IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN:2319-765X. Volume 10, Issue 3 Ver. IV (May-Jun. 2014), PP 77-81
www.iosrjournals.org

Nonsplit domsaturation number of a graph

Y. Therese Sunitha Mary * and R. Kala **
**Department of Mathematics, St. Xavier's College*
Palayamkottai 627 002, Tamil Nadu, India
***Department of Mathematics, Manonmaniam Sundaranar University*
Tirunelveli 627 012, Tamil Nadu, India

The nonsplit domsaturation number of a graph G , $ds_{ns}(G)$ is the least positive integer k such that every vertex of G lies in a nonsplit dominating set of cardinality k .In this paper, we obtain certain bounds for $ds_{ns}(G)$ and characterize the graphs which attain these bounds.

I. Introduction

By a graph $G=(V,E)$ we mean a finite, undirected graph without loops or multiple edges.The order and size of G are denoted by p and q respectively. For graph theoretical terms we refer to Harary [6] and for terms related to domination we refer Haynes et al.[7]

A subset D of V is said to be a dominating set in G if every vertex in $V-D$ is adjacent to at least one vertex in D.

Kulli and Janakiram introduced the concept of nonsplit domination in graphs [9]. A dominating set D of a graph G is a *nonsplit dominating set* if $<V-D>$ is connected. The *nonsplit domination number* $\gamma_{ns}(G)$ of G is the minimum cardinality of a nonsplit dominating set. A nonsplit dominating set with cardinality $\gamma_{ns}(G)$ is called a γ_{ns}*-set.*

Acharya[1] introduced the concept of domsaturation number of a graph. The least positive integer k such that every vertex of G lies in a dominating set of cardinality k is called the domsaturation number of G and is denoted by $ds(G)$. A detailed study of this parameter was already done by Arumugam and Kala[2]. In this paper, we define nonsplit domsaturation number of a graph . We determine the value of this parameter for several classes of graphs , obtain bounds for this parameter and also characterize the graphs which attain these bounds.

II. Main Results

Example 2.1 *(i) If* $G \cong K_p$ *then* $ds_{ns}(G)=1$.

(ii) If $G \cong K_{m,n}(2 \le m \le n)$ then $ds_{ns}(G)=2$.

Proposition 2.2 *For any connected graph* $G, \gamma_{ns}(G) \le p-1$. *Further equality holds if and only if* G *is a star.*

Proof. Every set $S \subseteq V(G)$ with $|S|=p-1$ is a nonsplit dominating set of G and so $\gamma_{ns}(G) \le p-1$. If G is a star, clearly $\gamma_{ns}(G)=p-1$. Suppose $\gamma_{ns}(G)=p-1$. If G is not a star, then G has an edge $e=uv$ such that both u and v are non - pendent vertices. Now $V(G)-\{u,v\}$ is a nonsplit dominating set of G and so $\gamma_{ns}(G) \le p-2$ which is a contradiction. Hence G is a star.

Corollary 2.3 *For any graph* G , $\gamma_{ns}(G)=p-1$ *if and only if* G *is a galaxy.*

Proposition 2.4 *For any graph* G , $\gamma_{ns}(G) \le ds_{ns}(G) \le min\{\gamma_{ns}(G)+\Delta(G), p-1\}$ *and these bounds are sharp.*

Proof. Lower bound is obvious. Suppose $ds_{ns}(G) = \gamma_{ns}(G) + \Delta(G) + k$, where $k \geq 1$. Then there exists a vertex $v \in V(G)$ such that the minimum cardinality of a nonsplit dominating set A containing v is $\gamma_{ns}(G) + \Delta(G) + k$. If S is any γ_{ns} - set, then $v \notin S$. Also $S \cap N(v) \neq \emptyset$. As $|A| = \gamma_{ns}(G) + \Delta(G) + k$, by choice of v, $< V - (S \cup \{v\}) >$ has $\Delta(G) + k - 1$ isolated vertices so that $|N(v)| \geq \Delta(G) + k$, which is a contradiction. Hence $ds_{ns}(G) \leq \gamma_{ns}(G) + \Delta(G)$. Always $ds_{ns}(G) \leq p - 1$ and so $ds_{ns}(G) \leq min\{\gamma_{ns}(G) + \Delta(G), p-1\}$.

If $G \cong C_p$, $ds_{ns}(G) = \gamma_{ns}(G) = p - 2$ and so the lower bound is sharp. If $G \cong B(2,2)$, then $ds_{ns}(G) = 5$ and $min\{\gamma_{ns}(G) + \Delta(G), p-1\} = min\{7,5\} = 5$. Thus the upper bound is also sharp.

Theorem 2.5 *Let G be a connected graph. Then $ds_{ns}(G) = p - 1$ if and only if $G \cong G_i (1 \leq i \leq 2)$ where $G_i (1 \leq i \leq 2)$ are given in Fig. 1.*

(2, -1)(3, 2) [dotscale = 1](-2.3, 0)(-2, -2)(0, -1.5)(-4.3, -1.5)(6, 0)(4, -2)(3, -4)(5, -4.3)(6, -2)(5.5, -4.1)(7.5, -4.3)(8, -1.7)(9.2, -3.4)(9.5, -1)(4.6, 1)(7, 1) (-4.3, -1.5)(-2.3, 0)(-2, -2) (-2.3, 0)(0, -1.5) (6, 0)(4, -2)(3, -4) (4, -2)(5, -4.3) (6, 0)(6, -2)(5.5, -4.1) (6, -2)(7.5, -4.3) (6, 0)(8, -1.7)(9.2, -3.4) (8, -1.7)(9.5, -1) (4.6, 1)(6, 0)(7, 1) [dotscale = .65](-1.5, -1.9)(-1, -1.8)(-.5, -1.64) [dotscale = .65](5.1, 1)(5.8, 1)(6.5, 1) [dotscale = .65](9.275, -2.9)(9.35, -2.2)(9.425, -1.5) [dotscale = .65](3.5, -4.075)(4, -4.15)(4.5, -4.225) [dotscale = .65](6, -4.15)(6.5, -4.2)(7, -4.25) [dotscale = .65](6.5, -1.9)(7, -1.8)(7.5, -1.7)

Proof. If $ds_{ns}(G) = p - 1$ then there exists at least one vertex $v \in V(G)$ such that the only minimal nonsplit dominating set containing v is of cardinality $p - 1$.

Case(i) : v is a pendent vertex.

In this case, we have $\gamma_{ns}(G) = p - 1$ by choice of v. Hence by Proposition 2.1 $G \cong G_1$.

Case (ii) : v is a non-pendent vertex.

Let $N(v) = \{v_1, v_2, \ldots, v_k\} (k \geq 2)$. If there exists an edge $(v_i, v_j) \in < N(v) >$, $(1 \leq i, j \leq k)$ then $V(G) - \{v_i, v_j\}$ is a nonsplit dominating set containing v and so $< N(v) >$ is independent.

We now claim that every vertex in $V(G) - N[v]$ is a pendent vertex. Suppose there exists $u \in V(G) - N[v]$ such that $d(u) \geq 2$. Since G is connected, there exists a $u - v$ path P with length at least 2. Let $w \in N(u) \cap P$. Then $V(G) - \{u, w\}$ is a nonsplit dominating set containing v and hence $G \cong G_2$.

Converse is obvious.

The following is immediate.

Corollary 2.6 *Let G be any graph. Then $ds_{ns}(G) = p - 1$ if and only if every component of G is isomorphic to any one of the graphs in Fig. 1.*

Theorem 2.7 *For any tree T, $ds_{ns}(\overline{T}) = \gamma_{ns}(\overline{T}) = 2$ if and only if T is not isomorphic to $B(r, s)$ where at least one of r or s equals 1.*

Proof. Suppose $T \cong B(r, s)$ where $r = s = 1$. Then $T \cong P_4$ and $\gamma_{ns}(\overline{P_4}) = 2$. But $ds_{ns}(\overline{P_4}) = 3$. Hence T ® $B(r, s)$ where $r = s = 1$. If $T \cong B(r, s)$ with exactly one of $\{r, s\}$ having value 1, then there is no γ_{ns}-set of $\overline{T}$ of cardinality 2 containing u. These contradictions exhibit that T is not isomorphic to $B(r, s)$ where at least one of r and s equals 1.

Conversely assume that T is a tree not isomorphic to $B(r, s)$ where at least one of r and s equals 1. If $T \cong K_{1,p-1}$ then $\gamma_{ns}(\overline{T}) = 2 = ds_{ns}(\overline{T})$. If T ® $K_{1,p-1}$, then there exists at least 2 pendent vertices u and

v with distinct supports u_1 and v_1 respectively such that $deg(u_1) \le p-3$ and $deg(v_1) \le p-3$.

Case(i) : $deg(u_1) = p-3$ and $deg(v_1) = p-3$.

If u_1 and v_1 are adjacent then $T \cong T_1$ where T_1 is given in Fig. 2.

(3, -1)(3, 1) [dotscale = 1.5](0, 0)(2, 0)(-1.5, -1)(-1.5, 1)(3.5, 1)(3.5, -1) (-1.5, 1)(0, 0)(2, 0)(3.5, 1) (-1.5, -1)(0, 0) (2, 0)(3.5, -1)

$\{v, v_1\}, \{v_2, v_1\}, \{u, u_1\}, \{u_2, u_1\}$ are all minimum nonsplit dominating sets of $\overline{T}$ and so $\gamma_{ns}(\overline{T}) = ds_{ns}(\overline{T}) = 2$. If u_1 and v_1 are non-adjacent then $T \cong P_5$ and so $\gamma_{ns}(\overline{T}) = ds_{ns}(\overline{T}) = 2$.

Case(ii) : $deg(u_1) = p-3$ and $deg(v_1) \ne p-3$.

If u_1 and v_1 are adjacent, then $T \cong T_2$ where T_2 is given in Fig. 3.

(-1, -1)(3, 0) [dotscale = 1.5](-2, 0)(2, 0)(2, -1.5)(-1, -1.5)(-3, -1.5)(3.5, -.7)(.2, -1.5) (2, -1.5)(2, 0)(-2, 0)(-1, -1.5) (-2, 0)(-3, -1.5) (2, 0)(3.5, -.7) (-2, 0)(.2, -1.5) [dotscale = .65](-.7, -1.5)(-.4, -1.5)(-.1, -1.5)

Since $d(v_1) \ne p-3$, $d(v_1) \ge 4$. For every $u' \in N(u_1)$, $\{u_1, u'\}$ is a γ_{ns}- set of $\overline{T}$ and for every $v' \in N(v_1)$, $\{v_1, v'\}$ is a γ_{ns}-set of $\overline{T}$ and so $\gamma_{ns}(\overline{T}) = ds_{ns}(\overline{T}) = 2$. If u_1 and v_1 are non-adjacent then $T \cong T_3$ where T_3 is given in Fig. 4.

(-1.5, -1)(3, .5) [dotscale = 1.5](-2, 0)(0, 0)(2, 0)(2, -1.5)(-1, -1.5)(-3, -1.5) (2, -1.5)(2, 0)(0, 0)(-2, 0)(-1, -1.5) (-2, 0)(-3, -1.5) [dotscale = .65](-1.5, -1.5)(-2, -1.5)(-2.5, -1.5)

As above $deg(u_1) \ge 3$. For every $u' \in N(u_1)$, $\{u', u_1\}$ is a γ_{ns}- set of $\overline{T}$. Also $\{u_1, v\}$ and $\{v_1, u\}$ are γ_{ns}- set of $\overline{T}$ and so $\gamma_{ns}(\overline{T}) = ds_{ns}(\overline{T}) = 2$.

Case(iii) : $deg(u_1) \ne p-3$ and $deg(v_1) = p-3$.

This is analogous to case(ii).

Case(iv) : $deg(u_1) \ne p-3$ and $deg(v_1) \ne p-3$.

If u_1 and v_1 are adjacent then $deg(u_1) \ge 4$ and $deg(v_1) \ge 4$ and for every $u' \in N(u_1)$, $\{u_1, u'\}$ is a γ_{ns}-set of $\overline{T}$ and for every $v' \in N(v_1)$, $\{v_1, v'\}$ is a γ_{ns}-set of $\overline{T}$ so that $ds_{ns}(\overline{T}) = \gamma_{ns}(\overline{T}) = 2$.

Suppose u_1 and v_1 are non-adjacent. Then $deg(u_1) \ge 3$ and $deg(v_1) \ge 3$. For every $x \in V(T)$ with $d(u_1, x) \ne 2$, $\{x, u_1\}$ is a γ_{ns}-set of $\overline{T}$ containing x and if $d(u_1, x) = 2$, $\{x, u\}$ is a γ_{ns}-set of $\overline{T}$ containing x. The γ_{ns}-sets containing neighbours of u_1 and v_1 are as above. Thus $ds_{ns}(\overline{T}) = \gamma_{ns}(\overline{T}) = 2$.

Theorem 2.8 *There exists a graph* G *for which* $ds_{ns}(G) - ds(G)$ *can be made arbitrarily large.*

Proof. Let $P_{p-k} = \{u_1, u_2, \ldots, u_{p-k}\}$ be a path on $p-k$ vertices where $1 \le k \le p-1$ and let $S = \{v, v_1, v_2, \ldots, v_{k-1}\}$. Join the vertex v to each of the vertices in P_{p-1} and to each vertex in $S - \{v\}$. The resulting graph G is of order p and $\gamma(G) = 1$. Also $\{v, u_i\}(1 \le i \le p-k)$ and $\{v, v_j\}(1 \le j \le k-1)$ are minimal dominating sets containing u_i, u_j respectively so that $ds(G) = 2$.

S is a minimum nonsplit dominating set of G and so $\gamma_{ns}(G) = k$. If $k = p-1$ or $p-2$, $ds_{ns}(G) = k$. Suppose $k \le p-3$. $S \cup \{u_1\}$, $S \cup \{u_{p-k}\}$, $(S - \{v\}) \cup \{u_2, u_5, \ldots\}$, $(S - \{v\}) \cup \{u_1, u_3, u_6, \ldots\}$ and $(S - \{v\}) \cup \{u_1, u_4, u_7, \ldots\}$ are all nonsplit dominating sets of G and so

$ds_{ns}(G) = k + \left\lfloor \frac{p-k}{3} \right\rfloor$ or $k + \left\lceil \frac{p-k}{3} \right\rceil$ according as $p-k \equiv 0,1(mod\, 3)$ or $p-k \equiv 2(mod\, 3)$. Thus $ds_{ns}(G) - ds(G) = k + \left\lfloor \frac{p-k}{3} \right\rfloor - 2$ or $k + \left\lceil \frac{p-k}{3} \right\rceil - 2$ where k can be chosen arbitrarily large.

Theorem 2.9 *For any connected graph* G, $ds_{ns}(G) + diam(G) \le 2p-2$ *and equality holds if and only if* $G \cong P_p (p \le 5)$.

Proof. Since G is connected, $diam(G) \le p-1$. Always $ds_{ns}(G) \le p-1$ and so $ds_{ns}(G) + diam(G) \le 2p-2$. Suppose $ds_{ns}(G) + diam(G) = 2p-2$. Then $ds_{ns}(G) = p-1$ and $diam(G) = p-1$. Since $ds_{ns}(G) = p-1$, by Theorem ?? we observe that $diam(G) \le 4$ and so $p \le 5$. For any graph on p vertices other than P_p we have $ds_{ns}(G) + diam(G) \ne 2p-2$ and so $G \cong P_p (p \le 5)$. Converse is obvious.

Theorem 2.10 *For any connected graph* G *with at least two pendent vertices,* $ds(G) + ds(\overline{G}) \le ds_{ns}(G) + ds_{ns}(\overline{G}) \le p+2$. *Also the bounds are sharp.*

Proof. For any graph G, $ds(G) \le ds_{ns}(G)$, $ds(\overline{G}) \le ds_{ns}(\overline{G})$ and so $ds(G) + ds(\overline{G}) \le ds_{ns}(G) + ds_{ns}(\overline{G})$. Always $ds_{ns}(G) \le p-1$. To establish the upper bound it is enough to prove that $ds_{ns}(\overline{G}) \le 3$. Let $P = \{u_1, u_2, \ldots, u_m\}$ be the set of pendent vertices of G and $S = \{v_i \ (1 \le i \le m)\}$ be the set of corresponding supports (not necessarily distinct). If $m \ge 3$ and there exists an index i such that $\{V(G) - \{u_i, v_i\}\}$ has two distinct supports then $A = \{u_i, v_i\}$ is a nonsplit dominating set of $\overline{G}$. If w is the unique support in $< V(G) - \{u_i, v_i\} >$ then $\{u_i, v_i, w\}$ is a nonsplit dominating set of $\overline{G}$. Otherwise v_i is the only support of G and $A = \{u_i, v_i\}$ is a nonsplit dominating set of $\overline{G}$. For every other vertex x, $A \cup \{x\}$ is nonsplit dominating set of $\overline{G}$. Hence $ds_{ns}(\overline{G}) \le 3$.

Suppose $m = 2$. Let the two pendent vertices be u and v with supports u_1 and v_1 respectively.

Case(i) : $u_1 = v_1$

Let $D = V(G) - \{u, v, u_1\}$. If $D = \phi$ then $\{u, v_1\}$ and $\{v, v_1\}$ are nonsplit dominating sets of $\overline{G}$. If $D \ne \phi$ then $\{u, v_1\}$, $\{v, v_1\}$ and $\{u, v_1, x\}$ [where $x \in V(G) - \{u, v, u_1\}$] are nonsplit dominating sets of $\overline{G}$.

Case(ii) : $u_1 \ne v_1$.

If $(u_1, v_1) \notin E(G)$ then $\{v, u_1\}, \{u, v_1\}$ and $\{u, v_1, x\}$ [where $x \in V(G) - \{u, v, u_1, v_1\}$] are nonsplit dominating sets of $\overline{G}$. Suppose $(u_1, v_1) \in E(G)$ and let $B = V(G) - \{u, v, u_1, v_1\}$. If $B = \phi$ then $\{v, v_1, u_1\}$ and $\{v_1, u_1, u\}$ are nonsplit dominating sets of $\overline{G}$. Suppose $B \ne \emptyset$. If $|B| \ge 2$ then $\{v, v_1\}, \{u, u_1\}, \{x, u_1\} (x \in B)$ are nonsplit dominating sets of $\overline{G}$. If $|B| = 1$ and $B = \{w\}$ then $\{u_1, v_1, w\}$, $\{u, u_1, v_1\}, \{u_1, v_1, v\}$ are nonsplit dominating sets of $\overline{G}$. Hence $ds_{ns}(\overline{G}) \le 3$. Thus $ds_{ns}(G) + ds_{ns}(\overline{G}) \le p+2$.

Lower bound is attained for K_2 and upper bound for P_4. Hence the bounds are sharp.

Definition 2.11 *Let* $G = (V, E)$ *be a graph. The maximum order of a partition of* V *into nonsplit* *dominating sets of G is called the nonsplit domatic number of G and is denoted by* $d_{ns}(G)$.

Definition 2.12 *A graph G with $d_{ns}(G) = \delta(G)+1$ is said to be nonsplit domatically full.*

Theorem 2.13 *If G is a k -regular graph which is nonsplit domatically full then* $\gamma_{ns}(G) = ds_{ns}(G)$.

Proof. Since G is nonsplit domatically full, $d_{ns}(G) = k+1$. Let $\{D_1, D_2, \ldots, D_{k+1}\}$ be a nonsplit domatic partition of G. Any set D_i either contains a vertex u or exactly one of its neighbours. Hence, each D_i is independent. Also, for all $1 \le j \le k+1$, $i \ne j$, every vertex in D_i is adjacent to exactly one vertex in D_j. Hence all sets D_i are of equal cardinality and $|D_i| = \gamma_{ns}(G)$. Hence $\gamma_{ns}(G) = ds_{ns}(G)$.

Remark 2.14 *The converse of theorem 2.13 is not true. The 3-regular graph G given in Fig.5 is not nonsplit domatically full.*
(-5, -1)(3, 2) [dotscale = 1.5](-3.5, 0)(4, 0)(-1.5, 1)(1.5, 1)(-1.5, -1)(1.5, -1) (-3.5, 0)(4, 0) (-3.5, 0)(-1.5, 1)(1.5, 1)(4, 0) (-3.5, 0)(-1.5, -1)(1.5, -1)(4, 0) (-1.5, 1)(-1.5, -1) (1.5, 1)(1.5, -1)

We observe that $ds_{ns}(G) = \gamma_{ns}(G) = 2$.

References

[1]. B. D. Acharya, The strong domination number of a graph and related concepts, J.Math. Phys.Sci. 14(1980), No. 5, 471-475.
[2]. S. Arumugam and R. Kala, Domsaturation number of a graph, Indian J. Pure appl. Math., 33(2002), No. 11, 1671-1676.
[3]. J. R. Carrington, F. Harary and T. W. Haynes, Changing and unchanging the domination number of a graph, J. Combin. Math. Combin. Comput., 9(1991), 57-63.
[4]. E. J. Cockayne, Domination of undirected graphs - A survey. In theory and Applications of Graphs. LNM 642, Springer - Verlag, (1978), 141-147.
[5]. E. J. Cockayne and S. T. Hedetniemi, Towards a theory of domination in graphs, Networks, 7, (1977), 247-261.
[6]. F. Harary, Graph theory, Addison Wesley, Reading Mass (1969).
[7]. T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Fundamentals of Domination in graphs, Marcel Dekker, Inc., (1998).
[8]. T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Domination in Graphs - Advanced Topics, Marcel Dekker, (1998).
[9]. V. R. Kulli and B. Janakiram, The nonsplit domination number of a graph, Indian J. Pure Appl. Math., 31 (2000), No. 4, 545-550.
[10]. Y. Therese Sunitha Mary and R. Kala, The nonsplit domination number of a graph, Proceedings of the National Conference on Mathematical and Computational Models, December 2007, PSG College of Technology, Coimbatore, India, 218-224.