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Generalized Contraction Principle in Complex valued Metric spaces

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Abstract: *In this paper, we introduce the notion of Generalized contractive type mappings in complex valued metric space and establish fixed point theorem for these mappings.*
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I. Introduction

The existence and uniqueness of fixed point theorems of operators or mappings has been a subject of great interest since the work of Banach in 1992[2].The Banach contraction mapping principle is widely recognized as the source of metric fixed-point theory. A mapping $T:X\to X$, where (X,d) is metric space , is said to be contraction mapping if for all $x,y\in X$, $d(Tx,Ty)\leq \lambda d(x,y)$, $0<\lambda<1$. (1)
According to the Banach contraction mapping principle,any mapping T satisfying (1) in Complete metric space will have a unique fixed-point. This principle includes different directions in different spaces adopted by mathematicians; for example ,metric space, G-metric spaces, partial metric spaces,cone metric spaces have already been obtained. A new space called the complex valued metric space which is more general than well-known metric space has been introduced by Azam et.al. Azam proved some fixed-point theorems for mappings satisfying a rational inequality. In2012, Rouzkard and Imdad [3] extended and improved the common fixed-point theorems which are more general than the result of Azam et.al. [1]. The idea of complex valued metric spaces can be exploited to define complex valued normed spaces and complex valued Hilbert spaces; additionally,it offers numerous research activities in mathematical analysis.

II. Basic Facts and Definitions

We recall some notations and definitions which will be utilized in our discussion.
Let **C** be a set of complex numbers and $z_1,z_2\in$ **C.** Define a partial order $\preccurlyeq$ on **C** as follows:
$z_1\preccurlyeq z_2$ if and only if $Re(z_1)\leq Re(z_2)$, $Im(z_1)\leq Im(z_2)$.
It follows that $z_1\preccurlyeq z_2$ if one of the following conditions is satisfied:
(i) $Re(z_1)=Re(z_2)$, $Im(z_1)< Im(z_2)$.
(ii) $Re(z_1)<Re(z_2)$, $Im(z_1)=Im(z_2)$.
(iii) $Re(z_1)<Re(z_2)$, $Im(z_1)<Im(z_2)$.
(iv) $Re(z_1)=Re(z_2)$, $Im(z_1)=Im(z_2)$.
In (i), (ii) and(iii), we have $|z_1|<|z_2|$ .In(iv), we have $|z_1|=|z_2|$ .So $|z_1|\leq|z_2|$. In particular, we will write $z_1\precnsim z_2$ if $z_1\neq z_2$ and one of (i), (ii) and(iii) is satisfied. In this case $|z_1|<|z_2|$ .We will write $z_1\prec z_2$ if and only if (iii) is satisfied.
Take into account some fundamental properties of the partial order $\preccurlyeq$ on **C** as follows.
(i) $0\preccurlyeq z_1\precnsim z_2$, then $|z_1|<|z_2|$.
(ii) If $z_1\preccurlyeq z_2$, $z_2\prec z_3$, then $z_1\prec z_3$.
(iii) If $z_1\preccurlyeq z_2$ and $\lambda\geq 0$ is a real number , then $\lambda z_1\preccurlyeq \lambda z_2$.

Definition1.[3] The "max" function for the partial order relation "$\preccurlyeq$" is defined by the following.
(i) $\max\{z_1,z_2\}=z_2$ if and only if $z_1\preccurlyeq z_2$.
(ii) If $z_1\preccurlyeq \max\{z_2,z_3\}$,then $z_1\preccurlyeq z_2$ or $z_1\preccurlyeq z_3$.
(iii) $\max\{z_1,z_2\}=z_2$ if and only if $z_1\preccurlyeq z_2$ or $|z_1|\leq|z_2|$.
Using Definition1 one can have the following lemma.

Lemma2 [3]Let $z_1,z_2,z_3....\in$ **C** and the partial order relation $\preccurlyeq$ is defined on **C.** Then the following conditions are easy follow.
(i) If $z_1\preccurlyeq \max\{z_2,z_3\}$, then $z_1\preccurlyeq z_2$ if $z_3\preccurlyeq z_2$.
(ii) If $z_1\preccurlyeq \max\{z_2,z_3,z_4\}$, then $z_1\leq z_2$ if $\max\{z_3,z_4\}\preccurlyeq z_2$.
(iii) If $z_1\preccurlyeq \max\{z_2,z_3,z_4,z_5\}$, then $z_1\leq z_2$ if $\max\{z_3,z_4,z_5\}\preccurlyeq z_2$.

Now we give the definition of complex valued metric space which has been introduced by Azam et. al. [1]

Definition3. Let X be non empty set. If a mapping $d:X\times X\to \mathbf{C}$ satisfies
(i) $0\preccurlyeq d(x, y)$ for all $x,y \in X$ and $d(x, y)=0$ if and only if $x=y$,
(ii) $d(x, y)= d(y, x)$ for all $x,y\in X$,
(iii) $d(x, y) \preccurlyeq d(x, z)+ d(z, y)$ for all $x,y,z \in X$.
Then d is called a complex valued metric on X and the pair(X, d) is called complex valued metric space .
Let $\{x_n\}$ be a sequence in complex valued metric space X and $x \in X$.If for every $\varepsilon \in \mathbf{C}$with $0\prec\varepsilon$ there $N\in\mathbf{N}$ such that , for all $n>N$, $d(x_n, x)\prec\varepsilon$, then x is called the limit of $\{x_n\}$ and is written as $\lim_{n\to\infty} x_n=x$ as $n\to\infty$ .If for every $\varepsilon \in \mathbf{C}$ with $0\prec\varepsilon$ there $N\in\mathbf{N}$ such that , for all $n>N$, $d(x_n, x_m)\prec\varepsilon$, then $\{x_n\}$ is called a Cauchy sequence in X. If every Cauchy sequence is convergent in X, then X is called a complete complex valued metric space.

Lemma2. [1] Let(X, d) is called complex valued metric space and let$\{x_n\}$ be a sequence in X. Then
(i) $\{x_n\}$ converges to x if and only if $|d(x_n, x)|\to 0$ as $n \to\infty$.
(ii) $\{x_n\}$ is Cauchy sequence if and only if $|d(x_n, x_m)|\to 0$ as $\to\infty$.

III. Main Results

In this paper, we prove Generalized contraction principle in complex valued metric space as follows:
Theorem1.1. Let $T:X\to X$ be self mappings of a complex valued metric space(X,d) satisfying
$d(Tx,Ty) \preccurlyeq k\, M(x,y)$ where $k\in[0,1)$ (2)
where $M(x, y)=\max\{d(x, y), d(Tx, y), d(Ty, y),\frac{[d(Tx,y)+d(Ty,x)]}{2}\}$.[5]

Then T has a unique fixed point.
Proof. Let $x_0\in X$ be arbitrary point and define a sequence $\{x_n\}$ as $Tx_n=x_{n+1}$.Then putting $x=x_n$, $y=x_{n-1}$ we get
$d(x_{n+1},x_n) =d(Tx_n,Tx_{n-1}) \preccurlyeq k\, M(x_n,x_{n-1})$ (3)
where $M(x_n,x_{n-1})=\max\{d(x_n,x_{n-1}),d(Tx_n, x_n), d(Tx_{n-1}, x_{n-1}),\frac{[d(Tx_n,x_n)+d(Tx_{n-1},x_n)]}{2}\}$.
$M(x_n,x_{n-1})=\max\{d(x_n,x_{n-1}),d(x_{n+1},x_n), d(x_n, x_{n-1}),\frac{[d(x_{n+1},x_n)+d(x_n,x_n)]}{2}\}$.
$\preccurlyeq\max\{d(x_n,x_{n-1}),d(x_{n+1},x_n)\}$.
Now from (2) we get
$d(x_{n+1},x_n) =d(Tx_n,Tx_{n-1}) \preccurlyeq k \max\{d(x_n,x_{n-1}),d(x_{n+1},x_n)\}$.
$|d(x_{n+1},x_n)|\le k\,|\max\{d(x_n,x_{n-1}),d(x_{n+1},x_n)\}|$
$\le k \max\{|d(x_{n+1},x_n)|, |d(x_n,x_{n-1})|\}$.
We shall take two cases.
Suppose $|d(x_{n+1},x_n)|>|d(x_n,x_{n-1})|$.Since $|d(x_{n+1},x_n)|>0$, we obtain
$|d(x_{n+1},x_n)|\le k|d(x_{n+1},x_n)|$ a contradiction. Therefore, we get
$\max\{|d(x_{n+1},x_n)|, |d(x_n,x_{n-1})|\}= |d(x_n,x_{n-1})|$.Then $d(x_{n+1},x_n) \preccurlyeq k\, d(x_n,x_{n-1})$ (4)
Again $d(x_n,x_{n-1})\preccurlyeq k d(x_{n-1},x_{n-2})$,then from (3) $d(x_{n+1},x_n) \preccurlyeq k^2 d(x_{n-1},x_{n-2})$.
Continuing in the same manner, we have $d(x_{n+1},x_n) \preccurlyeq k^n d(x_1,x_0)$. (5)
Then for all $n, m \in N$ and repeated use of triangular inequality for $m\ge 1$and from (5),we have
$d(x_{n+m},x_n) \preccurlyeq d(x_{n+m},x_{n+m-1}) +.............+ d(x_{n+1},x_n)$
$\preccurlyeq\sum_{p=n}^{n+m-1} k^n d(x_1,x_0)$
$\preccurlyeq\sum_{p=n}^{\infty} k^n d(x_1,x_0)$
Therefore, $|d(x_{n+m}, x_n)|\le\sum_{p=n}^{\infty} k^n\,|d(x_1,x_0)|$.
Since $k\in[0,1)$, if we take limit as $n \to\infty$ then $|d(x_{n+m},x_n)|\to 0$.
So, $\{x_n\}$ is complex valued Cauchy sequence .By completeness of (X,d) there exists $z \in X$ such that$\{x_n\}$ is complex valued convergent to z.
Next we prove Tz=z. Assume on contrary that $Tz\neq z$. Then by (1), put $x=z$, $y=x_{n+1}$
$d(Tz,Tx_{n+1}) \preccurlyeq k\, M(z , x_{n+1})$
where $M(z , x_{n+1})= \max\{d(z,x_{n+1}),d(Tz,z), d(Tx_{n+1}, x_{n+1}),\frac{[d(Tz,x_{n+1})+d(x_n,z)]}{2}\}$.
As $\{x_n\}$ is convergent to z,therefore,$\lim_{n\to\infty} |d(z,x)|=\lim_{n\to\infty}|d(x,x)| = 0$.
Thus letting $n\to\infty$,$d(Tz, z)\preccurlyeq k\, d(Tz, z)$ that is $|d(Tz,z)| \le |d(Tz,z)|$which is contradiction.
So,Tz=z that is, z is fixed point of T.
Uniqueness. Let u ($u\neq z$) be another fixed point of T, then from (2) we have
$d(u, z)=d(Tu, Tz)\preccurlyeq k\, M(u,z)$
where $M(u, z)= \max\{d(u, z),d(Tu, u), d(Tz, z),\frac{[d(Tu,z)+d(Tz,u)]}{2}\}$.

=d(u, z)

$|d(u, z)| \leq k\ |d(u,z)|$ which is a contradiction. Hence u=z that is, T has a unique fixed point.

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