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Variational Finite Element Approach to Study Radial Heat Distribution Problem in Human Limbs

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***Abstract*:** *This paper is based on bio heat transfer equation and demonstrates its application in solving one dimensional physiological heat distribution problems pertaining to limbs. The biological properties are assumed to vary along the radial direction. The dermal region is made up of three layers, namely epidermis, dermis, and subcutaneous tissues. The model incorporates significant variations of physical and physiological parameters like blood mass flow rate, rate of metabolic heat generation, and thermal conductivity in each layer. Numerical results have been obtained for various cases of practical interest.*
Keywords: *Rate of metabolism 1, blood mass flow rate 2, thermal conductivity 3, heat generation 4, finite element method 5***.**

I. Introduction

The one dimensional models for heat distribution in limbs have wide scope as the core temperature of the human limbs varies extensively at lower atmospheric temperatures. For falling atmospheric temperature, the body core shrinks rapidly and isotherm shells in the limb change their respective position gradually. This may be due to the fact that the arterial blood has cooled down while flowing towards the extremities. Also the two opposite side of inner core of a human limb may be at different temperature. This may be because one side of the limb contains major blood vessels (arteries) with blood coming from main trunk at body core temperature and the outer side of the limb contains veins with blood returning from the extremities at lower temperature.

Perl[1] combined differential forms of Fick’s perfusion principle with heat conduction and matter diffusion equations and metabolic term to obtain equation. He used equation to solve its simple cases by taking all parameters as constant throughout the region. Perl and Hirsch[2] used this equation to test the transient response for measuring local tissue blood flow on dog and rabbit kidney. Trezek and Cooper[3] computed thermal conductivity of tissue by taking all parameters as constant. Cooper and Trezek [4] obtain solution of equation in SST region by taking all parameters as constant. Patterson[5] made experimental attempts, to determine temperature profiles in skin and subcutaneous region.

This paper employs a variational finite element approach to study the temperature distribution in a normal cross-sectional region of a limb. Due to unsymmetric situations of large blood vessels passing through the core of the limb the inter-face has angular variation. The peripheral part of limb is directly exposed to atmosphere..Different types of variations of parameters have been considered for different natural subregions such as stratum corneum, stratum germinitivum, dermis and underlying tissue (Montagana [6], Jarrett [7]and Gray [8]). Finite element formulation provides necessary flexibility in taking care of different behavior of distinctly different subregions.

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II. Material and Methods

The heat flow in SST region is given by the following partial differential equation

$$div\ (K\ grad\ T) + m_b\ C_b\ (T_b - T) + S = \rho\bar{c}\frac{\partial T}{\partial t} \qquad (2.1)$$

Where T and T_b are Temperature and Body core temperature ,S is Rate of metabolic heat generation ,m_b is Blood mass flow rate in the tissue , ρ is density of the tissue.C_b=Specific heat of blood ,K is thermal conductivity of the tissue. Above equation has been modified and extensivelyused by Saxena [9], Saxena and Arya [10], Saxena and Bindra [11,12] in the thermal study of human skin andsubcutaneous tissue.

In this, skin and underlying tissues of cylindrical regions such as limbs of a human are divided into four annualar layers. They can be considered as multi layered regions. Each layer has different physical and physiological properties. The outer body surface is exposed to the environment and heat loss from the body surface takes place due to conduction ,convection, radiation and evaporation. Here we employ for a human limb with circular symmetry. The properties and temperature distribution are assumed to be uniform along θ and Z directions. Thus the equation reduces in one dimensional unsteady state case for each layer to the following cylindrical from.

$$\frac{1d}{r\,dr}\left(k^{(i)}r\,\frac{dT_j}{dr}\right)+M_i\,\left(T_b-T_i\right)+S_i=\rho\overline{C}\,\frac{\partial T}{\partial t},\qquad i=1\ to\ 4$$

Where $K^{(i)}$, M_i, S_i and T_i denote the values of K, M, S and T in i the sub-region.

Boundary and Interface Conditions:

In view of continuity of temperature and temperature gradient in various sublayers, the following boundary and interface conditions can be formulated

(i) $T^{(1)} = T^{(2)}$ at $r = a_1$

(ii) $K_1\,\frac{dT^{(1)}}{dr} = K_2\,\frac{dT^{(2)}}{dr}$ at $r = a_1$

(iii) $T^{(2)} = T^{(3)}$ at $r = a_2$

(iv) $K_2\,\frac{dT^{(2)}}{dr} = K_3\,\frac{dT^{(3)}}{dr}$ at $r = a_2$

(v) $T^{(3)} = T^{(4)}$ at $r = a_3$

(vi) $K_3\,\frac{dT^{(3)}}{dr} = K_4\,\frac{dT^{(4)}}{dr}$ at $r = a_3$

It is assumed that at the outermost layer ($r = a_0$) the heat is lost to the environment by conduction, convection, radiation and evaporation. Therefore at this layer we take

$$K_1\,\frac{dT}{dr} = -h\,\left(T-T_A\right)+LE$$

Where h is coefficient of convection,L is latent heat of evaporation and E is the Rate of sweat evaporation T_A is the atmospheric temperature.At the innermost layer the temperature will be same as that of the body core. Hence the boundary conditions will be$T_4 = T_b$ at $r = a_4$

III. Solution of the Problem:

The variational form is defined in the region as I_1, I_2, I_3 and I_4 respectively for stratumcorneum, stratumgermativum, dermis and subdermal parts. Assigning the values to T as T_i (i = 0, 1, 2, 3, 4) called as nodal values. The distance between the nodal values is given by a_i (I = 0, 1, 2, 3, 4) from the outermost layer of epidermis to innermost layer of subdermal layer.Let $T^{(i)}$ (i = 1, 2, 3, 4) denote the linear values of T (r) for $a_i < r < a_{i-1}$

Now, applying the shape function to approximate the solution of the problem.

$$T^{(i)} \approx A_i + B_i\,r \qquad \text{for} \qquad a_i < r < a_{i-1}$$

Where,

$$A_i = \frac{T_i - T_{i-1}}{a_i - a_{i-1}},\qquad B_i = \frac{a_{i-1}\,T_i - a_i T_{i-1}}{a_{i-1} - a_i}\quad for\qquad i = 2,3,4$$

Thus the equation in one dimensional unsteady state case for each layer to the following cylindrical form

$$\frac{1}{r}\,\frac{d}{dr}\left(Kr\,\frac{dT}{dr}\right)+M\left(T_b-T\right)+S=\rho\overline{C}\,\frac{\partial T}{\partial t},$$

After camparing with Euler's Lagrange's equation, we get the varitional forms

$$I = \frac{1}{2}\int_{a_i}^{a_i-1}\left[K_1 r\left(\frac{\partial T^i}{\partial t}\right)^2 + rm_b C_b\,\left(T^i-T_b\right)^2 - 2s^{(i)}\,rT^{(i)} + \bar{\rho c}r\,\frac{\partial T(i)^2}{\partial T}\right]dr$$

$$+\frac{1}{2}h\left(T^{(c)}{}_{(1)}-T_a\right)^2 + LET_{(i)}$$

We have,

I = $I_1 + I_2 + I_3 + I_4$

Clearly,

I = f(T_0, T_1, T_2, T_3, T_4)

Now minimizing I with respect to parameters T_0, T_1, T_2, T_3, therefore

$$\frac{\partial I}{\partial T_0}+\frac{\partial I}{\partial T_1}+\frac{\partial I}{\partial T_2}+\frac{\partial I}{\partial T_3}=0$$

Finally taking La-place transform, we get four non homogenous simultaneous equations as

$x_1 T_0 + y_1 T_1 = n_1$

$x_2 T_0 + y_2 T_1 + z_2 T_2 = n_2 = n_2$

$y_3 T_1 + z_3 T_2 + w_3 T_3 = n_3$

$z_4 T_2 + w_4 T_4 = n_4$

solving for $\overline{T_0}, \overline{T_1}, \overline{T_2}, \overline{T_3}$ by using matrix method, we get the values of $\overline{T_0}, \overline{T_1}, \overline{T_2}, \overline{T_3}$ in the form of polynomials

$$\left[\begin{array}{cccc|c} x_1 & y_1 & 0 & 0 & \eta_1 \\ x_2 & y_2 & z_2 & 0 & \eta_2 \\ 0 & y_3 & z_3 & \omega_3 & \eta_3 \\ 0 & 0 & z_4 & \omega_4 & \eta_4 \end{array}\right]$$

$$\left[\begin{array}{ccccc} 1 & y_1/x & 0 & 0 & \dfrac{\eta_1}{x_1} \\ 0 & 1 & \dfrac{x_1 z_2}{y_2 x_1 - x_2 y_1} & 0 & \dfrac{\eta_2 x_1 - x_2\eta_1}{y_2 x_1 - x_2 y_1} \\ 0 & 0 & 1 & \dfrac{w_3(y_2 x_1 - x_2 y_1)}{z_3 y_2 x_1 - z_3 x_2 y_1 - x_1 z_2 y_3} & \eta_3 y_2 x_1 - \eta_3 x_2 y_1 \quad \dfrac{-y_3\eta_2 x_1 + y_3 x_2 \eta_1}{z_3 y_2 x_1 - z_3 x_2 y_1 - x_1 z_3 y_3} \\ 0 & 0 & 0 & 1 & 1 \quad \dfrac{\eta_4 z_3 y_2 z_1 - \eta_4 z_3 x_2 y_1 - \eta_4 x_1 z_2 y_3 + \eta_2 z_4 x_1 y_3 - \eta_1 z_4 y_3 x_2}{w_4 z_3 y_2 x_1 - w_4 z_3 x_2 y_1 - w_4 x_1 z_2 y_3 - z_4 w_3 y_2 x_1 + z_4 w_3 x_2 y_1} \end{array}\right]$$

$$\overline{T_1} = \frac{X_i(p)}{Y_1(p)}$$

Where x_i (p) is a polynomial of degree less than n that of Y_j (P)i.e.,(n-1)

The value of nodal temperature $T_0 T_1 T_2 T_3$, can be obtained by taking the inverse La-place transform

$$T_1 = \sum_{n=1}^{N} \frac{X_1(p_N)}{y_1(p_N)} \mathbf{e}^{pnt}$$

Therefore, we can see that nodal values are dependent on time.

IV. Tables

The numerical result have been obtained with the help of following values –

Thermal Conductivity (cal/cm min ºC)	Heat Transfer Coefficient h (cal/cm^2 min ºC)	Specific Heat of Tissues c (cal/gm ºC)
K_1=0.060, K_2=0.045, K_3=0.030	0.009	0.830
Blood Density of Tissues ρ (gm/cm^3)	Latent Heat L (cal/gm)	Body Core Temperature T_b (ºC)
1.090	579.0	37

The numerical result have been computed for three case of atmospheric temperatures T = 15 °C , 23°C and 33°C. The following sets of numerical values have been taken and graphical representation obtained for temperature distribution in SST region.The three different set of values for thickness of layers in SST region are taken for a_0 , a_1 , a_2 , a_3 and a_4.

Thickness of Skin	a_0	a_1	a_2	a_3	a_4
Set I	8	7.5	7	6	5.5
Set II	7.5	7	6	5	4.5
Set III	7	6	6.5	5	5.5

V. Conclusion

Different $T^{(i)}$ (i = 0 , 1, 2, 3) vs time t for different thickness of skin and different values of atmospheric temperatures have been calculated On comparing the nodal temperature on outer surface and in each sub region, it is observed that these nodal temperatures vary considerably with the change in atmospheric temperatures and rate of sweat evaporation..

Obviously sharpness of gradient is more marked in the epidermal sub- layer which is directly exposed to environment. Moreover blood flatter in the dermal and sub dermal sub layers.

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