}(x)}{dx^6}\right]_{x=0}-\left[\frac{d^5N_{k,n}(x)}{dx^5}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=1}\bigg]\alpha_i$$

$$=\int_0^1\left[-\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{d\theta_0}{dx}-\theta_0{}^2N_{k,n}(x)+\{x(\cos x+\sin x)-11(\cos x-\sin x)-x^2(1-2\sin x\cos x)\}\right]N_{k,n}(x)$$

$$+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^9\theta_0}{dx^9}\right]_{x=1}-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^9\theta_0}{dx^9}\right]_{x=0}-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^8\theta_0}{dx^8}\right]_{x=1}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^8\theta_0}{dx^8}\right]_{x=0}$$

$$+\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^7\theta_0}{dx^7}\right]_{x=1}-\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^7\theta_0}{dx^7}\right]_{x=0}-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^6\theta_0}{dx^6}\right]_{x=1}+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^6\theta_0}{dx^6}\right]_{x=0}$$

$$+\left[\frac{d^5N_{k,n}(x)}{dx^5}\frac{d^5\theta_0}{dx^5}\right]_{x=1}-\left[\frac{d^5N_{k,n}(x)}{dx^5}\right]_{x=0}\times(-5)-\left[\frac{d^6N_{k,n}(x)}{dx^6}\right]_{x=1}\times(-3\sin1-5\cos1)+\left[\frac{d^6N_{k,n}(x)}{dx^6}\right]_{x=0}\times(-4)$$

$$+\left[\frac{d^7N_{k,n}(x)}{dx^7}\right]_{x=1}\times(-4\sin1+2\cos1)-\left[\frac{d^7N_{k,n}(x)}{dx^7}\right]_{x=0}\times3-\left[\frac{d^8N_{k,n}(x)}{dx^8}\right]_{x=1}\times(\sin1+3\cos1)+\left[\frac{d^8N_{k,n}(x)}{dx^8}\right]_{x=0}\times2$$

$$+\left[\frac{d^9N_{k,n}(x)}{dx^9}\right]_{x=1}\times(2\sin1)-\left[\frac{d^9N_{k,n}(x)}{dx^9}\right]_{x=0}\times(-1) \tag{22}$$

The above equation (22) is equivalent to matrix form

$$(D+B)A=G \tag{23a}$$

where the elements of *A, B, D, G* are $a_i, b_{i,k}, d_{i,k}$ and $g_k$ respectively, given by

$$d_{i,k}=\int_0^1\left[\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{dN_{i,n}(x)}{dx}+2\theta_0 N_{i,n}(x)N_{k,n}(x)\right]dx-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^9N_{i,n}(x)}{dx^9}\right]_{x=1}+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^9N_{i,n}(x)}{dx^9}\right]_{x=0}$$

$$+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^8N_{i,n}(x)}{dx^8}\right]_{x=1}-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^8N_{i,n}(x)}{dx^8}\right]_{x=0}-\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^7N_{i,n}(x)}{dx^7}\right]_{x=1}$$

$$+\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^7N_{i,n}(x)}{dx^7}\right]_{x=0}+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^6N_{i,n}(x)}{dx^6}\right]_{x=1}-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^6N_{i,n}(x)}{dx^6}\right]_{x=0}$$

$$-\left[\frac{d^5N_{k,n}(x)}{dx^5}\frac{d^5N_{i,n}(x)}{dx^5}\right]_{x=1} \tag{23b}$$

$$b_{i,k}=\sum_{j=1}^{n}\alpha_j\int_0^1(N_{i,n}(x)N_{j,n}(x)N_{k,n}(x))\,dx \tag{23c}$$

$$g_k=\int_0^1\left[-\frac{d^{10}N_{k,n}(x)}{dx^{10}}\frac{d\theta_0}{dx}-\theta_0{}^2N_{k,n}(x)+\{x(\cos x+\sin x)-11(\cos x-\sin x)-x^2(1-2\sin x\cos x)\}\right]N_{k,n}(x)dx$$

$$+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^9\theta_0}{dx^9}\right]_{x=1}-\left[\frac{dN_{k,n}(x)}{dx}\frac{d^9\theta_0}{dx^9}\right]_{x=0}-\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^8\theta_0}{dx^8}\right]_{x=1}+\left[\frac{d^2N_{k,n}(x)}{dx^2}\frac{d^8\theta_0}{dx^8}\right]_{x=0}$$

$$+\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^7\theta_0}{dx^7}\right]_{x=1}-\left[\frac{d^3N_{k,n}(x)}{dx^3}\frac{d^7\theta_0}{dx^7}\right]_{x=0}-\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^6\theta_0}{dx^6}\right]_{x=1}+\left[\frac{d^4N_{k,n}(x)}{dx^4}\frac{d^6\theta_0}{dx^6}\right]_{x=0}$$

$$+\left[\frac{d^5 N_{k,n}(x)}{dx^5}\frac{d^5\theta_0}{dx^5}\right]_{x=1} -\left[\frac{d^5 N_{k,n}(x)}{dx^5}\right]_{x=0} \times(-5) -\left[\frac{d^6 N_{k,n}(x)}{dx^6}\right]_{x=1} \times(-3\sin 1-5\cos 1)$$

$$+\left[\frac{d^6 N_{k,n}(x)}{dx^6}\right]_{x=0} \times(-4) +\left[\frac{d^7 N_{k,n}(x)}{dx^7}\right]_{x=1} \times(-4\sin 1+2\cos 1) -\left[\frac{d^7 N_{k,n}(x)}{dx^7}\right]_{x=0} \times 3$$

$$-\left[\frac{d^8 N_{k,n}(x)}{dx^8}\right]_{x=1} \times(\sin 1+3\cos 1) +\left[\frac{d^8 N_{k,n}(x)}{dx^8}\right]_{x=0} \times 2 +\left[\frac{d^9 N_{k,n}(x)}{dx^9}\right]_{x=1} \times(2\sin 1)$$

$$-\left[\frac{d^9 N_{k,n}(x)}{dx^9}\right]_{x=0} \times(-1) \tag{23d}$$

The initial values of these coefficients $\alpha_i$ are obtained by applying Galerkin method to the BVP neglecting the nonlinear term in (18a). That is, to find initial coefficients we solve the system

$$DA = G \tag{24a}$$

whose matrices are constructed from

$$d_{i,k} = \int_0^1 \frac{d^{10} N_{k,n}(x)}{dx^{10}}\frac{dN_{i,n}(x)}{dx}dx -\left[\frac{dN_{k,n}(x)}{dx}\frac{d^9 N_{i,n}(x)}{dx^9}\right]_{x=1} +\left[\frac{dN_{k,n}(x)}{dx}\frac{d^9 N_{i,n}(x)}{dx^9}\right]_{x=0} +\left[\frac{d^2 N_{k,n}(x)}{dx^2}\frac{d^8 N_{i,n}(x)}{dx^8}\right]_{x=1}$$

$$-\left[\frac{d^2 N_{k,n}(x)}{dx^2}\frac{d^8 N_{i,n}(x)}{dx^8}\right]_{x=0} -\left[\frac{d^3 N_{k,n}(x)}{dx^3}\frac{d^7 N_{i,n}(x)}{dx^7}\right]_{x=1} +\left[\frac{d^3 N_{k,n}(x)}{dx^3}\frac{d^7 N_{i,n}(x)}{dx^7}\right]_{x=0}$$

$$+\left[\frac{d^4 N_{k,n}(x)}{dx^4}\frac{d^6 N_{i,n}(x)}{dx^6}\right]_{x=1} -\left[\frac{d^4 N_{k,n}(x)}{dx^4}\frac{d^6 N_{i,n}(x)}{dx^6}\right]_{x=0} -\left[\frac{d^5 N_{k,n}(x)}{dx^5}\frac{d^5 N_{i,n}(x)}{dx^5}\right]_{x=1} \tag{24b}$$

$$g_k = \int_0^1 \left[-\frac{d^{10} N_{k,n}(x)}{dx^{10}}\frac{d\theta_0}{dx} +\left\{x(\cos x+\sin x)-11(\cos x-\sin x)-x^2(1-2\sin x\cos x)\right\}\right] N_{k,n}(x)dx$$

$$+\left[\frac{dN_{k,n}(x)}{dx}\frac{d^9\theta_0}{dx^9}\right]_{x=1} -\left[\frac{dN_{k,n}(x)}{dx}\frac{d^9\theta_0}{dx^9}\right]_{x=0} -\left[\frac{d^2 N_{k,n}(x)}{dx^2}\frac{d^8\theta_0}{dx^8}\right]_{x=1}$$

$$+\left[\frac{d^2 N_{k,n}(x)}{dx^2}\frac{d^8\theta_0}{dx^8}\right]_{x=0} +\left[\frac{d^3 N_{k,n}(x)}{dx^3}\frac{d^7\theta_0}{dx^7}\right]_{x=1} -\left[\frac{d^3 N_{k,n}(x)}{dx^3}\frac{d^7\theta_0}{dx^7}\right]_{x=0}$$

$$-\left[\frac{d^4 N_{k,n}(x)}{dx^4}\frac{d^6\theta_0}{dx^6}\right]_{x=1} +\left[\frac{d^4 N_{k,n}(x)}{dx^4}\frac{d^6\theta_0}{dx^6}\right]_{x=0} +\left[\frac{d^5 N_{k,n}(x)}{dx^5}\frac{d^5\theta_0}{dx^5}\right]_{x=1}$$

$$-\left[\frac{d^5 N_{k,n}(x)}{dx^5}\right]_{x=0} \times(-5) -\left[\frac{d^6 N_{k,n}(x)}{dx^6}\right]_{x=1} \times(-3\sin 1-5\cos 1) +\left[\frac{d^6 N_{k,n}(x)}{dx^6}\right]_{x=0} \times(-4)$$

$$+\left[\frac{d^7 N_{k,n}(x)}{dx^7}\right]_{x=1} \times(-4\sin 1+2\cos 1) -\left[\frac{d^7 N_{k,n}(x)}{dx^7}\right]_{x=0} \times 3 -\left[\frac{d^8 N_{k,n}(x)}{dx^8}\right]_{x=1}$$

$$\times(\sin 1+3\cos 1) +\left[\frac{d^8 N_{k,n}(x)}{dx^8}\right]_{x=0} \times 2 +\left[\frac{d^9 N_{k,n}(x)}{dx^9}\right]_{x=1} \times(2\sin 1) -\left[\frac{d^9 N_{k,n}(x)}{dx^9}\right]_{x=0} \times(-1) \tag{24c}$$

Once the initial values of $\alpha_i$ are obtained from eqn. (24a), they are substituted into eqn. (23a) to obtain new estimates for the values of $\alpha_i$. This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters into eqn. (19), we obtain an approximate solution of the BVP (18). Numerical results for example 3 are shown in the following Table 3. On the contrary the maximum absolute error were found by Siddiqi *et al* [7] is $4.415\times10^{-10}$. We have shown the exact and approximate solutions in Fig. 3 of example 3 for $n=13$.
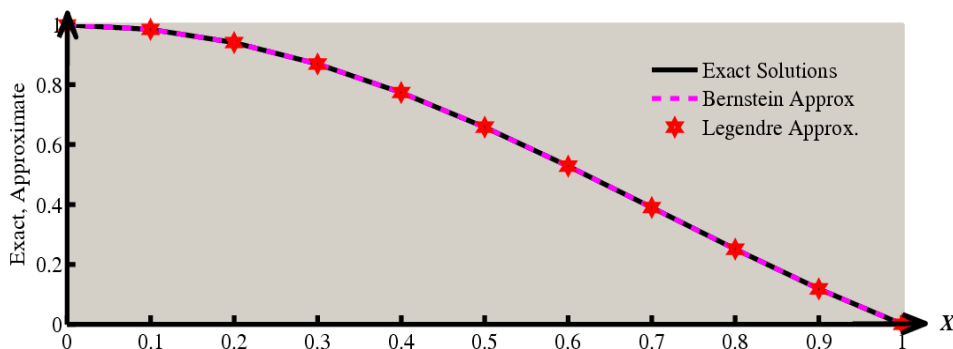
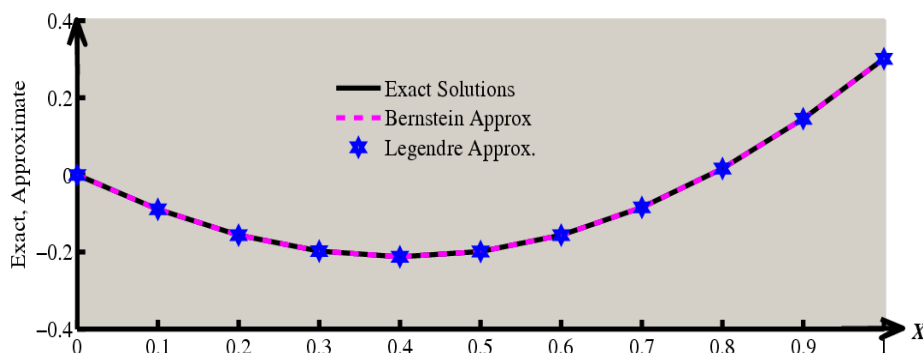Fig. 2: Graphical representation of exact and approximate solutions of example 2 using 15 polynomials.



Fig. 3: Graphical representation of exact and approximate solutions of example 3.

Table 3: Numerical Results for Example 3 using 6 iterations

| $x$ | Exact Results | 13, Bernstein Polynomials | | 13, Legendre Polynomials | |
|---|---|---|---|---|---|
| | | Approximate | Abs. Error | Approximate | Abs. Error |
| 0.0 | 0.0000000000 | 0.0000000000 | 0.0000000E+000 | 0.0000000000 | 5.3209018E-027 |
| 0.1 | -0.0895170749 | -0.0895170749 | 9.7560848E-015 | -0.0895170749 | 9.7977182E-015 |
| 0.2 | -0.1562794494 | -0.1562794494 | 1.0380585E-014 | -0.1562794494 | 1.0602630E-014 |
| 0.3 | -0.1979448847 | -0.1979448847 | 2.9445890E-013 | -0.1979448847 | 2.9437563E-013 |
| 0.4 | -0.2126570607 | -0.2126570607 | 8.5739749E-013 | -0.2126570607 | 8.5667584E-013 |
| 0.5 | -0.1990785116 | -0.1990785116 | 1.3202772E-012 | -0.1990785116 | 1.3194446E-012 |
| 0.6 | -0.1564158849 | -0.1564158849 | 1.2643497E-012 | -0.1564158849 | 1.2642110E-012 |
| 0.7 | -0.0844371500 | -0.0844371500 | 7.2382378E-013 | -0.0844371500 | 7.2435113E-013 |
| 0.8 | 0.0165195052 | 0.0165195052 | 1.9494822E-013 | 0.0165195052 | 1.9536109E-013 |
| 0.9 | 0.1455452472 | 0.1455452472 | 2.6478819E-014 | 0.1455452472 | 2.6062486E-014 |
| 1.0 | 0.3011686789 | 0.3011686789 | 0.0000000E+000 | 0.3011686789 | 0.0000000E+000 |

## V.    Conclusions

In this paper, we have used Bernstein and Legendre polynomials as basis functions for the numerical solutions of eleventh-order linear and nonlinear BVPs in the Galerkin method. It is clear from the tables that the numerical results obtained by our method are superior to other existing methods. Also we get better results for Bernstein polynomials than the Legendre polynomials. It may also notice that the numerical solutions are identical with the exact solution even Bernstein and Legendre polynomials are used in the approximation which are shown in Figs. [1-3].

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