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The Transitivity, Primitivity and Faithfulness of Wreath Products of Permutation groups

Chun P.B, Choji N.M Adidi C.S, Ajai P.T

Abstract: *Suppose C and D are permutation groups on Γ and Δ respectively. The wreath product of C by D denoted by C wr D is the semi – direct product of C by Δ so that W = {$(f, d)|f\in P, d\in D$}, with multiplication in W defined as; (f_1,d_1) (f_2,d_2) = $[(f_1f_2d_1^{-1}), (d_1d_2)]$ for all $f_1, f_2\in P$ and $d_1d_2\in D$.*
These communication (paper) provides with clarity the conditions under which wreath products of such permutation groups are transitive, primitive and faithful and also provides a very good example to demonstrate such conditions.

Keywords: *Group actions, Transitive Permutation groups, Primitivism and faithfulness of W on ΓXΔ, wreath product, stabilizer and centre of wreath products.*

I. Introduction

Wreath products of permutation groups has become an interesting area of study in recent times. These was first reported by Audu M.S in [1]. Ezenwanne I.U in [2] also discussed extensively on transitivity, primitivity and wreath products of permutation groups. Apine.E [3] considered permutation groups of prime-power. Ahmad, Suleiman (2006) [4] provided interesting examples buttressing the transitivity, primitivism and faithfulness of wreath products of permutation group.

II. Notations

C^{Δ} : The set of all maps of Δ into the permutation group C.
ΓxΔ : Direct products of two sets Γand Δ
W (α,δ): Stabilizer of any point (α,δ) in ΓXΔ
Z(W): Centre of W
CwrD: The wreath product of C by D

III. Preliminary

We shall state and prove several theorems, define certain notions and make obvious remarks which shall lead to the statement and prove of our claims.

Definition1.1: The wreath product of C by D donated by W = CwrDis the semi-direct product of P by Δ so that W = {$(f, d)|f\in P, d\in D$}, with multiplication in W defined as; (f_1,d_1) (f_2,d_2) = $[(f_1f_2d_1^{-1}), (d_1d_2)]$ for all $f_1, f_2\in P$ and $d_1d_2\in$ D. Hence forth, we would write (fd) instead of (f,d) for elements of W.

Theorem 1.2

Let C and D be permutation groups on Γ and D respectively. Let C^{Δ} be the set of all map∈ of D into the permutation group C
that is C^{Δ} = {f:Δ $\longrightarrow$ C}. For any f_1,f_2 in C, let f_1f_2 in C^{Δ}, be defined for all δ in Δ by

$$(f_1f_2)\, \delta = f_1(\delta)\, f_2(\delta)$$

Thus composition of functions is point wise and operation is placed on the right. With respect to this operation of multiplication, C^{Δ} acquires the structure of a group.

Proof:

i. C^{Δ} is a non –empty and is closed with respect to multiplication. For suppose $f_1f_2\in C^{\Delta}$ then $f_1(\delta)$ $f_2(\delta)\in C$. Hence $f_1(\delta)$**.**$f_2(\delta)\in C$. This implies that $(f_1f_2)\delta \in C$ and so $f_1f_2\in C^{\Delta}$

ii. Since multiplication in C is associative so also is the multiplication in C^{Δ}

iii. The identity element in C^{Δ} is the map e:Δ $\longrightarrow$C given by e(δ) = 1 for all δ∈Δ and 1eC.

iv. Every element f∈ C^{Δ} is defined for δ∈Δ by f(δ) = f(δ)1. Thus C^{Δ} is a group with respect to the multiplication defined above. (we denote this group by P).

LEMMA 1.3

Suppose that D acts on P as follows $f^d(\delta) = f(\delta d^{-1})$ for all δ∈ Δ, d ∈ D. Then D acts on P as a group.

Proof:

Take $f, f_1 f_2\in P$ and d, $d_1, d_2\in D$.

i. $(f^{d1})^{d}{}_{2}(\delta) = f^{d1}(\delta\, d_2^{-1})$
$= f(\delta\, d_2^{-1}d_1^{-1})$
$= f^{d1d2}(\delta)$

ii. $f^{1}(\delta) = f(\delta\, 1^{-1})$
$= f(\delta)$

iii. $(f_1f_2)^{d}(\delta) = f_1f_2(\delta d^{-1})$
$= f_1(\delta\, d^{-1})f_2(\delta\, d^{-1})$
$= f_1^{d}(\delta)f_2^{d}(\delta)$

Thus D act on p as a group.

Theorem 1.4

Let D act on P as group. The set of all or all ordered pairs (f,d) with fcP, deD acquires the structure of a group when we define all f_1, f_2 Є P and d_1, d_2ЄD. $(f_1,d_1)(f_2, d_2) = (f_1f_2\, d^{-1}, d_1d_2)$

Proof.

i. Closure property follows from the definition of multiplication

ii. Take f_1, f_2, f_3ЄP and d_1,d_2,d_3ЄD. Then

$[(f_1d_1)(f_2, d_2)](f_3, d_3) = (f_1f_2d^{-1}, d_1d_2)(f_3, d_3)$
$= (f_1f_2{}^{d}{}_{1}{}^{-1}f_3\,(d_1\, d_2)^{-1}, d_1d_2\, d_3)$
$= (f_1f_2d_1^{-1}f_3{}^{d}{}_{2}{}^{-1}\, d_1^{-1}, d_1d_2d_3)$
$= (f_1f_2d_1^{-1}f_3{}^{d}{}_{2}{}^{-1}, d_1d_2d_3)$

$= (f_1d_1)[(f_2,d_2)(f_3,d_3)] = (f_{1,}d_1)(f_2f_3d^{-1}, d_1d_2)(f_3, d_3)$
$= (f_1f_2f_3{}^{d}{}_{2}{}^{-1}f_3{}^{d}{}_{2}{}^{-1}d_1^{-1}, d_1d_2d_3)$
$= (f_1f_2{}^{d}{}_{2}{}^{-1}f_3d_1^{-1}, d_1d_2d_3) = [(f_1d_1)(f_2, d_2)](f_3, d_3)$

Thus multiplication is associative.

iii. we know that every f Є P, f^{1}=f. now for every d Є D, the map $f \longrightarrow f^{1}$ is an antomorphism of P. Also if e is the identity element in P then
e^{d} = e. also $(f^{1})^{d} = (f^{d})^{1}$

Now, $(f,d)(e,1) = (fe^{d-1}, d.1)$
$= (f(e^{-1})^{d}, d)$
$= (f,d)$

The identity element exists.

iv. $(f,d)((f^{-1})^{d}, d^{-1}) = (f((f^{-1})^{d})^{d-1}, dd^{-1})$
$= (f((f^{-1})^{dd-1}, dd^{-1})$
$= (f(f^{-1})1, dd^{-1})$
$= (e,1)$

Thus when D acts on P, the set of all ordered pairs (f,d) with fЄP and dЄP is a group if we define$(f_1d_1)(f_{21,}d_2)=(f_1f_2{}^{d-1}, d_1d_2)$

Theorem 1.5

Let D acts on P as $f^{1}(\delta)= f(\delta d^{-1})$ where fЄP, dЄP and δЄΔ . Let W be the group of all of all juxtaposed symbols fd, with fЄP , dЄP and multiplication given by
$(f_1d_1)(f_2d_2) = (f_1f_2{}^{d}{}_{1}{}^{-1})(d_1d_2)$
Then W is a group called the semi-direct product of P by D with the defined action. Proof (similar to the proof of Lemma 1.3.)

Remark 1.6.

1. We notice that if C and D are finite groups, then a wreath product W determined by an action of D on a finite set is a finite group of order ./W/ $=|C|^{|\Delta|}\,|D|$
2. P is a normal subgroup of W and D and it is a subgroup of W.
3. The action of W on Γ x Δ is given by (α,ß) fd =(αf(ß), ßd) where α ЄP and ß ЄΔ.

TRANSITIVITY OF W ON ΓXΔ 1.7

Suppose that we take two arbitrary point $(\alpha_1\delta 1)$ and $(\alpha_2\delta_2)$ in ΓX Δ. Then W will be transitive on ΓXΔ if and only if $(\alpha f(\delta_1), \delta_1 d) = (\alpha_{2,}\delta_2)$. That is if and only $\alpha_1 f(\delta_1) = \alpha_{2,}\delta_1 d = \delta_{2.}$ Thus such f, d exists if and only C and D are transitive on Γ and Δ respectively which is necessary the condition for W to be transitive on ΓXΔ

THE STABILIZER W(α,δ) OF A POINT (α_1,δ) IN ΓXΔ 1.8

Furthermore, under the action of W on ΓXΔ, the stabilizer of any point (α,δ) in ΓXΔ denotedby W (α,δ) is given by

W(α,δ) = {fdЄW|(α,δ) fd= (α,δ)}
={fdЄW(α,f(δ), δd) =(α,δ)}

={fЄW |(α,f(δ)= α, δd= δ}
= F(δ) αDδ

Where F(δ) α is the set of all f(δ) that stabilizes α and Dδ is the stabilizer of δ under the action of D on Δ

FAITHFULNES OF W ON ΓXΔ 1.9.

We recall that W is faithful on ΓXΔ if and only if the identity of W is its only element that fixes every point of ΓXΔ. Now the identity element of W is 1 and thus if W is to be faithful on ΓXΔ then for any (α,δ) in ΓXΔ ; the stabilizer of W on ΓXΔ, W(α,δ) must be f(δ) αD δ= 1 Hence f(δ) α =1 and Dδ=1 for all αЄΓ, δ ЄΔ and . αf (δ) = α, δd = δ imply that f (δ)

=1 and d =1. Thus we deduce that W would be faithful on ΓXΔ, if the stabilizer of any αЄΓand δ ЄΔ are the identity elements in P and D respectively. Therefore we conclude that W is faithful on ΓXΔ, if P or C and D are faithful on Γ or Δ respectively.

THE PRIMITIVITY OF W ON ΓXΔ 2.0

We recall that w would be primitive on ΓXΔ, if and only if given any (α,δ) in ΓXΔ, W (α,δ) the stabilizer of (α,δ) is a maximal subgroup of W. Now,W(α,δ) = F(δ) αDδ where where F(δ)α is the set of those f in P such that f(δ) α fixes α and Dδ is the stabilizer of δ under the action D on Δ. As f(δ)α does not include those f in P which do not stabilize α. We have that F(δ) αDδ<PDδ< PD = W. and also, in general W(α,δ) is not a maximal subgroup of W. Thus W would be inprimitive on ΓXΔ in a natural way.

However, if |Γ| =1 that is Γ= { α}, then $C_Γ$ =Cα = C. in Particular αf (δ) = α for all f in P. Thus F(δ) α = P hence F(δ) α Δδ. And if in addition, D were primitive on Δ then Δδ would be maximal in D and hence PDδ =F(δ) α Dδ=W(α, δ) would be maximal in W that is W would be primitive on ΓXΔ. Again if |Δ| =1 , say Δ={δ} , then Dδ=D and W(α,δ)= F(δ) α Dδ = F(δ) αΔ. And if in addition, C were primitive on Γ , then Cα, would be maximal in C={F(δ) | for all fЄP} and correspondingly F(δ) α would be maximal in P and hence W(α, δ) would be maximal in W, that is W would be primitive on ΓXΔ

In conclusion, we have shown that W is in primitive on ΓXΔ in a natural way, unless |Γ| =1 and D is primitive on Δ or |Δ| =1 and C is primitive on Γ.

THE CENTRE OF W 2.1

We denote the centre of W as Z(W) and define Z(W) = {fd |(fd)($f_1d_{1)}$ (f_1d_1)(fd), for all f, ЄP, d_1ЄD}. Hence fdЄZ(W) if and only if {ff_1^{d-1} dd_1 = $f_1 f^{d-1} d_1d$ for all f_1ЄP, d_1ЄD…….(1.a)} solve for f and d. put $d_{1=}$ 1 then (1.a.) becomes

$ff_1^{d-1}d = f_1d$ for all f_1 ЄP--------------------------------(1.b.)

put f_1 = 1 then (1.a) becomes

$fdd_1= f^{d-1}d_1d$ for all dЄD------------------------(1.c)

from (1.a) it follows that for fd to be in Z(W) it is necessary that dЄZ (W).

CLAIM 2.2

If C ≠ 1 ,fdЄZ (W) and dЄZ (D), then δd = δ for all δЄ Δ--------------(1.d)

To show this, let δЄ Δ and choose f_1ЄP such that $f_1(δ)$ = C≠ 1, cЄC and $f_1(δ^1)$= 1 for all $δ^1$ ≠ δ-------------------1e.

Then from (1.b), we have that

$f_1f =ff_1^{d-1}$ and so f_1 (δ) f(δ) = f(δ)f, (δd) =f(δ), if δd ≠δ. Hence f_1 (δ) = 1. But this is false by (1.e) and hence we must have δd = δ for all δЄ Δ. Accordingly, our Claim is correct.

Furthermore, (1.b) implies that for all δЄ Δ,

$f_{1\,(}$δ) f(δ) = f(δ)f_1(δd)
= f(δ)f_1(δ)

Hence f(δ) Є Z(C) for all δЄ Δ---(1.f)

Also (1.c) implies that f(δ d)= f(δ)

For all δЄ Δ, d_1Є Δ (since dЄZ(D)---(1.g)

Now (1.g) shows that f is constant over orbits of D in Δ.

IV. Conlusion

Thus from (1.4.), (1.f) and (1.g) we conclude that provided C≠ (1), f dЄZ(W) if and only if

i. dЄZ(D) nK, where K = { dЄD| δd = d for all δЄΔ }.

ii. fЄ{$Δ_i$Z(C)} where $Δ_i$ are orbits in Δ however, if C={1}, then clearly Z(W) = Z(D). with the above notion we conclude that

Z(W) = { Z(D), if C = 1
(IIZ_1) (Z(D) nk); other wise

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