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Abstract: *Spectrum of pi is broadened in this article and made to be applicable to regular polygons. By this, several equations of area and volumes are generalized. Also, a new method is suggested to find the value of arc trigonometric functions like arcsin.*
Keywords: *apothem, arcsin, cylinder, pi, pyramid,volume.*

I. Introduction

Angles are defined with a little new view in the II nd section. The words apothem and hypotenuse are newly included into the family of radius. An advanced definition is furnished for pi. Equations for perimeter and areas of regular polygons formed at this section resembles the equations of circle for perimeter and area.

Several triangles come into mind while studying the trigonometric functions w.r.t a circle which leads to confusion to the students. To avoid such confusion, required triangle for trigonometric functions is designated as determinant triangle in section III.

Horizon of trigonometric function is expanded in the IVth section. In this section trigonometric functions are derived for equilateral triangle. Sine graph is drawn at section V w.r.t the sine function of equilateral triangle.

And at sections VI and VII cosine function is derived and corresponding graph is drawn for equilateral triangle.

Validity of trigonometric function for negative values is checked at section VIII.

At sections IX and X expressions for tangent function and its graph are derived and drawn.

A generalized proof for trigonometric identities are furnished at sections XI and XII.

Trigonometric functions are derived w.r.t square at section XIII. Here pi-en takes an integral value i.e., 4.

And the corresponding graph is also drawn.

General equations are derived for sides of the determinant triangles of regular polygons.

A new method is proposed at section XV to find the value of arc trigonometric functions.

II. Identical lines in a regular polygon.

In a regular polygon of n number of sides, infinite number of lines can be drawn from its centre to the perimeter. All regular polygons are dividable into congruent triangles w.r.t the sides of the regular polygon, Fig (1). And each such triangle is further dividable into two congruent right triangles. i.e., Line $OP_1, OP_2, OP_3 \cdots$ divide the triangle $AOB, BOC, COD \cdots$, each into two right congruent triangles AOP_1, $BOP_2, COP_3 \cdots$. In the right triangle P_1OB, OB which is the line joining the centre and a vertex is the longest hypotenuse among all hypotenuses that can be drawn from O to the neighborhood of apothem OP_1. In the triangle AOB, OP_1 is apothem and P_1B is half of the side AB and OB is largest hypotenuse. All lines within this right triangle are unique in length. Also all lines formed by joining O and up to the neighborhood of apothem are hypotenuses. Further, all hypotenuses and apothems are collectively called as radii.

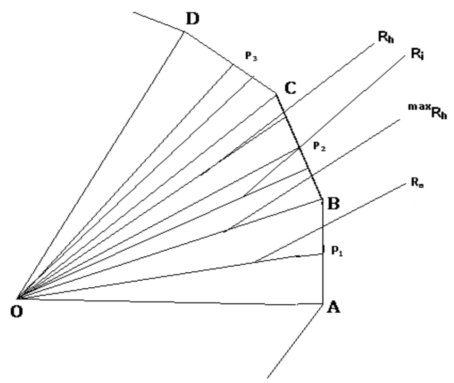


Fig.(1): Showing the corresponding or equivalent lines

Among all lines of these right triangles three border lines are significant lines. They are longest hypotenuse, apothem and half of a side. For example opposite side of the right triangle P_1OB is one of such lines. Also, all lines have corresponding lines in other triangles. For example, $OA = OB = OC = \cdots$ and $OP_1 = OP_2 = OP_3 = \cdots$ and so on.
i.e., OA corresponds to OB , OC, etc and OB corresponds to OA, OC etc and so on.
Generally, all radii can be denoted by R_i^θ and all apothem can be denoted by R_a and hypotenuses by R_h^θ , where i denotes a radius in the set of radii and θ defines the positional uniqueness also h designates the line as hypotenuse.

Mathematically magnitute of , $R_h^\theta = \frac{\text{Apothem}}{\cos\theta}$

$$= \frac{R_a}{\cos\theta} ---------- (1)$$

Generally, limit of variation of θ can be shown by the inequality as, $\left(0 < \theta \le \frac{\pi}{n}\right]$

For example, in an equilateral triangle , angle varies as, $\left(0 < \theta \le \frac{\pi}{3}\right]$, and if $\theta = \frac{\pi}{3}$, we have,

$$R_i^\theta = \frac{R_a}{\cos\left(\frac{\pi}{n}\right)} ------(2) = \frac{R_a}{\cos\left(\frac{\pi}{3}\right)} = 2\,R_a$$

And for a square, $R_h^{\frac{\pi}{4}} = \sqrt{2}R_a$
And so on.

Further, $\lim_{\theta\to 0} R_h^\theta = \lim_{\theta\to 0}\frac{R_a}{\cos\theta}$
$= R_a$; since $\cos 0 = 1$
Therefore, $R_i^\theta \in \{R_h, R_a\}$

2.0.1.Definitions of significant lines in a regular polygon.

Radii: Unique Lines w.r.t angle , drawn from centre of a regular polygon to the perimeter of the polygon are radii. Actually, said lines from a vertex to the centre of the side are unique only in a chosen right triangle and rest are corresponding radii.

A set of unique hypotenuses and apothem, is furnished below

Mathematically, $R_i^\theta \in \{\text{unique } R_h\}$; Where, $\left(0 < \theta \le \frac{\pi}{n}\right]$

$\in \left\{\text{unique } R_h^\theta\right\}$ $\qquad \in \left\{\text{unique } \left(\frac{R_a}{\cos\theta}\right)\right\}$ Also, $\lim_{\theta\to 0} R_h = R_a$

For example when regular polygon is a square, Fig(2) and $n = 4$, θ will be $= \frac{\pi}{4}$,
In the $\triangle OP_1B$ unique radii of square are all unique lines from OB to OP_1, among them OB and OP_1 are significant lines which are largest hypotenuse and apothem.

$$R_i^\theta \in \left\{\text{unique } \left(\frac{R_a}{\cos\theta}\right) \text{ and } R_a\right\}; \text{Here, } \left[0 \le \theta \le \frac{\pi}{3}\right] ------(3)$$

All such lines in the remaining right triangles are corresponding radii.

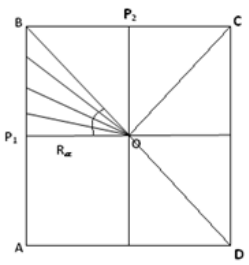


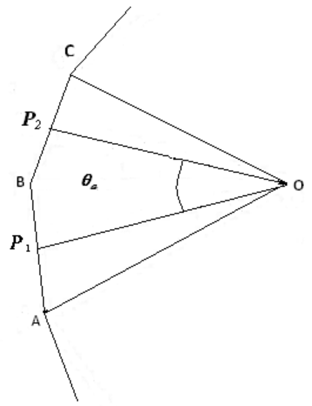
Fig.(2): radii in a square from B to P_1

2.0.2. Apothem-

It is known that apothem is the line joining the centre of a regular polygon to the midpoint of its side. Or it is the perpendicular drawn from the center of the regular polygon, on to one of its sides. Also it is the perpendicular bisector of a side and the corresponding angle subtended by the side at the centre of the regular polygon. It is denoted by R_a.

2.0.3.Di-apothem-

Di-apothem is any two apothems of a regular polygon. They are identifiable by their angle.
If di − apothem is denoted by D_a^θ, then Length of Di − apothem, $D_a^\theta = 2R_a$
Angle between apothems of diapothem are variable from 0 to π, w. r. t regular polygon.
considering from clockwise and anticlockwise direction whichever is least, is the angle of di − apothem.
Apothems that are immediately adjecent to one another will have least angle between them,
i. e., The least angles between apothems of diapothem can be given by, $\angle D_{an}^1 = \frac{2\pi}{n}$; fig: (2)
Least angle of the apothen is equal to the angle subtended by a side at the centre.



Fig(3): showing the angle of di-apothem

When n is an odd number angle of di apothems, $\angle D_{an}^{\left(1 \text{ to } \frac{(n-1)}{2}\right)}$ will be,

$$1 \times \frac{2\pi}{n}, 2 \times \frac{2\pi}{n}, 3 \times \frac{\pi}{n}, \cdots, \frac{(n-1)}{2} \times \frac{2\pi}{n}$$

Here, angle of maximum angled di − apothem of odd number of sides is, $\angle D_a^{\left(\frac{n-1}{2}\right)} = \frac{(n-1)}{2} \times \frac{2\pi}{n} < \pi$

Instead it will approximately be equal to π when n→∞,

i. e., $\lim_{n\to\infty} \angle D_a^{\left(\frac{n-1}{2}\right)} = \lim_{n\to\infty} \frac{(n-1)}{2} \times \frac{2\pi}{n} = \pi$

And when n is an even number, angles of diapothems, $\angle D_a^{\left(1 \text{ to } \frac{n}{2}\right)}$ will be, $1 \times \frac{2\pi}{n}, 2 \times \frac{2\pi}{n}, 3 \times \frac{\pi}{n}, \cdots, \frac{n}{2} \times \frac{2\pi}{n}$

By the above expression it is clear that angle of maximum angled di-apothem of regular polygon having even number of sides will always be equal to π

i. e, angle of maximum angled diapothem, $\angle D_a^{\left(\frac{n}{2}\right)} = \pi$, when n is an even number.

2.0.4.Specific angle Di-apothems-

Di-apothems that possess maximum angle, minimum angle or any identical or remarkable angle are called Specific angle Di-apothems.

2.0.5. Maximum angled Di-apothem -

This is one of the Specific Di-apothems that has maximum angle between the two apothems. Also, it is denoted by D_a^{max}. The length of $D_a^{max} = 2R_a$

And, Angle of D_a^{max} when n is even number is $= \angle D_a^{\left(\frac{n}{2}\right)} = \pi$

Also, Angle of D_a^{max} when n is odd number is $= \angle D_a^{\left(\frac{n-1}{2}\right)} = \frac{(n-1)}{2} \times \frac{2\pi}{n} = \frac{\pi(n-1)}{n}$

2.0.6. Minimum angled Di-Apothem-

Two apothems that have least angle between them are called Minimum angled Di-Apothem. Immediately adjacent apothems will have least angle between them.

Although, here Maximum angled Di-apothem is taken to relate it with diameter of a circle. Each apothem can be designated w.r.t its corresponding side. Corresponding side is the side from which corresponding apothem is drawn.

Apothem is one of the significant reference lines which is helpful in finding the relation between perimeter and the line within the regular polygon.

When, the number of sides of the regular polygon tends to infinity, then the apothem becomes conventional radius of the circle i.e., all lines from Maximum angled hypotenuse to apothem merge with apothem . And the angle of Maximum angled Di-apothem will be equal to a straight angle.

2.0.7. Di-angle bisectors or Hypotenuse of maximum-length or angle -

The line joining the centre and the vertex divides the interior angle at the vertices of a regular polygon and also divide the angle formed by the apothems that are immediately adjacent lines on both sides. Hence they divide two angles. Also, it divides the angle between corresponding hypotenuses on either side of it. Therefore they can be called as di-angle bisectors. Here OB is a di-angle-bisector. it bisects the angles B and P_1OP_2 Fig(1). The apothems that are immediately adjacent on both sides are P_1O and P_2O. Also it divides the angles of corresponding hypotenuses adjacent on either sides. This di-angle bisector can also be called as hypotenuse of maximum length or angle, because they have maximum angle and hence they are longest hypotenuses.

2.0.8. Duo-di-angle bisectors or Duo-Di-Hypotenuse of maximum-length or angle

Two di-angle bisectors or Di-Hypotenuse of maximum-length or angle in a regular polygon can be called as Duo-di-angle bisector or Duo-Di-Hypotenuse of maximum-length or angle.

Out of these Duo-di-angle bisectors, specific Duo-di-angle bisectors are isolatable.

Maximum angled Duo-di-angle bisectors are sub set of specific Duo-di-angle bisectors.And these Duo-di-angle bisectors are nothing but longest di-hypotenuses with maximum angles.

This can be denoted by D^{h}_{max}

In Fig.(2), BOD is a Duo di-angle bisector or duo di-hypotenuse of maximum length or angle. Also this line is conventionally called as diagonal.

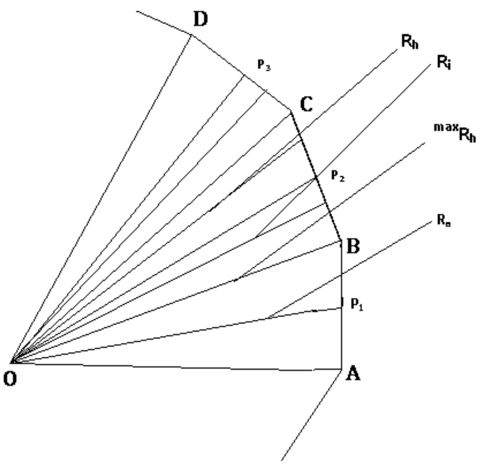


Fig.4. Showing the different kinds of lines.

Further, it can also be called as diameter of maximum length

When the sides of a regular polygon are increased indefinitely or when the sides of a polygon tends to a point , all hypotenuses and the apothem fuse to form the conventional radius which is equal to apothem.

Diameters:

Diameters are lines in a figure which are very helpful in understanding the figure. In that sense di-radius with maximum angles are diameters or regular polygons. And it can be denoted by D^{θ}_{n}

Angle between the radii of a diameter, $D^{\theta_1}_{n} = \frac{n}{2} \times \frac{2\pi}{n}$

when n is an even number $D^{\theta_1}_{n}$ will be equal to π

Conversely, any two radii that have an angle equal to π , in a regular polygon of even number of sides, are diameters.

Angle between the radii when n is an odd number is $D_n^{\theta_2} = \frac{(n-1)}{2} \times \frac{2\pi}{n}$

Conversely, all radii that have an angle equal to $\frac{(n-1)}{2} \times \frac{2\pi}{n}$, in a regular polygon of odd number of sides, are diameters.

The diameters that have maximum length are di-hypotenuse or duo di-angle bisectors. And the diameters that have minimum length are di-apothems.

Moreover, When n tends to infinity, the consequent figure will be a circle, hence all diameters merge to form a conventional diameter which is always equal to di − apothem.

Mathematically, $\lim_{n\to\infty} \frac{\pi(n-1)}{n} = \pi$

And the length of diameter $\lim_{n\to\infty} D_n^{\theta} = \lim_{n\to\infty} \frac{2R_a}{[\cos\theta]}$; here absolute value of $\cos\theta$ is considered.

$= \frac{2R_a}{\left[\lim_{n\to\infty} \cos\theta\right]}$

As n tends to infinity θ tends to π,

$= \frac{2R_a}{[\cos\pi]}$

$= 2R_a$

$=$ Twice the radius

2.0.9. Definition of angle of a regular polygon (Two dimensional):- The angle is the measure of rotation of a given ray about its initial point. Here Initial side is positive side of X-axis and vertex of the angle is the origin of the co-ordinate axis.

The angle can also be defined as the space subtended by the segment of the perimeter of a regular polygon, up to its in-center limited between the two radii. It is usually measured from anti-clockwise direction. The measurement of an angle is conventional as in degrees and scientific as in radians. The measurement of an angle does not comply with the concept of angle because length and amount of space bounded by the specified limits of angle are immaterial to the measurement of an angle. Hence radius of angle of a circle used for representation of an angle will always be a constant.

As usual, all regular polygons in this case are considered as a set of concentric regular polygons. Its corresponding radii of each polygons are equal and constants. Whereas, in case of a circle, all radii and diameters are equal and constant. Hence hands of the angle will also be equal. In case of a regular polygons other than a circle, hands of the angle may be different and periodically equal.

Also, angle subtended by equal segments of a perimeter in a regular polygon are not equal always, except in case of a circle.

For example, the angle subtended by some segment at the vertex which includes both the adjacent sides of the vertex will not be equal to the same length on other side of the vertex.

Segment (arc) of circumference of a regular polygon has variable radii except the segment of (arc) a circle. Hence radius of curvature of segment of circumference is variable whereas the radius of curvature of segment of a circle is equal w.r.t the corresponding circle.

Therefore, angle of a regular polygon is measured from the positive X –axis in anti-clockwise direction.

Circular and polygonal angle are denotable as shown in Fig (5)

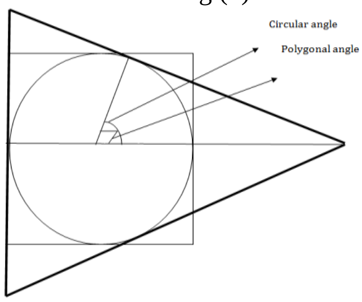


Fig.5

2.1.0. The relation between apothem and perimeter of a regular polygon.

Till now conventional π is used as the unit of angle. In this section an effort is made for generalized representation of the angle.

Let ABCD … be a regular polygon of n number of sides Fig(6), O be its center, OP_1 be its apothem drawn on to the side. Let l be the length of each side.

Here, OP_1 is the apothem of the corresponding side AB, and R_a is its length. Also let l be the length of each side.

A, B, C ⋯ denote vertices and ∠A, ∠B, ∠C ⋯ denote the corresponding interior angles at the vertices of the regular polygon.

The angle subtended by the side at the centre of the regular polygon is θ_n, n indicates the number of sides. by which total angle is divided by to get the angle. i.e., $\angle OAB = \frac{2\pi}{n} = \theta_n$

And, $\angle P_1OB = \frac{\theta_n}{2}$, which is the angle between apothem and largest hypotenuse.

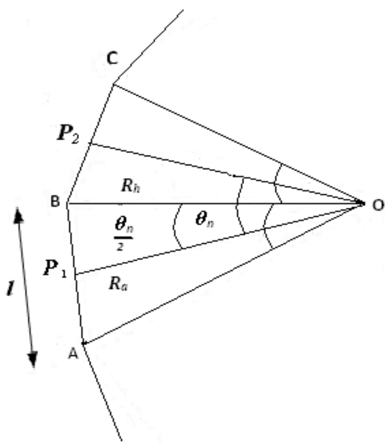


Fig.(6): A typical regular polygon of n number of sides

$$i.e., \angle AOB = \angle BOC = \angle COD = \cdots = \frac{Total\ angle\ of\ the\ circle}{number\ of\ sides\ of\ the\ regular\ polygon}$$

$$= \frac{Total\ angle\ of\ the\ circle}{number\ of\ sides\ of\ the\ regular\ polygon}$$

$$= \frac{2\pi}{n}\ or\ \frac{360^o}{n}\ or \frac{400\ grd}{n}$$

$$If\ \frac{2\pi}{n}\ or\ \frac{360^o}{n}\ or \frac{400\ grd}{n}\ is\ denoted\ by\ \theta_n$$

$$Then\ \theta_n = \frac{2\pi}{n}\ or\ \frac{360^o}{n}\ or \frac{400\ grd}{n}$$

$$Further\ \frac{1}{2}\angle AOB = P_1OB$$

$$Accordingly, \frac{\theta_n}{2} = \frac{\pi}{n}\ or\ \frac{180}{n}\ or \frac{200\ grd}{n}$$

Further, $\angle OP_1B = \angle OP_2C = \angle OP_3D = \cdots = \angle OP_nZ = A\ right\ angle$

2.1.1.Theorem-1

The ratio of perimeter of a plane convex regular polygon to its maximum angled Di-apothem is a constant. This universal constant can be called as PIEN (pi,en) and symbolically it can be written as $\boldsymbol{\pi_n^a}$

Note- The ratio of perimeter of a regular polygon to its Apothem or to its Di-apothem is also constant.

Here, superscript a implies that π_n is derived w.r.t apothem and n indicates the number of sides of the regular polygon to which pi refers to. For example, when the regular polygon is an equilateral triangle π_n^a is written as π_3^a, for a square π_4^a, and so on. Also, n which denotes the number of sides of a regular polygon is

$\in N \geq 3, which\ is\ the\ set\ = \{3,4,5,\cdots,n\}$

and θ_n is the angle subtended by a side of the regular polygon. Here also the subscript n indicates the number of sides of the regular polygon. n starts from 3 because the minimum number of sides required to make a closed plane figure is 3. The reason for retaining the subscript n of θ is to recognize the divisor of the total angle of the circle from which θ is obtained.

2.1.2. Derivation of mathematical expression for π_n^a

Since apothems are perpendiculars drawn on the sides of the regular polygon from its centre, the angles $\angle OP_1A, \angle OP_1B, \angle OP_2B,\ \angle OP_2C \ldots are\ right\ angles\ Fig. 6.$

Let P be the perimeter of a plane convex regular polygon of n number of sides.

Then. P$= nl - - - - - - - - - - -(4)$

With reference to the Fig.(1), the angle between the side and apothem is a right angle.

$Then\ length\ of\ a\ side, l = 2\left(R_a\ tan\left(\frac{\theta_n}{2}\right)\right)\ ; a\ general\ equation\ w.r.t. the\ triangle\ OAB,$

$above\ equation\ \ can\ be\ written\ as$

$l = 2\left(R_a\ tan\left(\frac{\theta_n}{2}\right)\right)$

Substituting this relation of l in (4) we have

$P = n(2R_a)\ tan\left(\frac{\theta_n}{2}\right)$

Since $2R_a$ is equal to the Di-apothem of maximum angle, $which\ is\ denoted\ by\ D_{max}^a,$ above equation can be written as

$P = n\ D_{max}^a tan\left(\frac{\theta_n}{2}\right)$

According to the theorem 1.

$\pi_n^a = \frac{Perimeter\ of\ regular\ polygon}{Maximum\ angled\ Diapothem}$

$= \frac{nD_{max}^a\ tan\left(\frac{\theta_n}{2}\right)}{D_{max}^a} \pi_n^a = n\ tan\left(\frac{\theta_n}{2}\right) - - - - - (5)$

$When\ the\ unit\ of\ angle\ is\ \pi, (5)\ will\ be, \pi_n^a = n\ tan\left(\frac{\pi}{n}\right)$

$When\ the\ unit\ of\ angle\ is\ expressed\ in\ degrees, \pi_n^a = n\ tan\left(\frac{180}{n}\right)$

And so on

Thus derived.

Table-1:

Table showing the different values of π_n^a for different values of n

n	$n\ tan\left(\frac{\theta_n}{2}\right)$	$\frac{n\ tan\left(\frac{\theta_n}{2}\right)}{\pi}$
1	2	3
3	5.196152423	1.653986686
4	4	1.273239545
5	3.632712640	1.156328347
6	3.464101615	1.102657791
7	3.371022332	1.073029735
8	3.313708499	1.054786175
9	3.275732108	1.042697915
10	3.249196962	1.034251515
10^2	3.142626604	1.000329117
10^3	3.141602989	1.00000329
10^4	3.141592757	1.000000033
10^5	3.141592655	
⋮		
10^n		Value tend to 1

It is true that as the number sides of a regular polygon increase indefinitely, the regular polygon approximates to a circle. Hence the value of π_n^a tends to conventional π.

$Mathematically, \qquad \lim_{n\to\infty} n\ tan\left(\frac{\theta_n}{2}\right) = conventional\ \pi$

Note: Conventional π means the present value of π.

$Therefore, conventional\ \pi = \ \pi_\infty^a$

Note: The symbol π when written without superscript and subscript may be considered as conventional pi.

2.1.3.The relation between Di-hypotenuse of maximum angle or length (Maximum angled Duo-di-angle bisectors) and perimeter of a regular polygon.

Theorem-2:
The ratio of perimeter of a plane convex regular polygon to its longest di-hypotenuse (Maximum angled Duo-di-angle bisectors) is a constant.
This universal constant can be called as PIEN H-MAX (pi,en, ā ch max) and symbolically it can be written as π_n^{h-max}
Derivation of mathematical expression for π_n^{h-max}**:**
Let P be the perimeter of a regular polygon of n number of sides
$Then\ the\ mathematical\ equation\ for\ the\ perimeter\ of\ regular\ polygon\ will\ be\ , P = nl$
ΔAOB is an isosceles triangle. Fig(2) here OA and OB are typical longest hypotenuses

$$Then, l = 2R_h \sin\left(\frac{\theta_n}{2}\right)$$

Then (5) can be written as

$$P = n(2R_h) \sin\left(\frac{\theta_n}{2}\right)$$

Now, according to the theorem 2.

$$\pi_n^{h-max} = \frac{Perimeter\ of\ regular\ polygon}{longest\ di - hypotenuse}$$

$$= \frac{nD_{max}^h \sin\left(\frac{\theta_n}{2}\right)}{D_{max}^h}$$

$$\pi_n^{h-max} = n \sin\left(\frac{\theta_n}{2}\right) ------(6)$$

$$When\ the\ unit\ of\ angle\ is\ \pi, (6)\ will\ be, \pi_n^{h-max} = n \sin\left(\frac{\pi}{n}\right)$$

$$When\ the\ unit\ of\ angle\ is\ degree, \pi_n^{h-max} = n \sin\left(\frac{180}{n}\right)$$

Table2: Table showing the different values of π_n^{h-max} for different values of n

n	$n \sin\left(\frac{\theta_n}{2}\right)$	$\frac{n \sin\left(\frac{\theta_n}{2}\right)}{\pi}$
1	2	3
3	2.598076211	0.826993343
4	2.828427125	0.900316316
5	2.938926261	0.935489284
6	3	0.954929659
7	3.037186174	0.966766385
8	3.061467459	0.974495358
9	3.07818129	0.979815536
10	0.983631643	0.983631643
10^2	3.141075908	0.999835515
10^3	3.141587486	0.999998355
10^4	3.141592602	0.999999984
10^5	3.141592653	
⋮		
10^n		Value tend to 1

2.1.4.Theorem -3: A general theorem

The ratios of perimeters of a plane convex regular polygons to their diameters are constants.
This can be denoted by $\pi_n^{R_i}$. It can also be written as π_n^{θ} , since θ is a significant value that is required to find the length of the required lines and represents the uniqueness of its position.
Mathematical description for the above theorem.

$We\ already\ know\ that\ the\ length\ of\ the\ unique\ line\ in\ a\ right\ triangle\ in\ a\ regular\ polygon\ is,$

$R_i = \frac{R_a}{\cos\theta}$

$Then\ opposite\ side\ of\ the\ triangle\ is, l_i\ = R_i \times \sin\theta$

$= \frac{R_a}{\cos\theta} \times \sin\theta$

$l_i = R_a \times \tan\theta$

$Then, \frac{l}{2} = l_i + \left(\frac{l}{2} - l_i\right)$

$Accordingly, \frac{l}{2} = R_a \tan\theta + \left\{R_a \tan\left(\frac{\theta_n}{2}\right) - R_a \tan\theta\right\}$ $\qquad \frac{l}{2} = R_a \tan\left(\frac{\theta_n}{2}\right) And, l$

$= 2\,R_a \tan\left(\frac{\theta_n}{2}\right)$

$As\ usual, Perimeter\ of\ a\ regular\ polygon, P = nl$

Substituting the expression for l in the above equation we have,

$P = 2nR_a \tan\left(\frac{\theta_n}{2}\right)$

$We\ know\ that, D_n^\theta = \frac{2R_a}{\cos\theta}$

In view of theorem 3 we have,

$$\frac{P}{D_n^\theta} = \frac{2nR_a \tan\left(\frac{\theta_n}{2}\right)}{\frac{2R_a}{\cos\theta}}$$

$$\frac{P}{D_n^\theta} = n \tan\left(\frac{\theta_n}{2}\right)(\cos\theta)$$

$Since\ \frac{P}{D_n^\theta} = \boldsymbol{\pi_n^\theta}, above\ equation\ can\ be\ written\ as,$

$\boldsymbol{\pi_n^\theta} = n \tan\left(\frac{\theta_n}{2}\right)(\cos\theta)$

$We\ know\ that, n \tan\left(\frac{\theta_n}{2}\right) = \pi_n^a$

$Then, \boldsymbol{\pi_n^\theta} = n \tan\left(\frac{\theta_n}{2}\right)(\cos\theta) - - - - - - - (7)$

$\boldsymbol{\pi_n^\theta} = \pi_n^a(\cos\theta) - - - - - - - (8)$

$When\ \theta = 0; i.e., for\ lower\ bound\ of\ \theta$

$\boldsymbol{\pi_n^\theta} = \pi_n^a(\cos\theta)$

$\boldsymbol{\pi_n^\theta} = \pi_n^a$

$And, when\ \theta = \left(\frac{\theta_n}{2}\right)\ or\ \left(\frac{\pi}{n}\right), i.e., for\ upper\ bound\ of\ \theta$

$\boldsymbol{\pi_n^\theta} = n\left(\sin\left(\frac{\theta_n}{2}\right)\right)$

Case Study:

$1. When\ n = 3, \quad it\ is\ an\ equlateral\ triangle; and\ let\ \ \theta = 0$

$Then, \pi_n^\theta = n \tan\left(\frac{\theta_n}{2}\right)(\cos\theta)$

$When\ \theta = 0;\ \pi_n^\theta\ will\ be = \pi_n^a$

$Also, by\ substituting\ the\ value\ of\ \theta\ and\ the\ value\ of\ \frac{\theta_n}{2}\ in\ radians, in\ the\ above\ equation\ we\ get,$

$\pi_n^0 = 3 \tan\left(\frac{\pi}{3}\right)(\cos 0)$

$\pi_3^0 = 3\sqrt{3}$

$2. When\ n = 3; and\ let\ \ \theta = 30^\circ$

$\pi_n^\theta = n \tan\left(\frac{\theta_n}{2}\right)(\cos\theta)$

$= 3 \tan\left(\frac{180}{3}\right)(\cos 30)$

$= 3\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)$

$\pi_n^{30^\circ}$ =4.5

This means that the length of the perimeter of this triangle i.e., equal to 4.5 times that of hypotenuse that can be drawn at 30°.

$2. Let, n = 3\ ; and\ \ \theta = 45^\circ$

$Then, \boldsymbol{\pi_n^\theta} = n\ tan\left(\frac{\theta_n}{2}\right)(cos\ \theta)$

$= 3\ tan\left(\frac{180}{3}\right)(cos\ 45)$

$= \frac{3}{\sqrt{2}}$

$3. When\ n = 3\ ; and\ let\ \ \theta = 60^\circ$

$Then, \boldsymbol{\pi_n^{60^\circ}} = 3\ tan\left(\frac{180}{3}\right)(cos\ 60)$

$= \pi_3^0\left(\frac{1}{2}\right)$

$= \ 3\sqrt{3}\left(\frac{1}{2}\right)$

$= \frac{3\sqrt{3}}{2}$

$= 3\ sin\ 60$

This result is also obtainable directly from (8) as following

$\boldsymbol{\pi_n^{60^\circ}} = 3\ tan\left(\frac{180}{3}\right)(cos\ 60)$

$= 3\left(\frac{sin\ 60}{cos\ 60}\right)(cos\ 60)$

$= 3\ sin\ 60$

=2.598076

$When\ n = 3\ , the\ value\ of\ \pi_n^\theta\ varies\ from\ 3\sqrt{3}\ to\ \frac{3\sqrt{3}}{2}\ for\ \left[\frac{\pi}{4} \geq \theta \geq 0\right]$

$1. For\ example, when\ n = 4\ \ and\ \ \theta = 0\ , the\ value\ of\ semiperimeter\ will\ be$

$\pi_n^\theta = n\ tan\left(\frac{\theta_n}{2}\right)(cos\ \theta)$

$= n\ tan\left(\frac{\theta_n}{2}\right)$

$= \pi_n^a$

$\pi_n^a = \ n\ tan\left(\frac{\theta_n}{2}\right)$

$= n\ tan\left(\frac{\pi}{4}\right)$

$\pi_4^a = n$

$= 4$

$3.\ When\ n = 4\ ;\ \theta = 30^\circ$

$\boldsymbol{\pi_n^\theta} = n\ tan\left(\frac{\theta_n}{2}\right)(cos\ \theta)$

$= n\ cos\ 30$

$= 4 \times 0.866025$

$= 3.464102$

$2. When\ n = 4\ ;\ \theta = 45^\circ$

$When\ \theta_n\ will\ be = \ 90^\circ\ ; since\ \theta_n\ is\ the\ angle\ subtended\ by\ a\ side\ at\ the\ center.$

$\boldsymbol{\pi_n^\theta} = n\ tan\left(\frac{\theta_n}{2}\right)(cos\ \theta)$

$= \frac{4}{\sqrt{2}}$

$= 2\sqrt{2}$

$When\ n = 4\ , the\ value\ of\ \pi_n^\theta\ varies\ from\ 2\sqrt{2}\ to\ 4\ for\ \left[\frac{\pi}{4} \geq \theta \geq 0\right]$

Similarly, value of π_n^θ can be found for subsequent values of n

2.1.5. Applications of π_n^θ :

To find the perimeter of a regular polygon:

If the perimeter of a regular polygon is denoted by P_n where n denotes the number of sides of the polygon.

$Then, P_n = 2\ \pi_n^\theta\ R_i^\theta ------(9)$

Proof: $Given,\ P_n = 2\,\pi_n^\theta\ R_i^\theta$

$= 2n\ tan\left(\frac{\theta_n}{2}\right)(cos\ \theta)R_i^\theta$

$= n\left[\frac{2\left(\frac{l}{2}\right)}{R_a}\right](cos\ \theta)\ R_i^\theta$

$= n\left(\frac{l}{R_a}\right)\times\frac{R_a}{R_i^\theta}\times R_i^\theta$

$= nl$

Thus proved.

$At\ simplified\ terms\ and\ w.r.t\ apothem\ \ perimeter\ of\ a\ regular\ polygon\ can\ be\ written\ as$

$P_n = 2\pi_n^a R_a ------(10)$

2.1.6. To find the area of a regular polygon.

If the area of a regular polygon is denoted by $A_n, then\ A_n = \pi_n^\theta(cos\ \theta)\left(R_i^\theta\right)^2 --------(11)$

Proof:

$Given, A_n = \pi_n^\theta(cos\ \theta)\left(R_i^\theta\right)^2$

$= \left[n\ tan\left(\frac{\theta_n}{2}\right)(cos\ \theta)\right](cos\ \theta)\left(R_i^\theta\right)^2$

$= \left[n\left(\frac{l}{2R_a}\right)\left(\frac{R_a}{R_i^\theta}\right)\right](cos\ \theta)\left(R_i^\theta\right)^2$

$= \frac{nl(cos\ \theta)R_i^\theta}{2}$

$= \frac{nlR_a(cos\ \theta)}{2(cos\ \theta)}; since\ R_i^\theta = \frac{R_a}{cos\ \theta}$

$= \frac{nlR_a}{2}$

$\frac{l}{2}$ $is\ half\ of\ the\ base\ of\ the\ isosceles\ triangle\ formed\ when\ two\ closest\ vertices\ are\ joined\ to\ the\ center,$ $also\ it\ is\ half\ of\ a\ side\ of\ the\ regular\ polygon\ and\ R_a\ will\ be\ the\ height\ ofthis\ triangle, Hence\ \frac{l}{2}\ R_a$ $is\ the\ area\ of\ this\ triangle. Further\ , n\ is\ the\ numberof\ sides\ which\ is$ $also\ equal$ $to\ number\ of\ isosceles\ triangles\ in\ a\ regular\ polygon.$

$Consequently, product\ of\ n\ and\ \left(\frac{l}{2}\ R_a\right)\ will\ be\ the\ area\ of\ the\ regular\ polygon.$

$= n\left[\left(\frac{l}{2}\right)R_a\right]$

$A_n = n\times area\ of\ the\ triangle\ formed\ by\ the\ sides\ of\ the\ regular\ polygon\ with\ the\ centre$

Thus proved.

$At\ simplified\ terms\ (11)\ can\ be\ written\ as\ \ A_n = \pi_n^a\ R_a^2\ ------- (12)$

Example1: Apothem of an equilateral triangle is 1 m find the perimeter, area, also find perimeter and area w.r.t the radius $R_i^\theta = R_i^{30}, R_i^{45}, R_i^{60}$

Data: $R_a = 1\ m$

$\frac{\theta_n}{2} = \frac{\pi}{3}, since\ figure\ is\ an\ equilateral\ triangle. and\ \theta = 0$

Perimeter of the triangle can be found by using (9):

$i.e., P_n = 2\ \pi_n^\theta\ R_i^\theta$

$P_3 = 2\ \pi_3^a\ R_a\ \ ; since\ radius\ is\ an\ apothem\ and\ n = 3.$

$= 2\times 3\ \ tan\left(\frac{\pi}{3}\right)$

=10.3923 m

Area of the triangle can be found by using (11) :

$i.e, A_n = \pi_n^{\theta}(\cos\theta)\left(R_i^{\theta}\right)^2$
$Since\ given\ radius\ is\ an\ apothem, \theta\ will\ be\ = 0$
$Cosequently, above\ equation\ will\ be, A_n = \pi_n^{a}(R_a)^2$
$Also, given\ that\ R_a = 1m$
$Hence, A_n = \pi_n^{a}$
$= n\tan\left(\frac{\pi}{3}\right)\ m^2$
$= 5.196\ m^2$
$Further\ magnitude\ of\ R_i^{30} = \frac{R_a}{\cos\theta}$
$= \frac{1}{\cos 30}$
$R_i^{30} = 1.1547$ m
$Then, A_n = \pi_n^{\theta}(\cos\theta)\left(R_i^{\theta}\right)^2$
$= \pi_3^{30}(\cos 30)(R_i^{30})^2$
$= 3\tan\left(\frac{\pi}{3}\right)(\cos\theta)^2(R_i^{30})^2$
$= 3\ \times(1.732)\times(0.866)^2\times(1.1547)^2$
$= 5.196\ m^2$
Similarly, we get the same result for $R_i^{\theta} = R_i^{45}\ and\ R_i^{60}$
Example 2:
Find the perimeter and area of the pentagon whose line joining the centre and the vertex is 2 cm.
$Given\ that, \theta_n = \frac{\pi}{5}$; $since\ it\ is\ the\ line\ joining\ the\ centre\ and\ the\ vertex. Also, R_i^{\theta} = 2\ cm\ and\ it\ is\ a$ largest radius.
$= 2\pi_n^{h-max}R_i^{max}$
$= 2\times 5\times\sin\left(\frac{\pi}{5}\right)\times 2$
$= 11.75571$
Or
$R_i^{\theta} = \frac{R_a}{\cos\theta}$
$R_a = R_i^{\theta}\cos\theta$
$= 2\cos\left(\frac{\pi}{5}\right)$
$= 1.618034$
$P_3\ = 2\ \pi_5^{a}\ R_a$
$= 2\times 3.632713\times 1.618034$
$= 11.75571$
Similarly, area and perimeter can be found for all regular polygons.

2.1.7. Equation for the area of a circle
$Area\ of\ a\ regular\ polygon\ is\ generally\ given\ by\ \ A_n = \pi_n^{a}(R_a)^2$
$as\ n\to\infty, \pi_n^{a}\ will\ be = \pi, R_a = r$
$consequently, area\ of\ the\ circle\ will\ be = \pi r^2$

2.1.8.Perimeter of a regular polygon exceeding the perimeter of the in-circle.
$Perimeter\ of\ a\ regular\ polygon\ is,\ P_n = 2\ \ \pi_n^{\theta}R_i^{\theta}$
$= 2n\tan\left(\frac{\theta_n}{2}\right)(\cos\theta)R_i^{\theta}$
$And, perimeter\ of\ a\ circle\ is\ P = 2\pi R_a$
$Total\ perimeter\ \ of\ regular\ polygon\ exceeding\ the\ \ perimeter\ of\ a\ circle\ is\ , P_n - P$
$If\ this\ difference\ is\ denoted\ P_{(n-\infty)}\ we\ have, P_{(n-\infty)}\ = P_n - P$
$P_{(n-\infty)}\ = 2\pi_n^{\theta}R_i^{\theta} - 2\pi R_a$
$= 2R_a\left(\frac{\pi_n^{\theta}}{\cos\theta} - \pi\right)$
$=\ 2R_a\left[\left(n\tan\left(\frac{\theta_n}{2}\right)\right) - \pi\right]$

$P_{(n-\infty)} = 2(\pi_n^a - \pi)R_a$ −−−−−−−−−−−− (13)

Example: side of a square is 2 units and radius of a circle is 1 units find the difference of their perimeters.

Solution- $P_{(n-\infty)} = 2(\pi_n^a - \pi)R_a$

$= 2(4 - 3.1416)$

$= 1.7168\ units$

2.1.9.Area of a regular polygon exceeding the area of the in-circle.

$Area\ of\ a\ regular\ polygon\ is, A_n = \pi_n^\theta(\cos\theta)(R_i^\theta)^2$

$= \pi_n^a(R_a)^2$; $when\ \theta\ of\ cosine\ in\ the\ equation\ is\ cosidered\ as\ 0$

$And\ area\ of\ incircle\ is, A_\infty = \pi(R_a)^2$

$Then, A_n - A_\infty = \pi_n^a(R_a)^2 - \pi(R_a)^2$

$A_n - A_\infty = (\pi_n^a - \pi)(R_a)^2$ −−−−−−−− (14)

$Also, \lim_{n\to\infty}(\pi_n^a - \pi)(R_a)^2 = 0$

Example: Consider a square with apothem R_a and circle inscribed in it. Then the area of the shaded part is.

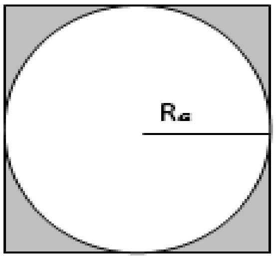


Fig 7: Area bounded between the circle and the square

$A_s = \pi_4^a(R_a)^2 - \pi(R_a)^2$; $If\ this\ area\ is\ denoted\ by, A_s$

$= (\pi_4^a - \pi)(R_a)^2$

$= (4 - \pi)(R_a)^2$

$\approx 0.858\ (R_a)^2$

1.2.0. Volume and surface area of a Isogonic tetrahedron (regular tetrahedron):

Regular tetrahedron can be considered as a pyramid or as semi-conics with a base of equilateral triangle.

$a - Edge\ length\ of\ the\ tetrahedron, Fig(8).$

$h - Height\ of\ the\ tetrahedron$

$r - Radius\ of\ the\ insphere\ that\ is\ tangent\ to\ the\ face$

$O - Incentre\ of\ this\ insphere$

$R_a - apothem\ of\ the\ base\ triangle.$

$A_3 - area\ of\ the\ base\ triangle.$

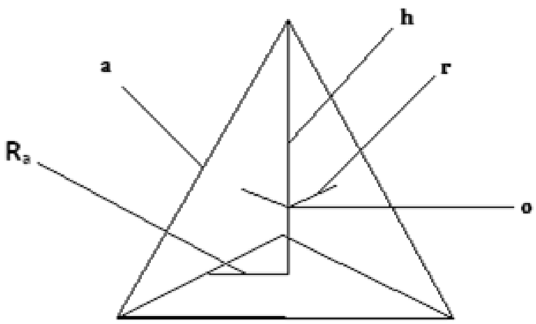


Fig 8: Tetrahedron shown with its different elements.

$Known\ mathematical\ data\ ;\ h = \frac{\sqrt{6}\,a}{3};\quad r = \frac{a}{\sqrt{24}}$

$Ratio\ of\ r\ and\ h\ is, \frac{r}{h} = \frac{\frac{a}{\sqrt{24}}}{\frac{\sqrt{6}\,a}{3}}$

$= \frac{1}{4}$

$Cosequently, r: h = 1: 4$

$Then\ Area\ of\ the\ base\ can\ be\ given\ by\ the\ equation\ , A_3 = \pi_3^a (R_a)^2 - - - - - (15)$

$It\ is\ known\ that\ volume\ a\ pyramid\ is\ equal\ to\ the\ product\ of\ area\ of\ the\ base\ by\ the\ altitude.$

$Accordingly, volume\ of\ this\ pyramid\ will\ be\ , V_3 = \frac{1}{3}\pi_3^a (R_a)^2 h - - - - - - (15a)$

$And, Volume\ of\ this\ tetrahedron\ can\ also\ be\ written\ as\ ,$

$V_3 = \frac{4}{3}\pi_3^a (R_a)^2\, r\ ; since\ h = 4r\ \ - - - - - (16)$

For further simplification, relation between $r\ and\ R_a\ is\ found\ as\ following.$

$Radius\ of\ the\ incircle\ that\ is\ tangential\ to\ the\ face\ is, r = \frac{a}{\sqrt{24}}$

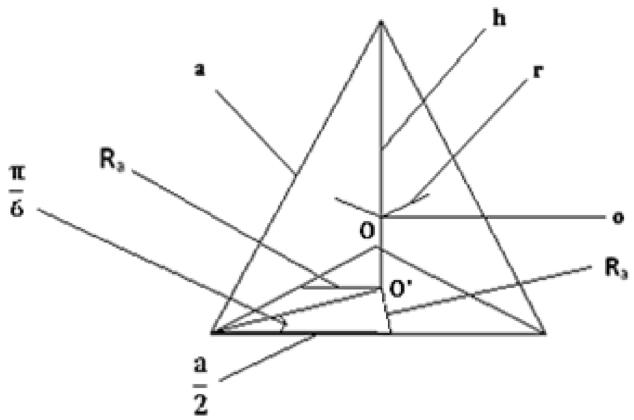


Fig. 9

$And, R_a = \frac{a}{2} tan \frac{\pi}{6}$

$R_a = \frac{a}{2\sqrt{3}}$

$Consequently, r: R_a = \left(\frac{\frac{a}{\sqrt{24}}}{\frac{a}{2\sqrt{3}}}\right)$

$r: R_a = 1: \sqrt{2}$

$Hence, R_a = \sqrt{2}\ r$

$Volume\ of\ this\ tetrahedron, V_3 = \frac{4}{3}\pi_3^a (R_a)^2\ r$

$= \frac{4}{3}\pi_3^a \left(\sqrt{2}\, r\right)^2\ r$

$= \frac{8}{3}\pi_3^a\ r^3$

$= \frac{8}{3} \times 3\sqrt{3}\, r^3$

$V_3 = 8\,\sqrt{3}\, r^3 - - - - - - - (17)$

Example.1: If radius of in-sphere of a regular tetrahedron is 1 units find the volume of the tetrahedron.

$Solution - V_3 = 8\,\sqrt{3}\, r^3$

$given, r = 1$
$Then\ , V_3 = 8\sqrt{3}$
$$Also, \frac{R_a}{h} = \frac{\frac{a}{2\sqrt{3}}}{\frac{\sqrt{6}\,a}{3}}$$
$$\frac{R_a}{h} = \frac{1}{2\sqrt{2}}$$
$h = 2\sqrt{2}R_a - - - - - - - - (18)$
$When\ (18)\ is\ replaced\ in\ (15a),$
$$i.e.,, V_3 = \frac{1}{3}\pi_3^a(R_a)^2 h$$
$$= \frac{1}{3}\pi_3^a(R_a)^2(2\sqrt{2}R_a)$$
$$= \frac{2\sqrt{2}}{3}\pi_3^a(R_a)^3$$
$$= \frac{2\sqrt{2}}{3} \times 3\sqrt{3}\ \ (R_a)^3$$
$= 2\sqrt{6}\ \ (R_a)^3 - - - -(19)$

2.2.1. The ratio of volume of Regular tetrahedron and its in-sphere

$$Volume\ of\ insphere\ is\ V_s = \frac{4}{3}\pi r^3$$
$Volume\ of\ tetrahedron\ that\ circumscribe\ the\ sphere\ and\ touch\ the\ surface\ at\ its\ centroid\ is$
$V_t = 8\sqrt{3}\,r^3$
$$\frac{V_t}{V_s} = \frac{\frac{8}{3}\pi_3^a\,r^3}{\frac{4}{3}\pi r^3}$$
$$= \frac{2\pi_3^a}{\pi}$$
$= 3.308$

Example.1: If the radius of the in-sphere of an isogonic tetrahedron is 1cm find the volume of the said tetrahedron.

$Solution: By\ \ (17)\ we\ have\ volume\ of\ the\ isogonic\ tetrahedron\ is, V_3 = 8\sqrt{3}\,r^3$
$= 8\sqrt{3}\ \ cubic\ units$

Example.2: If the slant edge length of the isogonic tetrahedron is $\sqrt{24}$ units find its volume and the volume of the in-sphere.

$$Solution: Given\ a = \sqrt{24}\,, r = \frac{a}{\sqrt{24}} = \frac{\sqrt{24}}{\sqrt{24}} = 1$$
$$Then\ the\ volume\ of\ the\ in-sphere\ is\ = \frac{4}{3}\pi\ cubic\ units$$
$Consequantly, V_3 = 8\sqrt{3}\,r^3$
$= 8\sqrt{3}$ cubic units

This is also solvable as following by using (19)

$Given\ that\ slant\ edge\ is\ \sqrt{24}\ units$
$Then\ the\ perimeter\ of\ the\ triangle\ of\ the\ faces\ will\ be\ = 3\sqrt{24}$
$$Then\ the\ apothem\ R_a\ will\ be = \frac{3\sqrt{24}}{3\sqrt{3}}\ , since\ 3\sqrt{3} = \pi_3^a$$
$Consequantly, R_a = \sqrt{2}$
$Given\ that, V_3 = 2\sqrt{6}\ \ (R_a)^3$
$When\ the\ value\ of\ R_a\ is\ substiutted\ in\ the\ given\ equation\ we\ have$
$V_3 = 8\sqrt{3}\ cubic\ units$

2.2.2. Surface area and volume of a closed cylinder of regular polygonal base and of height h.

$a)\ General\ equation\ for\ the\ surface\ area\ of\ a\ closed\ cylinder\ of\ a\ regular\ polygonal\ base\ is$

$derived\ as\ following$
$Area\ of\ the\ 2\ bases\ that\ are\ on\ either\ end\ of\ the\ \ polygonal\ cylinder\ is, =\ 2\pi_n^a(R_a)^2$
$Area\ of\ cylindrical\ surface\ is = perimeter\ of\ the\ base\ \times height$
$Mathematically, Area\ of\ cylindrical\ surface\ is = (2\pi_n^a R_a) \times h$
$Total\ surface\ area\ of\ such\ cylinder\ is = 2\pi_n^a(R_a)^2 + 2\pi_n^a R_a h$
$If\ surface\ area$ of $a\ cylinder\ of\ regular\ base\ is\ denoted\ by\ A_n^c, we\ have$
$A_n^c = 2\pi_n^a R_a(R_a + h) ------(20)$
b) Volume of the closed cylinder with regular polygonal base
$It\ is\ known\ that\ the\ area\ of\ the\ base\ is = \pi_n^a(R_a)^2$
$consequantly\ volume\ of\ such\ cylinder\ is = \pi_n^a(R_a)^2 h\ ; whence\ h\ is\ the\ height\ of\ the\ cylinder.$
Example: Find the area and volume of the prism with equilateral base with following data.
Length of side of the base =2 units
Height of the prism =4 units
Solution :
$Volume\ of\ the\ prism = \pi_n^a(R_a)^2 h$
$To\ find\ R_a, the\ relation, P_3 = 2\pi_3^a R_a, is\ used,$
$6 = 2 \times 3\sqrt{3} \times R_a\ ; since\ 6\ is\ the\ perimeter\ of\ the\ triangle\ by\ data.$

$$R_a = \frac{1}{\sqrt{3}}$$

$$Now, the\ volume\ of\ the\ prism\ is = 3\sqrt{3} \times \left(\frac{1}{\sqrt{3}}\right)^2 \times 4$$
$= 4\sqrt{3}\ cubic\ \ units$
$Surface\ area = 2\pi_n^a R_a(R_a + h)$
$$= 2 \times 3\sqrt{3} \times \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}} + 4\right)$$
$= 2\sqrt{3} + 24\ square\ units$
Check-This prism has 3 rectangular faces of length 4 units and breadth 3 units. Also it has 2 equilateral triangular faces of side 2 units
$Total\ surface\ area\ of\ the\ Prism\ is\ =\ 3 \times area\ of\ rectangle + 2 \times area\ of\ triangle$
$$= 3 \times 4 \times 2 + 2 \times \frac{2}{2} \times 2sin\left(\frac{\pi}{3}\right)$$
$= 24 + 2\sqrt{3}$
Thus checked.

2.2.3. Some more general examples:
$Surface\ area\ of\ a\ cylinder\ of\ square\ base\ is\ A_4^c = 2\pi_4^a R_a(R_a + h)$
$If\ h = 2R_a\ , we\ have,$
$= 2\pi_4^a R_a(R_a + 2R_a)$
$= 6\pi_4^a {R_a}^2$
$$= 6 \times 4 \times \left(\frac{a}{2}\right)^2 ; since\ \pi_4^a = 4\ and\ R_a =\ \frac{a}{2}$$
$= 6a^2 ----(21)$
Similarly for cylinder of pentagonal base
$A_5 = 2\pi_5^a R_a(R_a + 2R_a)$
$= 6\ \pi_5^a {R_a}^2 ------(22)$
Generally , for a base of n number of sides and the height equal to $2R_a$
$A_n = 6\ \pi_n^a {R_a}^2 ------(23)$
$Since\ 2R_a = a\ ; side\ of\ a\ cube, we\ have$
$$A_n = 6\ \pi_n^a \left(\frac{a}{2}\right)^2$$
$$= \frac{3}{2}\pi_n^a a^2$$
$When\ n \to \infty, above\ equation\ will\ be,$
$$= \frac{3}{2}\pi a^2$$
$Replacing\ 2R_a\ for\ a\ , we\ have$

$=6\,\pi {R_a}^2$

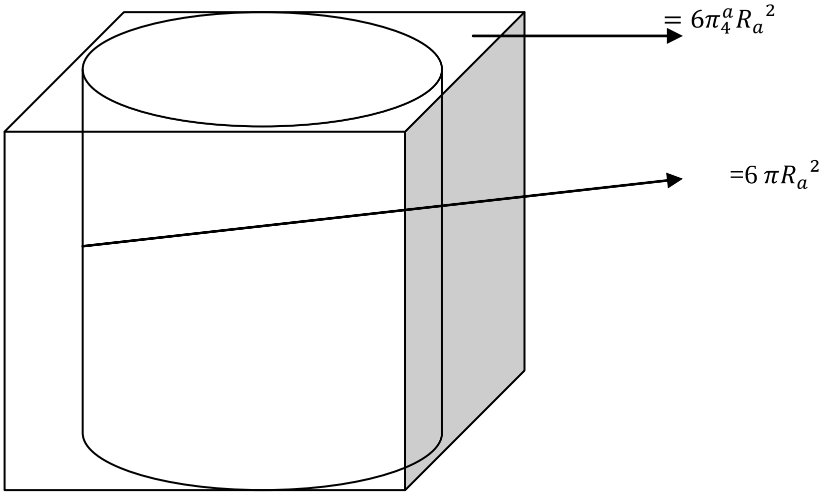


Fig.10:surface areas of cube and inscribed cylinder in the cube

$For\ example\ when\ radius\ of\ the\ incribed\ cylinder\ is\ 1\ , surface\ area\ of\ the\ cylinder\ will\ be$
$= 6\pi\ sq\ units$
$And\ the\ surface\ area\ of\ the\ circumscribing\ cube\ will\ be = 6\pi_4^a\ sq\ units$

Further, general equation with a base of n number and height h = br, i.e., height expressed in terms of r.

$A_n = 2(b+1)\ \pi_n^a r^2 - - - -(24)$

$Further, when\ n\ tends\ to\ infinity\ surface\ area\ of\ the\ figure\ will\ be,$

$A_\infty = 2(b+1)\pi r^2$

$= 2\pi r(r+h)\ ; since\ rb = h.$

2.2.4. Surface area and Volume of a regular hexahedron (cube)

This is the basic reference figure that is considered as standard for all kinds of measurements of volumes.

Here volume and surface area are expressed in the form of the equation of sphere.

$a - Edge\ length\ of\ the\ cube, Fig: 5.$
$h - Height\ of\ the\ cube$
r $- Radius\ of\ the\ insphere\ that\ is\ tangent\ to\ the\ face$
$O - Incentre\ of\ the\ insphere\ that\ is\ tangent\ to\ the\ face. In\ this\ case\ it\ is\ also\ the\ centre$
$of\ the\ cube.$
$R_a - apothem\ of\ the\ base\ square.$
$A_4 = area\ of\ the\ base\ square.$

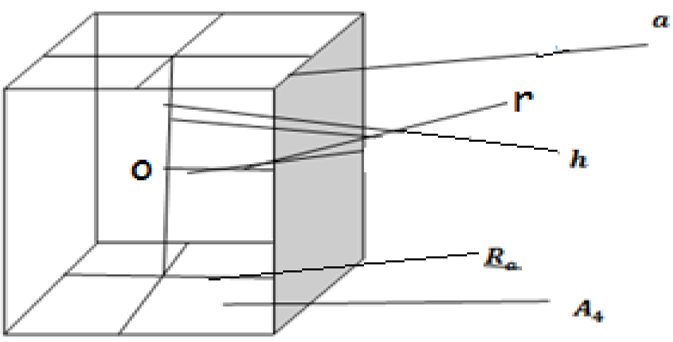


Fig.11.

A cube can also be considered as the cylinder of square base with height equal to a side.

2.2.5. Volume of a cube :
$We, know\ that\ volume\ of\ a\ cube\ is\ V_4 = a^3$
$Since, edge\ length\ a = 2r = 2\ R_a\ , above\ equation\ will\ be$
$V_4 = \ (2r)^3$
$= 8\ r^3$
$Also, V_4 = 8\ R_a^3 - - - - - (25)$

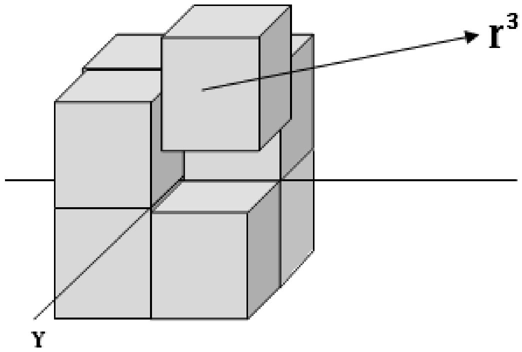


Fig.12.

$There\ are\ 8\ number\ of\ r^3s\ \ in\ a\ cube.$

$\mathbf{2.2.6.A\ general\ equation\ for\ the\ volume\ of\ a\ cylinder\ of\ regular\ polygonal\ base}$:
$V_n^c = A_n h$; $here\ superscript\ c\ indicates\ shape\ of\ the\ figure\ i.e, cylinder.$
Substituting the relation of A_n in the above equation we have
$V_n^c = \pi_n^a (R_a)^2 h - - - - - - (26)$
$When\ h = bR_a\ above\ equation\ will\ be$; $V_n^c = b\ \pi_n^a (R_a)^3 - - - - - (27)$
$In\ a\ special\ case\ i.e., when\ b = 2\ and\ n = 4; then\ \ (27) will\ be$
$V_4^c = 2 \times \ \pi_4^a (R_a)^3 - - - - - - - (28)$
$= 8(R_a)^3$
It is the volume of a cube.
Example :Find the volume of a regular pentagonal cylinder of whose side of base is 1 unit and height 3 units
$V_5^c = \pi_5^a (R_a)^2 h$
$= 5 \times tan\left(\frac{\pi}{5}\right) \times (R_a)^2 \times 3$
$V_5^c = 10.8981 \times (R_a)^2$
$By\ using\ (10) we\ can\ find\ R_a.$
$i.e., R_a = \frac{5}{2 \times 5 \times tan\left(\frac{\pi}{5}\right)}$
$= 0.6882$
$And\ value\ of\ 5 \times tan\left(\frac{\pi}{5}\right) = 3.6327$
$Then, V_5^c \ = 3.6327 \times (0.6882)^2 \times 3$
$= 5.1616\ cubic\ units$

2.2.7.Volume of a regular pyramid with square base and height equal to the side of the base:
.Conventional formula for finding the volume of pyramid is
$V_\infty^P = \frac{1}{3} Ah$; $where\ A\ is\ the\ area\ of\ the\ base\ and\ h\ is\ the\ height\ of\ the\ pyramid.$

$Substituting\ (12)\ for\ A\ in\ \ the\ above\ equation, we\ have, V_4^p$
$= \frac{1}{3}\pi_n^a (R_a)^2 h\ ; subscript\ of\ V\ is\ 4\ because$
$base\ is\ a\ square.$
$Since\ h = 2R_a\ in\ a\ cube\ we\ have, V_4^P = \frac{2}{3}\pi_4^a (R_a)^3 ----- (29)$

2.2.8. Comparison of volumes of cube and regular pyramid of square base with height equal to the side of the base:

$Volume\ of\ the\ cube\ is\ V_4 = Ah\ ; since\ cube\ is\ a\ cylinder\ of\ square\ base\ with\ height\ equal\ to\ the\ side.$
$Volume\ of\ the\ pyramid\ of\ square$ base $is\ V_4^p = \frac{1}{3}Ah$

$Now\ the\ ratio\ of\ \frac{V_4^p}{V_4} = \frac{\frac{1}{3}Ah}{Ah}$
$= \frac{\frac{2}{3}\pi_4^a (R_a)^3}{2\pi_4^a (R_a)^3}$
$= \frac{1}{3}$
$= 1:3$
$Further\ when\ n\ tends\ to\ infinity\ Pyramid\ will\ become\ a\ closed\ cone, V_\infty^p = \frac{2}{3}\pi (R_a)^3 --- (30)$
$Hence$ it $is\ equal\ to\ volume\ of\ the\ hemisphere.$

$2.2.9. A\ general\ equation\ for\ the\ volume\ of\ a\ regular\ pyramid\ with\ a\ regular\ polygonal\ base$:

$Conventional\ formula\ for\ the\ volume\ of\ pyramid\ is\ V = \frac{1}{3}Ah\ ;\ where\ A\ is\ the\ area\ of\ the\ base\ and$
$h\ indicates\ height\ of\ the\ pyramid.$
$Formula\ derived\ for\ volume\ of\ pyramid\ in\ this\ article\ is\ , V_n^p = \frac{1}{3}\pi_n^a (R_a)^2\ h ------(31)$
$Applying\ limits\ to\ \ (31), we\ get\ \ an\ equation\ to\ volume\ of\ cone\ i.e.,$
$\lim_{n\to\infty} V_n^p = \lim_{n\to\infty}\left(\frac{1}{3}\pi_n^a (R_a)^2\ h\right)$
$= \frac{1}{3}\pi r^2\ h ----- (32)$

$2.3.0. Surface\ area\ of\ a\ regular\ pyramid\ of\ polygonal\ base$:

$If\ surface\ area\ of\ the\ pyramid\ is\ denoted\ by\ A_n^p, then$
$A_n^p = Area\ of\ the\ base + total\ area\ of\ the\ slant\ surfaces$
$= \pi_n^a (R_a)^2 + \pi_n^a \left(\sqrt{(R_a)^2 + h^2}\right)$
$A_n^p = \pi_n^a \left((R_a)^2 + \left(\sqrt{(R_a)^2 + h^2}\right)\right) ------(33)$
$Example$: If side length of the base of the square pyramid is 2 units and height 3 units, find its surface area.
Solution: $A_n^p = \pi_n^a \left((R_a)^2 + \left(\sqrt{(R_a)^2 + h^2}\right)\right)$
$8 = 2 \times 4 \times R_a, ; here\ \pi_n^a = 4$
$R_a = 1$
$And, \pi_4^a = 4$
$Then\ A_4^p = 4\left(1 + \left(\sqrt{1 + 3^2}\right)\right)$
$= 16.649\ sqr\ units$

III. $TRIGONOMETRIC\ \ APPLICATIONS\ OF\ \boldsymbol{\pi_n}$

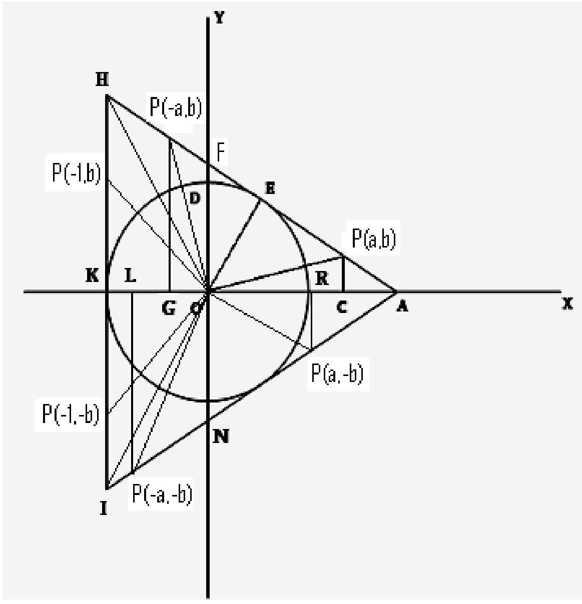


Figure.13: Trigonometric angles w.r.t equilateral triangle.

3.0.1. Particulars, elements and required data w.r.t figure 13:

$AHI - is\ an\ equilateral\ triangle\ circumscribing\ the\ circle\ of\ unit\ radius.$

O - Is the centre of the in-circle and origin of co-ordinate axes X and Y also the centroid of the triangle AHI.

OE- Is the in-radius of in-circle and apothem of circumscribing equilateral triangle which is denoted by R_a. Also it is the variable radius of this triangle . In this case $OE = R_a = 1\ unit\ or\ 1\ rad.$ $Accordingly, total\ length\ of\ the\ perimeter\ of\ the\ triangle\ is\ 2\pi_3^a\ units = 6\sqrt{3}\ units$ $hence\ \pi_3^a\ units = 3\sqrt{3}\ units\ or\ radians.$

OP- is the moving radius, moving from OA in anti-clockwise direction and varying in magnitude from 2 units to 1 unit in length. Its value is absolute positive.

$Determinant\ triangles - These\ are\ the\ triangles\ that\ have\ radii\ as\ their\ hypotinuses, absissas\ as\ their\ base\ and\ ordinate\ as\ their\ opposite\ side, for\ example\ OCP, OGP, OKP\ etc.$ $are\ typical\ determinant\ triangles.$

$z - denotes\ the\ magnitude\ of\ the\ position\ of\ the\ moving\ radius\ OP(a,b)\ w.r.t\ the\ circumferance\ of\ the\ equilateral\ triangle\ measured\ from + X - axis, in\ the\ anti\ clockwise\ direction\ Fig(13).$

$(-z) - denotes\ the$ magnitude of the position of the moving radius from +X axis in the clockwise direction.

$k - This\ denotes\ that\ a\ side\ is\ divided\ into\ k\ number\ of\ equal\ segments.$

$P(a,b)$
$- \ is\ a\ point\ on\ the\ circumferance, varying\ from\ A\ to\ F\ and\ its\ domain\ is\ definable\ as\ varying\ between$ $0 \le z \le \frac{4\pi_3^a}{9}\ or\ 0 \le z \le \frac{4}{\sqrt{3}}.$

$P(-a,b): - is\ a\ point\ on\ the\ circumferance, varying\ from\ F\ to\ H, its\ domain\ is\ definable\ as\ varying\ between$ $\frac{4\pi_3^a}{9} \le z \le \frac{2\pi_3^a}{3}\ or\ \frac{4}{\sqrt{3}} \le z \le 2\sqrt{3};$

$P(-1,b): - is\ a\ point\ on\ the\ circumference, varying\ from\ H\ to\ K, its\ domain\ is\ definable\ as\ varying\ between$ $\frac{2\pi_3^a}{3} \le z \le \pi_3^a\ \ or\ 2\sqrt{3} \le z \le 3\sqrt{3};\ this\ domain\ belongs\ to\ II\ nd\ quadrant. Note\ that\ the\ points$ $in\ this\ domain\ move\ towards\ X - axis\ \ perpendicularly.$

$P(-1,-b): - is\ the\ point\ varying\ from\ K\ to\ I, its\ domain\ is\ definable\ as\ varying\ between\ \pi_3^a \le z \le \frac{4\pi_3^a}{3}$

$value\ of\ z\ varies\ from\ 3\sqrt{3} \le z \le 4\sqrt{3}; this\ domain\ belongs\ to\ III\ rd\ quadrant\ and\ the$ $points\ move\ away\ from\ X - axis\ \ perpendicularly\ towards\ I. k\ varies\ from$

$1\ to\ \frac{3}{4}\ P(-a,-b)$

$-$ *is the point varying from I to N, its domain is definable as varying between* $\frac{4\pi_3^a}{3} \leq z \leq \frac{14\pi_3^a}{9}$; *this domain belongs to III rd quadrant.*

$P(a,-b)$ $-$ *It is the point on the circumferance varying from N to A, and this domain can be written as* $\frac{14\pi_3^a}{9} \leq z \leq 2\pi_3^a$; *This domain belongs to IV th quadrant*

OA $-$ *is the base of the* $\Delta\, OAB$ *and maximum radius within the triangle AHI, it's equal to* $2R_a$ *at this position.*

$\angle AOH$ $-$ *is the angle subtended by the side AH at the centre. And it is equal to* $\frac{2\pi_3^a}{3}$ *rad or* $rad_{3.}$

$AP(\pm a, \pm b)$

$-$ *is the part of the perimeter of equilateral triangle with unit apothem, measuredfrom A and it is equal to z w.r.t. the respective quadrant and domain.*

$\angle AOP(\pm a, \pm b)$ $-$ *is the angle subtended by the perimeter of the equlateral triangle at the centre of the triangle measured from OA in anti − clockwise direction. Magnitude of this angle is* $\theta_3 = z$ *rad ; segment of this angle is not uniform througout the perimeter of the triangle, it is defined within the limits of the domain as* $\theta_3 = z$ *within the domain of* $0 \leq z \leq 2\sqrt{3}$ *in the Ist quadrant.*

$\angle P(a,b)AC$ $-$ *This angle is constant for all values of* $P(a,b)$ *and OA is a constant equal to 2. And magnitude of angle* equal to $\pi/6$ *in the conventional radian measure.*

3.0.2. Opposite sides of the determinant triangles at different quadrants and domains:

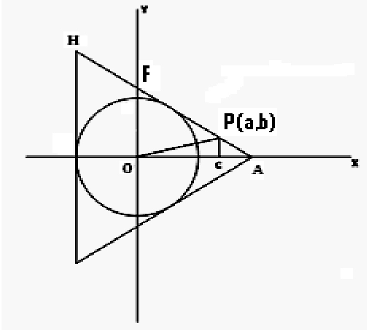


Fig.14:

P(a,b)C - is an opposite side of the determinant triangle OCP(a,b) varying between A and F in the I st quadrant

and in the domain $0 \leq z \leq \frac{4}{\sqrt{3}}$ *here z takes any value from* 0 *to* $2.3094\ldots$ *,; fig14 Then, the length of opposite side in the I st quadrant is,*

$= z \sin\left(\frac{\pi}{6}\right)$

$Then, P(a,b)C = \frac{z}{2}$; *since* $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$$P(a,b)C = \frac{z}{2} \quad -------- (34)$$

For example when $z = 0$; *Opposite side* $P(a,b)C = \frac{z}{2}$ *will be*

$P(a,b)C = 0$

When $z = \sqrt{3}$ *value of,* $P(a,b)C = \frac{z}{2}$ *will be*

$P(a,b)C = \frac{\sqrt{3}}{2}$

When $z = \frac{4}{\sqrt{3}}$ *we have*

$P(a,b)C = \frac{z}{2}$ *will be equal to*

$$P(a,b)C = \frac{\frac{4}{\sqrt{3}}}{2}$$

$$= \frac{2}{\sqrt{3}}$$

Check for the above result:

It is known that length of the base of the triangle AOF $(Fig. 14)$ *is* 2 *units and length of hypotenuse is* $\frac{4\pi_3^a}{9}$

$$= \frac{4}{\sqrt{3}} \ units$$

Then the magnitude of opposite side, $OF = \sqrt{\frac{16}{3} - 4}$

$$= \frac{2}{\sqrt{3}}$$

Thus checked.

3.0.3. *Further, in the II nd quadrant same domain extends upto H i.e,* $\frac{4}{\sqrt{3}} \leq z \leq 2\sqrt{3}$; *here z varies from* $\frac{4}{\sqrt{3}}$ *to* $2\sqrt{3}$; $Fig. 15$.

Then $GP(-a,b) = z \sin\left(\frac{\pi}{6}\right)$

$$GP(-a,b) = \frac{z}{2} ------ (35)$$

For example when the magnitude of $AP(-a,b)$ *is* $2\sqrt{3}$ *magnitude of opposite side will be*

$$GP(-a,b) = \frac{z}{2}$$

$$= \frac{2\sqrt{3}}{2}$$

$$= \sqrt{3}$$

This $\sqrt{3}$ *is the length of the opposite side of the triangle AKH, also half of the side HI.*

This is the maximum length of the opposite side in the I st and II nd domains.

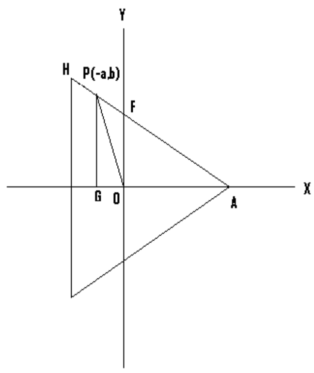


Fig.15.

3.0.4. *And in the II nd quadrant for the domain* $2\sqrt{3} \leq z \leq 3\sqrt{3}$, *here z varies from* $2\sqrt{3}$ *to* $3\sqrt{3}$; *and* ; $Fig. 16$.

The typical opposite side, $KP(-1,b) = \pi_3^a - z$

Here $z = AH + HP(-1,b)$

Since AH is a constant equal to $2\sqrt{3}$; $z = 2\sqrt{3} + HP(-1,b)$

Consequently, $HP(-1,b) = z - 2\sqrt{3}$

$$HP(-1,b) = z - 2\sqrt{3}$$

$HP(-1,b) = z - 2\sqrt{3}$

$HP(-1,b) = (z - 2\sqrt{3}) \quad ---------(36)$

$For\ example, if\ the\ values\ of\ z\ are\ 2\sqrt{3}, \frac{5\sqrt{3}}{2}, 3\sqrt{3}\ , the\ values\ of\ HP(-1,b)\ will\ be$

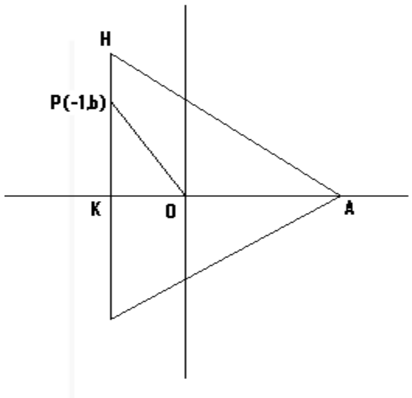


Fig.16.

$When\ 2 = 2\sqrt{3}\ ;\ \ HP(-1,b) = (z - 2\sqrt{3})$

$i.e., HP(-1,b) = (2\sqrt{3} - 2\sqrt{3})$

$= 0$

$Similarly\ when\ z = \frac{5\sqrt{3}}{2}\quad ;\ HP(-1,b) = (z - 2\sqrt{3})$

$= \frac{\sqrt{3}}{2}$

$And\ when\ z = 3\sqrt{3},\ \ ; HP(-1,b) = (z - 2\sqrt{3})$

$= \sqrt{3}$

$Now\ the\ opposite\ side\ KP(-1,b) = \left(3\sqrt{3} - \left(2\sqrt{3} + (z - 2\sqrt{3})\right)\right)$

$= \sqrt{3} - (z - 2\sqrt{3})$

$Opposite\ side\ KP(-1,b) = (3\sqrt{3} - z) \quad ------(37)$

$Similarly\ the\ values\ of\ K\mathrm{P}(-1,b)\ to\ the\ values\ of\ z = 2\sqrt{3}, 2.5\sqrt{3}, 3\sqrt{3}\ \ , \pi_3^a\ are,$

$For\ z = 2\sqrt{3}\ \ or\ \ k = \frac{3}{2}\ \ we\ have\ ;\ KP(-a,b) = (3\sqrt{3} - z)$

$= (3\sqrt{3} - 2\sqrt{3})$

$= \sqrt{3}$

3.0.5. $Further\ in\ the\ III\ rd\ quadrant, domain\ is\ definable\ as\ varying$ between $3\sqrt{3}\ \le\ z\ \le\ 4\sqrt{3}\ ; z\ varies\ 3\sqrt{3}\ to\ 4\sqrt{3}\ \ ;\ Fig. 17.$

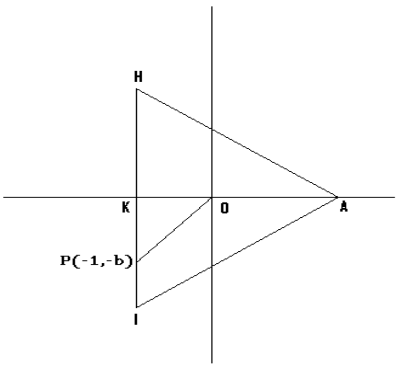


Fig.17.

$-KP(-1,-b) = -(z - \pi_3^a\ )$

$= (3\sqrt{3} - z)$

$-KP(-1, -b) = (3\sqrt{3} - z)$ $---------(38)$

Also, $\qquad = (3\sqrt{3} - z)$

For example when the value of $z = 4\sqrt{3}$

$-KP(-1, -b) = (3\sqrt{3} - 4\sqrt{3})$

$= -\sqrt{3}$

2.06. *Next in the III rd quadrant which consists the domain* $4\sqrt{3} \leq z \leq \frac{14}{\sqrt{3}}$ *; this domain varies between* $4\sqrt{3}$ *to* $\frac{14}{\sqrt{3}}$; *Fig.* 18.

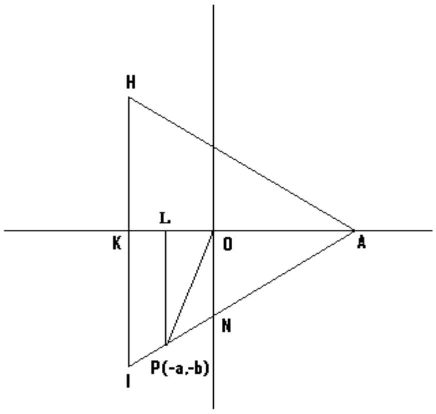


Fig.18.

Here $4\sqrt{3}$ *is the length of* AHI; $IP(-a, -b) = z - AHI = z - 4\sqrt{3}$

hence $AP(-a.-b) = AP(-a, -b) - IP(-a, -b) = (z - 4\sqrt{3}) - 2\sqrt{3}$

Then , opposite side , $-LP(-a, -b) = [(z - 4\sqrt{3}) - 2\sqrt{3}] \sin\left(\frac{\pi}{6}\right)$

$$= \frac{[(z - 4\sqrt{3}) - 2\sqrt{3}]}{2}$$

$$-LP(-a, -b) = \frac{[(z - 6\sqrt{3})]}{2} \quad -------- (39)$$

For example when, $z = \frac{14\sqrt{3}}{3}$

$$-LP(-a, -b) = \frac{\left[\left(\frac{14\sqrt{3}}{3} - 6\sqrt{3}\right)\right]}{2}$$

$$= -\frac{2}{\sqrt{3}}$$

3.0.7. *And finally the IV th quadrant that consists the domain,* $\frac{14}{\sqrt{3}} \leq z \leq 6\sqrt{3}$ *here z varies from* $\frac{14}{\sqrt{3}}$ *to* $6\sqrt{3}$; *fig.* 19.

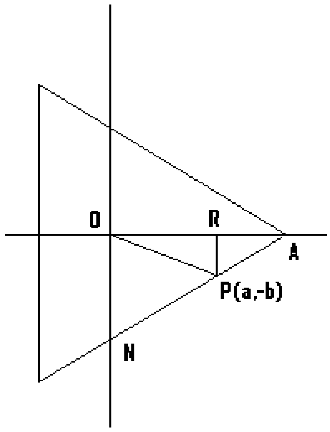


Fig.19.

$$-RP(a,-b) = \left[\left(z - \frac{14}{\sqrt{3}}\right) - \frac{4}{\sqrt{3}}\right] sin\left(\frac{\pi}{6}\right)$$
$$= \frac{\sqrt{3}\,z - 18}{2\sqrt{3}}$$
$$-RP(a,-b) = \frac{z - 6\sqrt{3}}{2} \quad --------(40)$$
$When\ z = 6\sqrt{3}$
$$-RP(a,-b) = \frac{6\sqrt{3}\ - 6\sqrt{3}}{2}$$
$$= 0$$

3.0.8. Adjacent sides of determinant triangles at different quadrants and domains:

$Adjacent\ side\ OC\ exists\ in\ the\ first\ quadrant\ and\ its\ domain\ is$

$0 \le z \le \frac{4}{\sqrt{3}}\ \ here\ z\ takes\ any\ value\ from\ 0\ to\ 2.3094\ \ldots.; Fig.14$

$Here\ OC\ is\ a\ typical\ adjacent\ side\ that\ varies\ between\ A\ and\ \ O.$

$A\ typical\ adjacent\ side, OC = OA - \left[A\ P(a,b)\ cos\left(\frac{\pi}{6}\right)\right];\ Fig.14.$

$$i.e., OC = 2 - \left[z\,cos\left(\frac{\pi}{6}\right)\right]$$
$$OC = \frac{4 - \sqrt{3}z}{2} \ ---------(41)$$

3.0.9. Values of adjacent side at some of significant positions:

$When\ z = 0$

$Applying\ (41) we\ have\ OC = \frac{4 - \sqrt{3}z}{2}$
$$= \frac{4 - 0}{2}$$
$$= 2$$
$When\ z = \sqrt{3}$
$$OC = \frac{1}{2}$$
$When\ z = \frac{4}{\sqrt{3}}$
$$OC = 0$$

3.1.0. $Further, in\ the\ II\ nd\ \ quadrant\ and\ domain, \frac{4}{\sqrt{3}} \le z \le 2\sqrt{3}, the\ \ typical\ adjacent\ side\ is - OG$ $that\ varies\ from\ O\ to\ K.; Fig(15).$

$i.e, -OG = 2 - \left[A\,P(-a,b)\,cos\left(\frac{\pi}{6}\right)\right]$

$= 2 - [A\,P(-a,b)]\frac{\sqrt{3}}{2}$

$= 2 - \frac{z\sqrt{3}}{2}$

$-OG = \frac{4 - z\sqrt{3}}{2} - - - - - - - - (42)$

$For\ example\ when\ z = \frac{4}{\sqrt{3}}\ ; (42) will\ be$

$-OG = \frac{4 - \left(\frac{4}{\sqrt{3}}\right)\sqrt{3}}{2}$

$-OG = 0$

$Also\ when\ z = 2\sqrt{3}$

$(42) will\ be\ equal\ to\ -1$

3.1.1. $In\ the\ same\ quadrant\ another\ domain\ i.e\ 2\sqrt{3} < z \leq 3\sqrt{3}\ or\ \pi_3^a$ $exists, and\ changes\ its\ direction\ by, \frac{\pi}{3}, w.r.t\ the\ side\ AH\ and\ X\ axis, these\ points\ move\ perpendicularly\ and\ collinearly\ upto\ X\ axis, consequently\ value\ of\ adjacent\ side\ remains\ constant. Fig. 16$

$i.e, -OK = 2 - (AH)\,cos\left(\frac{\pi}{6}\right)$

$= 2 - 2\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)$

$-OK = -1 - - - - - - - - (43)$

3.1.2. $Also\ in\ the\ III\ rd\ quadrant\ same\ condition\ continues, i.e, in\ the\ domain$

$3\sqrt{3} < z \leq \frac{4\pi_3^a}{3}$ $, points\ of\ this\ domain\ keep\ perpendicularity\ and\ collinearity\ with\ X\ axis\ up\ to\ the\ vertex\ I. Fig. 17.$

$-OK = 2 - (AI)\,cos\left(\frac{\pi}{6}\right)$

$Then, -OK = -1 - - - - - - - - (44)$

3.1.3. $Further, in\ the\ III\ rd\ quadrant\ base\ of\ the\ determinant\ triangle\ begins\ to\ increase\ from\ -1\ to\ 0\ in\ the\ domain\ 4\sqrt{3} \leq z \leq \frac{14}{\sqrt{3}}\ which\ extends\ up\ to\ N. Fig. 18$

$-OL = \left[\left(z - 6\sqrt{3}\right) cos\left(\frac{\pi}{6}\right)\right] + 2$

$-OL = \frac{\sqrt{3}z - 14}{2} - - - - - - (45)$

$When, z = 4\sqrt{3}$

$-OL = -1$

$When\ z = \frac{14}{\sqrt{3}}$

$-OL = \frac{\sqrt{3}z - 14}{2}$

$= \frac{\sqrt{3}\frac{14}{\sqrt{3}} - 14}{2}$

$= 0$

3.1.4. $Finally\ in\ the\ IV\ th\ quadrant\ and\ in\ the\ domain\ \frac{14}{\sqrt{3}} \leq z \leq 6\sqrt{3};\ fig. 19.$

$The\ value\ of\ the\ base\ of\ the\ determinant\ triangle\ will\ be,\ OR = OA - \left[A\,P(a,-b)\,cos\left(\frac{\pi}{6}\right)\right]$

$-OR = 2 - \left[\left(6\sqrt{3} - z\right) cos\left(\frac{\pi}{6}\right)\right]$

$-OR = \frac{\sqrt{3}z - 14}{2}\ - - - - - - - - - - - (46)$

$When, z = \frac{14}{\sqrt{3}}$

$-OR = \frac{\sqrt{3}z - 14}{2}$

$= \frac{\sqrt{3}\left(\frac{14}{\sqrt{3}}\right) - 14}{2}$

$= 0$

$And\ when\ z = 6\sqrt{3}$

$-OR = \frac{\sqrt{3}z - 14}{2}$

$= \frac{\sqrt{3}(6\sqrt{3}) - 14}{2}$

$= 2$

3.1.5. Hypotenuses (radii) of determinant triangles at different quadrants and domains:

$P(a,b)O\ -\ is\ an\ hypotenuse\ or\ variable\ radius\ of\ the\ determinant\ triangle\ OCP\ varying$

$between\ A\ and\ F\ \ in\ the\ I\ st\ quadrant\ and\ in\ the\ domain\ 0 \le z \le \frac{4}{\sqrt{3}}; Fig. 14.$

All hypotenuses are considered as having absolute values.

$Now\ a\ typical\ hypotenuse\ or\ radius, OP(a,b) = \sqrt{OC^2 + PC^2}$

From (32) and

$= \sqrt{\left(\frac{4 - \sqrt{3}z}{2}\right)^2 + \left(\frac{z}{2}\right)^2}$

$OP(a,b) = \sqrt{z^2 - 2\sqrt{3}z + 4} - - - - - (47)$

$When\ z = 0$

$The\ hypotenuse, OP(a,b) = \sqrt{z^2 - 2\sqrt{3}z + 4}$

$= 2$

$And\ when\ , z = \frac{4}{\sqrt{3}}$

$OP(a,b) = \sqrt{z^2 - 2\sqrt{3}z + 4}$

$= \sqrt{\left(\frac{4}{\sqrt{3}}\right)^2 - 2\sqrt{3}\left(\frac{4}{\sqrt{3}}\right) + 4}$

$= \frac{2}{\sqrt{3}}$

3.1.6. $Further, in\ the\ II\ nd\ \ quadrant\ and\ domain, \frac{4}{\sqrt{3}} \le z \le 2\sqrt{3}, the\ \ typical\ hypotenuse$

$is\ OP(-a,b)\ \ that\ varies\ from\ F\ to\ H.$; Fig. 15

$Then, OP(-a,b) = \sqrt{OG^2 + [GP(-a,b)]^2}$

From (35) and (42) we have

$= \sqrt{\left(\frac{4 - z\sqrt{3}}{2}\right)^2 + \left(\frac{z}{2}\right)^2}$

$OP(-a,b) = \sqrt{z^2 - 2\sqrt{3}z + 4} - - - - - - (48)$

$when, z = \frac{4}{\sqrt{3}},\qquad The\ value\ of\ the\ radius\ will\ be, OP(-a,b) = \sqrt{z^2 - 2\sqrt{3}z + 4}$

$= \sqrt{\left(\frac{4}{\sqrt{3}}\right)^2 - 2\sqrt{3}\left(\frac{4}{\sqrt{3}}\right) + 4}$

$= \frac{2}{\sqrt{3}}$

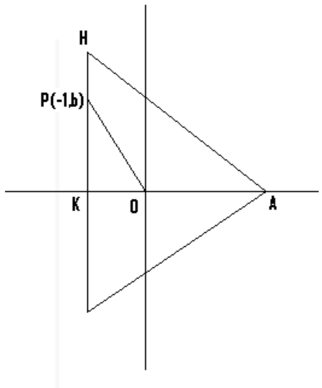
$When\ z = 2\sqrt{3}$

(48) will be

$OP(-a,b) = \sqrt{\left(2\sqrt{3}\right)^2 - 2\sqrt{3}\left(2\sqrt{3}\right) + 4}$

$= 2$

3.1.7. $And\ in\ the\ II\ nd\ quadrant\ for\ the\ domain\ \ 2\sqrt{3}\ \le z \le 3\sqrt{3}\ ;\ Fig16.$



Fig(20)

$A\ typical\ hypotenuse\ in\ this\ domain\ is\ OP(-1,b) = \sqrt{OK^2 + [OP(-1,b)]^2}$

$$= \sqrt{1 + \left[3\sqrt{3} - z\right]^2}$$

$OP(-1,b) = \sqrt{z^2 - 6\sqrt{3}z + 28}\ ------(49)$

$When, z = 2\sqrt{3}\ , the\ value\ of\ OP(-1,b)\ will\ be$

$OP(-1,b) = \sqrt{z^2 - 6\sqrt{3}z + 28}$

$= \sqrt{\left(2\sqrt{3}\right)^2 - 6\sqrt{3}(2\sqrt{3}\ ) + 28}$

$= 2$

$When\ z = 3\sqrt{3}$

$OP(-1,b) = \sqrt{z^2 - 6\sqrt{3}z + 28}$

$= \sqrt{\left(3\sqrt{3}\right)^2 - 6\sqrt{3}\left(3\sqrt{3}\right) + 28}$

$= 1$

3.1.8. $Further, in\ the\ III\ rd\ \ quadrant\ same\ condition\ continues\ i.e, in\ the\ domain\ 3\sqrt{3} <\ z \le\ 4\sqrt{3},$ $points\ move\ away\ from\ X - axis, perpendicularly.\ \ Fig.17.$

$The\ length\ of\ radius\ or\ hypotenuse, OP(-1.-b) = \sqrt{1 + \left(z - 3\sqrt{3}\right)^2}$

$OP(-1.-b) = \sqrt{z^2 - 6\sqrt{3}z + 28} ------(50)$

$When, z = 3\sqrt{3}$

$OP(-1.-b) = \sqrt{z^2 - 6\sqrt{3}z + 28}$

$= \sqrt{\left(3\sqrt{3}\right)^2 - 6\sqrt{3}\left(3\sqrt{3}\right) + 28}$

$= 1$

$When\ z = 4\sqrt{3}$

$OP(-1.-b) = \sqrt{z^2 - 6\sqrt{3}z + 28}$

$= \sqrt{\left(4\sqrt{3}\right)^2 - 6\sqrt{3}\left(4\sqrt{3}\right) + 28}$

$= 2$

3.1.9. $Moving\ to\ the\ next\ domain\ in\ the\ same\ IIIrd\ quadrant\ 4\sqrt{3}\ \leq\ z\ \leq \frac{14}{\sqrt{3}};\ Fig.18$

$OP(-a,-b) = \sqrt{(OL)^2 + [LP(-a,-b)]^2}$

From (45) and (39) we have

$$OP(-a,-b) = \sqrt{\left(\frac{\sqrt{3}z - 14}{2}\right)^2 + \left(\frac{\left[\left(z - 6\sqrt{3}\right)\right]}{2}\right)^2}$$

$OP(-a,-b) = \sqrt{z^2 - 10\sqrt{3}z + 76}\ ------(51)$

$When, z = 4\sqrt{3}$

$OP(-a,-b) = \sqrt{z^2 - 10\sqrt{3}z + 76}$

$= \sqrt{\left(4\sqrt{3}\right)^2 - 10\sqrt{3}\left(4\sqrt{3}\right) + 76}$

$= 2$

$When\ z = \frac{14}{\sqrt{3}}$

$OP(-a,-b) = \sqrt{z^2 - 10\sqrt{3}z + 76}$

$= \sqrt{\left(\frac{14}{\sqrt{3}}\right)^2 - 10\sqrt{3}\left(\frac{14}{\sqrt{3}}\right) + 76}$

$= \frac{2}{\sqrt{3}}$

3.2.0. $Finally\ in\ the\ IV\ th\ quadrant\ and\ the\ domain\ \frac{14}{\sqrt{3}} \leq z \leq 6\sqrt{3}\ ; Fig.19.$

$OP(\mathrm{a},-b) = \sqrt{OR^2 + [RP(a,-b)]^2}$

From (46) and (40) we have

$$Then\ , OP(a,-b) = \sqrt{\left(\frac{\sqrt{3}z - 14}{2}\right)^2 + \left(\frac{\sqrt{3}\,z - 18}{2\sqrt{3}}\right)^2}$$

$OP(a,-b) = \sqrt{z^2 - 10\sqrt{3}z + 76}\ -------(52)$

$When\ z = \frac{14}{\sqrt{3}}$

$OP(a,-b) = \sqrt{z^2 - 10\sqrt{3}z - 76}$

$= \sqrt{\left(\frac{14}{\sqrt{3}}\right)^2 - 10\sqrt{3}\left(\frac{14}{\sqrt{3}}\right) + 76}$

$= \frac{2}{\sqrt{3}}$

$When, z = 6\sqrt{3}$

$P(a,-b) = \sqrt{z^2 - 10\sqrt{3}z + 76}$

$= \sqrt{\left(6\sqrt{3}\right)^2 - 10\sqrt{3}\left(6\sqrt{3}\right) + 76}$

$= 2$

IV. Now, coming to the trigonometric functions with respect to π_n,

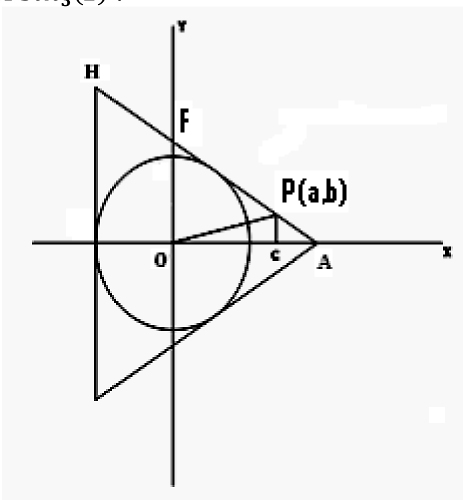
They are written as following to provide with uniqueness.

$$\boldsymbol{sin_n\,(x), cos_n(x), tan_n(x) etc.}$$

For example, for an equilateral triangle, the above expressions will be

$$\boldsymbol{sin_3\,(x), cos_3(x), tan_3(x) etc.}$$

4.0.1. Trigonometric Functions w.r.t. π_3^a:
4.0.2. Mathematical expression of $sin_3(z)$:



Fig(21)

$In\ the\ above\ figure\ AF\ is\ the\ domain\ defined\ by\ the\ expression\ of\ inequality\ , 0 \leq z \leq \frac{4}{\sqrt{3}},$
$where\ P(a,b) is\ the\ variable\ position\ of\ moving\ radius\ OP(a,b) measured\ from\ \ A\ in\ the\ anticlockwise\ direction.$

$Accordingly, sin_3(z) = \frac{Opposite\ side}{Hypotenuse}\ ; of\ the\ determinant\ triangle$

$= \frac{CP(a,b)}{OP(a,b)}$

$From\ (34) and\ (47) we\ have\ \ sin_3(z) = \frac{\left(\frac{z}{2}\right)}{\sqrt{z^2 - 2\sqrt{3}z + 4}}$

$sin_3(z) = \frac{z}{2\sqrt{z^2 - 2\sqrt{3}z + 4}} - - - - - - - - - - (53)$

Some of the significant values of sin (z) at significant positions are:

$When\ \theta_3 or\ z = 0$

$sin_3(0) = \frac{z}{2\sqrt{z^2 - 2\sqrt{3}z + 4}}$

$= \frac{0}{2\sqrt{z^2 - 2\sqrt{3}z + 4}}$

$= 0$

$Further, when\ z = \sqrt{3}$

$sin_3(\sqrt{3}) = \frac{z}{2\sqrt{z^2 - 2\sqrt{3}z + 4}}$

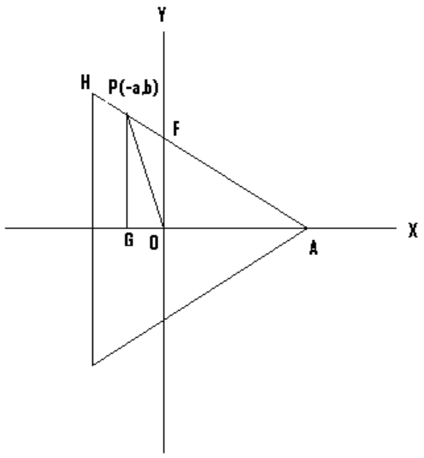
$= \frac{\sqrt{3}}{2}$

$Now, when\ z = \frac{4}{\sqrt{3}}$

$$sin_3\left(\frac{4}{\sqrt{3}}\right) = \frac{z}{2\sqrt{z^2 - 2\sqrt{3}z + 4}}$$
$$= \frac{\frac{4}{\sqrt{3}}}{2\sqrt{\left(\frac{4}{\sqrt{3}}\right)^2 - 2\sqrt{3}\left(\frac{4}{\sqrt{3}}\right) + 4}}$$
$$= 1$$

Further moving to the next domain

4.0.3. *Now, FH, Fig(15) is the domain defined by the expression of inequality,* $\frac{4}{\sqrt{3}} \le z \le 2\sqrt{3}$ *and it is in the II nd quadrant.*



Fig(21a)

$$sin_3(z) = \frac{GP(-a,b)}{OP(-a,b)}$$

From (35) *and* (48) $sin_3(z) = \frac{\frac{z}{2}}{\sqrt{z^2 - 2\sqrt{3}z + 4}}$

$$sin_3(z) = \frac{z}{2\sqrt{z^2 - 2\sqrt{3}z + 4}} ------(54)$$

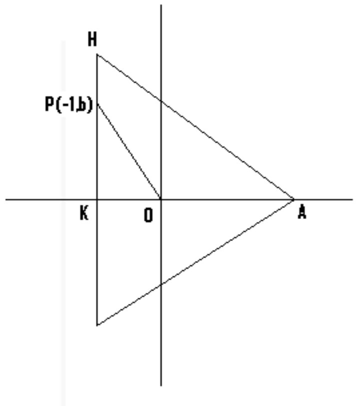
When $z = \frac{4}{\sqrt{3}}$

$$sin_3\left(\frac{4}{\sqrt{3}}\right) = \frac{z}{2\sqrt{z^2 - 2\sqrt{3}z + 4}}$$
$$sin_3(z) = 1$$

When , $z = 2\sqrt{3}$

$$sin_3(2\sqrt{3}) = \frac{z}{2\sqrt{z^2 - 2\sqrt{3}z + 4}}$$
$$= \frac{2\sqrt{3}}{2\sqrt{(2\sqrt{3})^2 - 2\sqrt{3}(2\sqrt{3}) + 4}}$$
$$= \frac{\sqrt{3}}{2}$$

4.0.4. *Moving to the next domain in the same II nd quadrant Fig*(16) , $2\sqrt{3} \le z \le 3\sqrt{3}$;



Fig(22)

$we\ have, sin_3(z) = \dfrac{KP(-1,b)}{OP(-1,b)}$

$From\ (37)\ and\ (49)\quad sin_3(z) = \dfrac{(3\sqrt{3}-z)}{\sqrt{z^2-6\sqrt{3}z+28}} -------(55)$

$When\ z = 2\sqrt{3}$

$Then\ , sin_3(2\sqrt{3}) = \dfrac{(3\sqrt{3}-z)}{\sqrt{z^2-6\sqrt{3}z+28}}$

$= \dfrac{(3\sqrt{3}-2\sqrt{3})}{\sqrt{(2\sqrt{3})^2-6\sqrt{3}(2\sqrt{3})+28}}$

$= \dfrac{\sqrt{3}}{2}$

$When, z = 3\sqrt{3}$

$sin_3(z) = \dfrac{(3\sqrt{3}-z)}{\sqrt{z^2-6\sqrt{3}z+28}}$

$= \dfrac{(3\sqrt{3}-3\sqrt{3})}{\sqrt{z^2-6\sqrt{3}z+28}}$

$= 0$

4.0.5. $Now\ \ in\ the\ III\ rd\ quadrant\ \ and\ domain\ 3\sqrt{3} \le z \le 4\sqrt{3}$

$Then, sin_3(z) = \dfrac{-KP(-1,-b)}{OP(-1,-b)}$

$From\ (38) and (50), sin_3(z) = \dfrac{(3\sqrt{3}-z)}{\sqrt{z^2-6\sqrt{3}z+28}} ----- (56)$

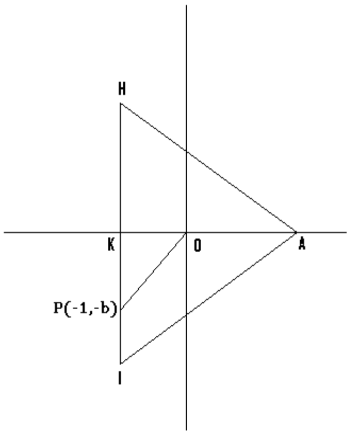
$When, z = 3\sqrt{3}\ ;\ The\ value\ of\quad sin_3(\theta)\ will\ be,$

$sin_3(3\sqrt{3}) = \dfrac{(3\sqrt{3}-z)}{\sqrt{z^2-6\sqrt{3}z+28}}$

$= \dfrac{3\sqrt{3}-3\sqrt{3}}{\sqrt{z^2-6\sqrt{3}z+28}}$

$= 0$

Fig(23)



$When, z = 4\sqrt{3}$

$sin_3(3\sqrt{3}) = \dfrac{(3\sqrt{3} - z)}{\sqrt{z^2 - 6\sqrt{3}z + 28}}$ $will\ be$

$sin_3(3\sqrt{3}) = \dfrac{(3\sqrt{3} - 4\sqrt{3})}{\sqrt{(4\sqrt{3})^2 - 6\sqrt{3}(4\sqrt{3}) + 28}}$

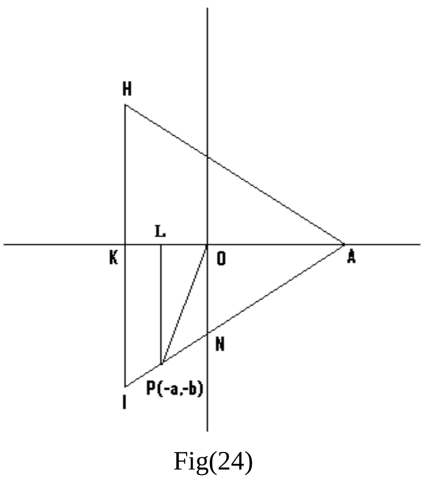
$= \dfrac{\sqrt{3}}{2}$

4.0.6. $In\ the\ same\ III\ rd\ quadrant\ and\ in\ the\ domain\ 4\sqrt{3} \leq\ z\ \leq \dfrac{14}{\sqrt{3}}, value\ of$

$sin_3(z), will\ be,$

$sin_3(z) = \dfrac{-LP(-a, -b)}{OP(-a, -b)}$

$Substituting\ \ (39)\ for - LP\ and\ (51) for\ OP(-a, -b) we\ have,$



Fig(24)

$$sin(z) = \frac{\dfrac{[(z - 6\sqrt{3})]}{2}}{\sqrt{z^2 - 10\sqrt{3}z + 76}}$$

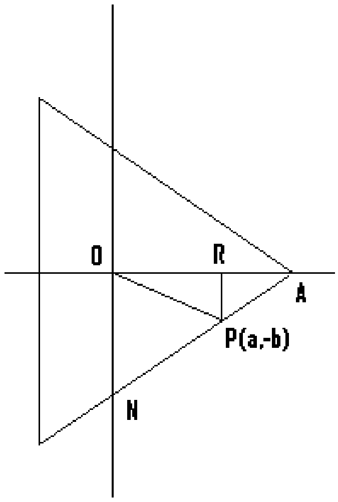
$$sin_3(z) = \frac{[(z - 6\sqrt{3})]}{2\sqrt{z^2 - 10\sqrt{3}z + 76}} - - - - - (57)$$

$$When\ z = \frac{14}{\sqrt{3}}$$

$$sin_3\left(\frac{14}{\sqrt{3}}\right) = \frac{\left[\left(\frac{14}{\sqrt{3}} - 6\sqrt{3}\right)\right]}{2\sqrt{\left(\frac{14}{\sqrt{3}}\right)^2 - 10\sqrt{3}\left(\frac{14}{\sqrt{3}}\right) + 76}}$$

$$= -1$$

3.07. $Final\ quadrant\ is\ IV\ th\ quadrant\ and\ domain\ is\ \frac{14}{\sqrt{3}} \leq\ z\ \leq 6\sqrt{3}\ , value\ of\ sin_3(z)\ is,$



Fig(25)

$$sin_3(z) = \frac{-RP(a, -b)}{OP(a, -b)}$$

$$From, (40) and\ (52), we\ have$$

$$sin_3(z) = \frac{\frac{z - 6\sqrt{3}}{2}}{\sqrt{z^2 - 10\sqrt{3}z + 76}}$$

$$sin_3(z) = \frac{z - 6\sqrt{3}}{2\sqrt{z^2 - 10\sqrt{3}z + 76}} - - - - (58)$$

$$When, z =\ 6\sqrt{3}, the\ value\ of\ sin_3(\theta)\ will\ be$$

$$sin_3(6\sqrt{3}) = \frac{6\sqrt{3} - 6\sqrt{3}}{2\sqrt{z^2 - 10\sqrt{3}z + 76}}$$

$$= 0$$

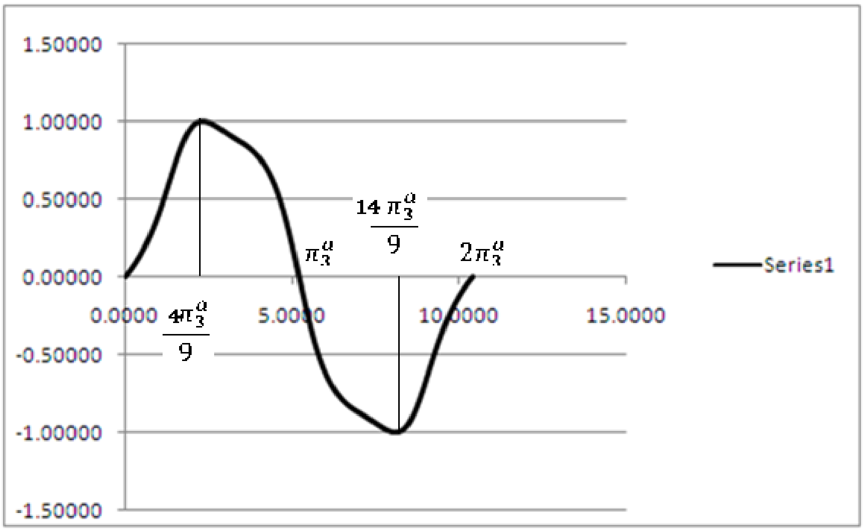
V. Graph of $\sin_3(z)$

Graph is drawn for the values of z , in the table furnished below . $2\pi_3^a$ or 6 $\sqrt{3}$ is divided into 360 units.

TABLE-I

Sl.No.	Equation	Z	Sin(z)
1.	$sin_3(z) = \frac{z}{2\sqrt{z^2 - 2\sqrt{3}z + 4}}$	0.0000	0.0000
2.		0.0289	0.0073
3.		0.0577	0.0148
4.		0.0866	0.0225
5.		0.1155	0.0304

Sl.No.	Equation	Z	Sin(z)
6.		0.1443	0.0385
.	.	.	.
.	.	.	.
.	.	.	.
		2.2517	0.9990
	$Domain, \quad 0 \leq z \leq \frac{4}{\sqrt{3}}$	2.2805	0.9998
80		2.3094	1.0000
81		2.3383	0.9998
82	$sin_3(z) = \frac{z}{2\sqrt{z^2 - 2\sqrt{3}z + 4}}$	2.3671	0.9991
83		2.3960	0.9981
.	.	.	.
.	.	.	.
.	.	.	.
118		3.4064	0.8733
119		3.4352	0.8697
120	$Domain, \ \frac{4}{\sqrt{3}} \leq z \leq 2\sqrt{3}$	3.4641	0.8660
121		3.4930	0.8624
122	$sin(z) = \frac{(3\sqrt{3} - z)}{\sqrt{z^2 - 6\sqrt{3}z + 28}}$	3.5218	0.8585
123		3.5507	0.8546
.		.	.
.		.	.
.		.	.
178		5.1384	0.05764
179		5.1673	0.02886
180	$2\sqrt{3} \leq z \leq 3\sqrt{3}$	5.1962	0.00000
181		5.2250	-0.02886
182	$sin_3(z) = \frac{(3\sqrt{3} - z)}{\sqrt{z^2 - 6\sqrt{3}z + 28}}$	5.2539	-0.05764
183		5.2828	-0.08628
.		.	.
.		.	.
.		.	.
238		6.8705	-0.8585
239		6.8993	-0.8624
240	$3\sqrt{3} \leq z \leq 4\sqrt{3}$	6.9282	-0.8660
241		6.9571	-0.8697
242	$sin_3(z) = \frac{[(z - 6\sqrt{3})]}{2\sqrt{z^2 - 10\sqrt{3}z + 76}}$	6.9859	-0.8733
243		7.0148	-0.8771
.		.	.
.		.	.
.		.	.
278		8.0252	-0.99911
279		8.0540	-0.99977
280	$4\sqrt{3} \leq z \leq \frac{14}{\sqrt{3}}$	8.0829	-1.00000
281		5.2250	-0.02886
282	$sin_3(z) = \frac{z - 6\sqrt{3}}{2\sqrt{z^2 - 10\sqrt{3}z + 76}}$	5.2539	-0.05764
283		5.2828	-0.08628
.		.	.
.		.	.
.		.	.
358		10.3346	-0.01480
359		10.3634	-0.00731
360	$\frac{14}{\sqrt{3}} \leq z \leq 6\sqrt{3}$	10.3923	0.00000



5.0.1. Mathematical expression of $sin_3(-z)$:

When the variable radius is rotated in the clockwise direction the valus of $\boldsymbol{sin_3(-z)}$ *will be as following*

Opposite side $RP(a,-b)$ *will be* $=\left(\frac{-z}{2}\right)$

$Then, sin_3(-z) = \frac{\left(\frac{-z}{2}\right)}{\sqrt{z^2-2\sqrt{3}z+4}}$; *here hypotinuse remains same since it is absolute value.*

$= \frac{-z}{2\sqrt{z^2-2\sqrt{3}z+4}}$

$= -\left(\frac{z}{2\sqrt{z^2-2\sqrt{3}z+4}}\right)$

$sin_3(-z) = -(\,sin_3(z))$

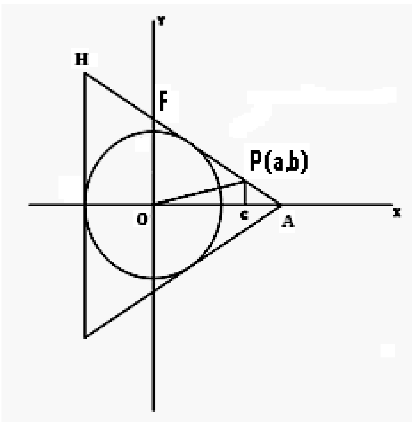
Similarly equations are drawable for all quadrants

VI. Mathematical expression of $cos_3(z)$:

$Value\ of\ , cos_3(z) = \frac{Adjacent\ side}{Hypotenuse}$; *of the determinant triangle*

$i.e.,\quad cos_3(z) = \frac{\pm OC}{OP(\pm a, \pm b)}$

6.0.1. $\boldsymbol{Value\ of\ , cos_3(z)\ in\ the\ Ist\ quadrant\ and\ in\ the\ domain\ 0 \le z \le \frac{4}{\sqrt{3}}}$,



Fig(26)

$$cos_3(z) = \frac{OC}{OP(a,b)}$$

From (41) and (47) we have

$$cos_3(z) = \frac{\frac{4 - \sqrt{3}z}{2}}{\sqrt{z^2 - 2\sqrt{3}z + 4}}$$

$$cos_3(z) = \frac{4 - \sqrt{3}z}{2\sqrt{z^2 - 2\sqrt{3}z + 4}} ------(59)$$

$When\ z = 0$

$$Accordingly, cos_3(z) = \frac{4 - \sqrt{3}z}{2\sqrt{z^2 - 2\sqrt{3}z + 4}}$$

$$cos_3(0) = \frac{4 - \sqrt{3} \times 0}{2\sqrt{0^2 - 2\sqrt{3} \times 0 + 4}}$$

$= 1$

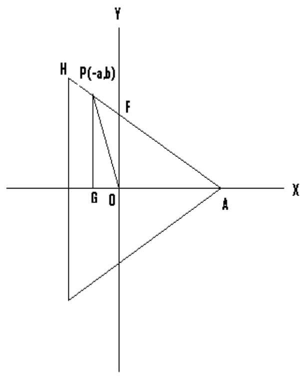
$Also\ , when, z = \frac{4}{\sqrt{3}}, we\ have$

$$cos_3(z) = \frac{4 - \sqrt{3}z}{2\sqrt{z^2 - 2\sqrt{3}z + 4}}$$

$$cos_3\left(\frac{4}{\sqrt{3}}\right) = \frac{4 - \sqrt{3}\left(\frac{4}{\sqrt{3}}\right)}{2\sqrt{\left(\frac{4}{\sqrt{3}}\right)^2 - 2\sqrt{3}\left(\frac{4}{\sqrt{3}}\right) + 4}}$$

$= 0$

6.0.2. $Now, moving\ to\ the\ next\ quadrant\ (II\ nd) and\ domain\ , \frac{4}{\sqrt{3}} \leq z \leq 2\sqrt{3}\ \ , we\ have,$



Fig(27)

$$cos_3(z) = \frac{-OG}{P(-a,b)}$$

$From\ (42) and\ (48) we\ have$

$$cos_3(z) = \frac{\frac{4 - z\sqrt{3}}{2}}{\sqrt{z^2 - 2\sqrt{3}z + 4}}$$

$$cos_3(z) = \frac{4 - z\sqrt{3}}{2\sqrt{z^2 - 2\sqrt{3}z + 4}} ----- (60)$$

$When\ z = \frac{4}{\sqrt{3}}\ , the\ value\ of\ cos_3(z)\ will\ be$

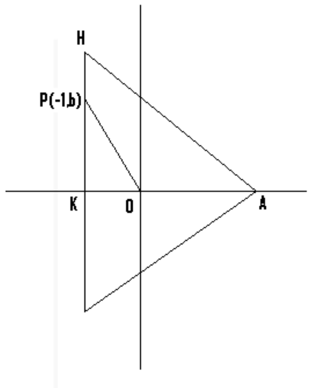
$$cos_3\left(\frac{4}{\sqrt{3}}\right) = \frac{4 - \left(\frac{4}{\sqrt{3}}\right)\sqrt{3}}{2\sqrt{z^2 - 2\sqrt{3}z + 4}}$$
$= 0$

$When, z = 2\sqrt{3}$

$$cos_3(z) = \frac{4 - z\sqrt{3}}{2\sqrt{z^2 - 2\sqrt{3}z + 4}}$$

$$cos_3(2\sqrt{3}) = \frac{4 - (2\sqrt{3})\sqrt{3}}{2\sqrt{(2\sqrt{3})^2 - 2\sqrt{3}(2\sqrt{3}) + 4}}$$
$= -\frac{1}{2}$

6.0.3. $Moving\ to\ the\ next\ domain\ in\ the\ same\ II\ nd\ quadrant\ , 2\sqrt{3} \le z \le 3\sqrt{3};$
From (43) and (49) we have



Fig(28)

$$cos_3(z) = \frac{-OK}{OP(-1,b)}$$

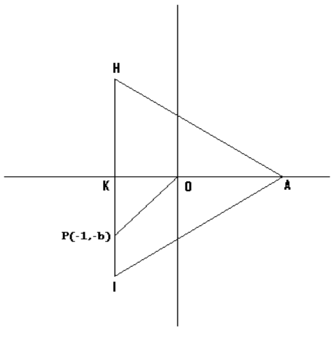
$$we\ have, cos_3(z) = \frac{-1}{\sqrt{z^2 - 6\sqrt{3}z + 28}} ---------(61)$$

$When\ , z = 2\sqrt{3}, the\ value\ of\ cos_3(z)\ will\ be$

$$cos_3(z) = \frac{-1}{\sqrt{z^2 - 6\sqrt{3}z + 28}}$$

$$= \frac{-1}{\sqrt{(2\sqrt{3})^2 - 6\sqrt{3}(2\sqrt{3}) + 28}}$$
$= -\frac{1`}{2}$

6.0.4. $Now\ in\ the\ III\ rd\ quadrant\ and\ domain\ 3\sqrt{3} \le z \le 4\sqrt{3}, value\ of\ cos_3 z\ will\ be$,



Fig(29)

$$cos_3(z) = \frac{-OK}{OP(-1,-b)}$$

From (44)*and* (50), *we have*

$$cos_3(z) = \frac{-1}{\sqrt{z^2 - 6\sqrt{3}z + 28}} ------(62)$$

When, $z = 3\sqrt{3}$; *value of* $cos_3(\theta)$ *will be*

$$cos_3(3\sqrt{3}) = \frac{-1}{\sqrt{(3\sqrt{3})^2 - 6\sqrt{3}(3\sqrt{3}) + 28}}$$

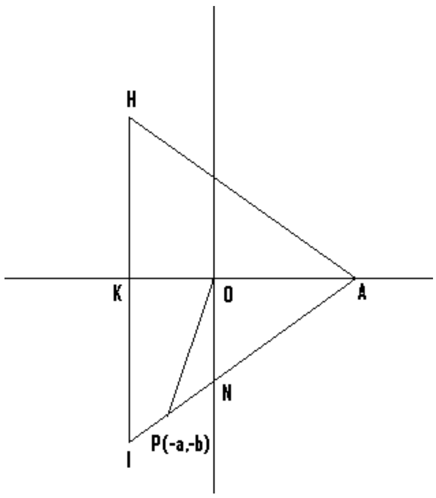
$$= -1$$

When $z = 4\sqrt{3}$, *we have*

$$cos_3(4\sqrt{3}) = \frac{-1}{\sqrt{(4\sqrt{3})^2 - 6\sqrt{3}(4\sqrt{3}) + 28}}$$

$$= -\frac{1}{2}$$

6.0.5. *In the same III rd quadrant and in the domain* $4\sqrt{3} < z \leq \frac{14}{\sqrt{3}}$,



Fig(30)

$$cos_3(z) = \frac{-OL}{OP(-a,-b)}$$

$$cos_3(z) = \frac{\frac{\sqrt{3}z - 14}{2}}{\sqrt{z^2 - 10\sqrt{3}z + 76}}$$

$$cos_3(z) = \frac{\sqrt{3}z - 14}{2\sqrt{z^2 - 10\sqrt{3}z + 76}} ------(63)$$

When the value of $z = 4\sqrt{3}$,, *the value of* $cos_3(z)$ *will be*

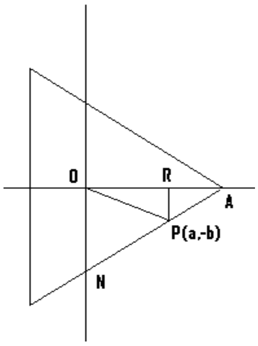
$$cos_3(4\sqrt{3},) = \frac{\sqrt{3}(4\sqrt{3},) - 14}{2\sqrt{(4\sqrt{3},)^2 - 10\sqrt{3}(4\sqrt{3},) + 76}}$$

$$= -\frac{1}{2}$$

6.0.6. *Final quadrant is IV th quadrant and domain is* $\frac{14}{\sqrt{3}} \leq z \leq 6\sqrt{3}$

$$cos_3(z) = \frac{-OR}{OP(a,-b)}$$

From (46)*and* (52)*we have*,



Fig(31)

$$cos_3(z) = \frac{\frac{\sqrt{3}z - 14}{2}}{\sqrt{z^2 - 10\sqrt{3}z + 76}}$$

$$cos_3(z) = \frac{\sqrt{3}z - 14}{2\sqrt{z^2 - 10\sqrt{3}z + 76}} - - - - - (64)$$

$When\ z = \frac{14}{\sqrt{3}}, the\ value\ of\ cos_3(z)\ will\ be,$

$$cos_3(z) = \frac{\sqrt{3}z - 14}{2\sqrt{z^2 - 10\sqrt{3}z + 76}}$$

$$cos_3(z) = \frac{\sqrt{3}\left(\frac{14}{\sqrt{3}}\right) - 14}{2\sqrt{z^2 - 10\sqrt{3}z + 76}}$$

$= 0$

$When\ z = 6\sqrt{3}, the\ value\ of\ cos_3(z)\ will\ be,$

$$cos_3(z) = \frac{\sqrt{3}(6\sqrt{3}) - 14}{2\sqrt{(6\sqrt{3})^2 - 10\sqrt{3}(6\sqrt{3}) + 76}}$$

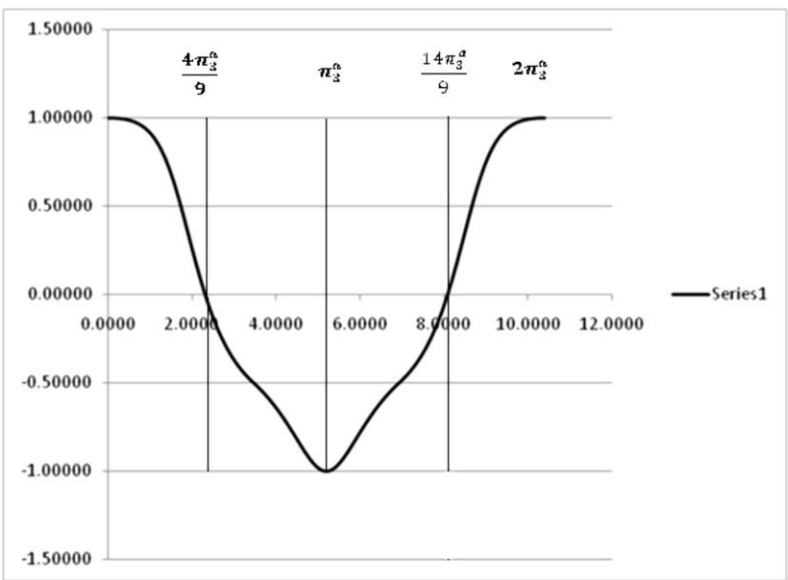
$= 1$

VII. Graph of $\cos_3(z)$

TABLE-II

Sl.No.	Equation	Z	cos(z)
1.	$cos_3(z) = \frac{4 - \sqrt{3}z}{2\sqrt{z^2 - 2\sqrt{3}z + 4}}$	0.0000	1.00000
2.		0.0289	0.99997
3.		0.0577	0.99989
4.		0.0866	0.99975
5.		0.1155	0.99954
6.		0.1443	0.99926
.	.	.	.
.	.	.	.
.	.	.	.
78	$Domain, \quad 0 \le z \le \frac{4}{\sqrt{3}}$	2.2517	0.04437
79		2.2805	0.02192
80		2.3094	0.00000
81	$cos_3(z) = \frac{4 - z\sqrt{3}}{2\sqrt{z^2 - 2\sqrt{3}z + 4}}$	2.3383	-0.02138
82		2.3671	-0.04221
83		2.3960	-0.06248
.	.	.	.
.	.	.	.
.	.	.	.

Sl.No.	Equation	Z	cos(z)
118		3.4064	-0.48713
119		3.4352	-0.49366
120	$Domain, \frac{4}{\sqrt{3}} \le z \le 2\sqrt{3}$	3.4641	-0.50000
121	$cos_3(z) = \frac{-1}{\sqrt{z^2 - 6\sqrt{3}z + 28}}$	3.4930	-0.50632
122		3.5218	-0.51276
123		3.5507	-0.51935
.		.	.
.		.	.
.		.	.
178		5.1384	-0.99834
179		5.1673	-0.99958
180	$2\sqrt{3} \le z \le 3\sqrt{3}$	5.1962	-1.00000
181	$cos_3(z) = \frac{-1}{\sqrt{z^2 - 6\sqrt{3}z + 28}}$	5.2250	-0.99958
182		5.2539	-0.99834
183		5.2828	-0.99627
.		.	.
.		.	.
.		.	.
238		6.8705	-0.51276
239		6.8993	-0.50632
240	$3\sqrt{3} \le z \le 4\sqrt{3}$	6.9282	-0.50000
241	$cos_3(z) = \frac{\sqrt{3}z - 14}{2\sqrt{z^2 - 10\sqrt{3}z + 76}}$	6.9571	-0.49366
242		6.9859	-0.48713
243		7.0148	-0.48040
.		.	.
.		.	.
.		.	.
278		8.0252	-0.04221
279		8.0540	-0.02138
280	$4\sqrt{3} \le z \le \frac{14}{\sqrt{3}}$	8.0829	0.00000
281	$cos_3(z) = \frac{\sqrt{3}z - 14}{2\sqrt{z^2 - 10\sqrt{3}z + 76}}$	8.1118	0.02192
282		8.1406	0.04437
283		8.1695	0.06733
.	.	.	.
.	.	.	.
.	.	.	.
358		10.3346	0.99989
359		10.3634	0.99997
360	$\frac{14}{\sqrt{3}} \le z \le 6\sqrt{3}$	10.3923	1.00000



VIII. Mathematical expression for $\cos_3(-z)$:

When the variable radius is rotated in the clockwise direction the valus of $\boldsymbol{cos_3}(-\boldsymbol{z})$ *will be as following*

Adjacent side OR in the IV th quadrant will be will be $= \dfrac{4-\sqrt{3}z}{2}$

$Then, cos_3(-z) = \dfrac{\frac{4-\sqrt{3}z}{2}}{\sqrt{z^2-2\sqrt{3}z+4}}$ *; here hypotinuse remains same since it is absolute value.*

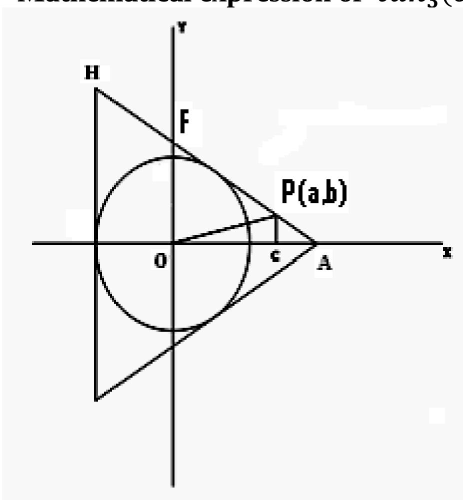
$= \dfrac{4-\sqrt{3}z}{2\sqrt{z^2-2\sqrt{3}z+4}}$

$= cos(-z)$

Hence trignometric identity $cos_3(-z) = cos_3(z)$ *holds*

Similarly equations are drawable for all quadrants

IX. Mathematical expression of $\boldsymbol{tan_3(\theta)}$:



Fig(32)

3.16. *In the above figure AF is the domain defined by the expression of inequality* , $0 \le z \le \dfrac{4}{\sqrt{3}}$, *where* $P(a,b)$ *is the variable position of moving radius* $OP(a,b)$ *measured from A in the anticlockwise direction. where k vary from 1 to ∞ will be*

$Accordingly, tan_3(\theta) = \dfrac{Opposite\ side}{Adjacent\ side}$ *; of the determinant triangle* $OCP(a,b)$

$Or = \dfrac{sin_3(z)}{cos_3(z)}$

By(41)*and* (47), *we have,* $\quad tan_3(z) = \dfrac{\frac{z}{2}}{\frac{4-\sqrt{3}z}{2}}$

$tan_3(z) = \dfrac{z}{4-\sqrt{3}z}$ − − − − − (65)

Some of the significant values at significant angles are:

When $z = 0$

$tan_3(z) = \dfrac{z}{4-\sqrt{3}z}$

$= \dfrac{0}{4-\sqrt{3}\times 0}$

$= 0$

$Next, when\ z = \sqrt{3}$

$tan_3(z) = \frac{z}{4 - \sqrt{3}z}$

$= \frac{\sqrt{3}}{4 - \sqrt{3}(\sqrt{3})}$

$= \sqrt{3}$

$Now, when\ z = \frac{4}{\sqrt{3}}$

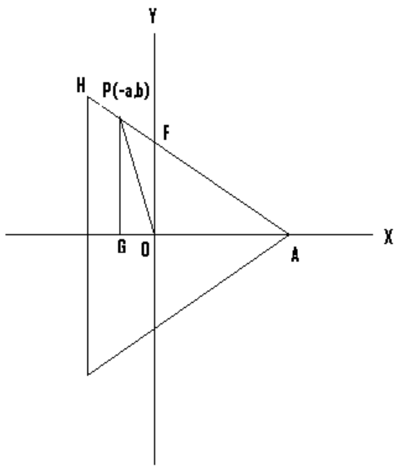
$tan_3(z) = \frac{z}{4 - \sqrt{3}z}$

$= \frac{\frac{4}{\sqrt{3}}}{4 - \sqrt{3} \times \frac{4}{\sqrt{3}}}$

$= \infty$

Further moving to the next domain

9.0.1. $Now, in\ the\ Fig(\ 25)\ FH\ is\ the\ domain\ defined\ by\ the\ expression\ of\ inequality, \frac{4}{\sqrt{3}} \leq z \leq 2\sqrt{3}$



Fig(33)

$tan_3(\theta) = \frac{sin_3(\theta)}{cos_3(\theta)}$

$from\ (55) and\ (61) we\ have$

$= \frac{\frac{z}{2\sqrt{z^2 - 2\sqrt{3}z + 4}}}{\frac{4 - z\sqrt{3}}{2\sqrt{z^2 - 2\sqrt{3}z + 4}}}$

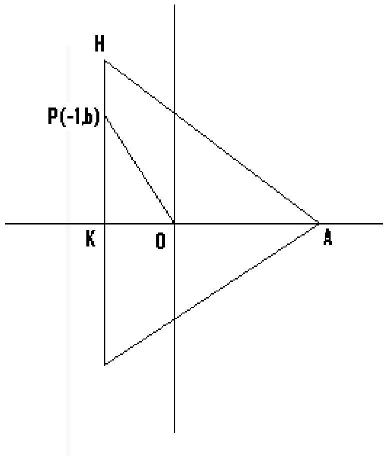
$tan_3(\theta) = \frac{z}{4 - z\sqrt{3}} - - - - - (66)$

$When\ z = 2\sqrt{3}\ ,$

$tan_3(2\sqrt{3}) = \frac{2\sqrt{3}}{4 - (2\sqrt{3})\sqrt{3}}$

$tan_3(2\sqrt{3}) = -\sqrt{3}$

9.0.2. $Moving\ to\ the\ next\ domain\ in\ the\ same\ II\ nd\ quadrant, 2\sqrt{3} \leq z \leq 3\sqrt{3}$



Fig(34)

$tan_3(z) = \frac{sin_3(\theta)}{cos_3(\theta)}$

$From\ (37) and\ (43), tan_3(\theta) = \frac{(3\sqrt{3} - z)}{-1}$

$tan_3(z) = -(3\sqrt{3} - z)$

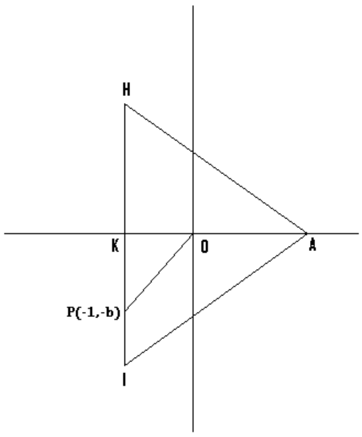
$tan_3(z) = (z - 3\sqrt{3}) - - - - - (67)$

$When\ z = 3\sqrt{3},$

$Then\ , tan_3(3\sqrt{3}) = (z - 3\sqrt{3})$

$= 0$

9.0.3. $Now\ in\ the\ III\ rd\ quadrant\ and\ domain\ 3\sqrt{3} \leq z \leq 4\sqrt{3}, value\ of\ tan_3\theta\ will\ be\ ,$



Fig(35)

$Similarly, tan_3(z) = (z - 3\sqrt{3})$

$tan_3(z) = (z - 3\sqrt{3}) - - - - - (68)$

$When, z = 4\sqrt{3}\ ;$

$tan_3(z) = (z - 3\sqrt{3})$

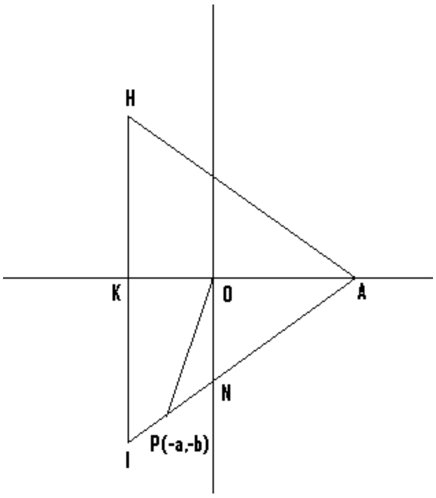
$= (4\sqrt{3} - 3\sqrt{3})$

$= \sqrt{3}$

9.0.4. $In\ the\ same\ III\ rd\ quadrant\ and\ in\ the\ domain\ 4\sqrt{3} \leq z \leq \frac{14}{\sqrt{3}},$

$Then\ value\ of\ tan_3(z), will\ be,$

.



Fig(36)

From (39) and (45)

$$tan_3(z) = \frac{\frac{\left[\left(z - 6\sqrt{3}\right)\right]}{2}}{\frac{\sqrt{3}z - 14}{2}}$$

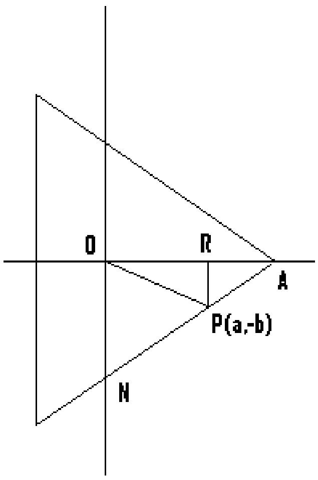
$$tan_3(z) = \frac{\left[\left(z - 6\sqrt{3}\right)\right]}{\sqrt{3}z - 14} - - - - - (69)$$

$$When\ z = \frac{14}{\sqrt{3}}$$

$$tan_3\left(\frac{14}{\sqrt{3}}\right) = \frac{\left[\left(\frac{14}{\sqrt{3}} - 6\sqrt{3}\right)\right]}{\sqrt{3}\left(\frac{14}{\sqrt{3}}\right) - 14}$$

$$= -\infty$$

9.0.5. $Final\ quadrant\ is\ IV\ th\ quadrant\ and\ domain\ is\ \frac{14}{\sqrt{3}} \le\ z\ \le 6\sqrt{3}$.



Fig(37)

From (40) and (46) we have

$$tan_3(z) = \frac{\frac{z - 6\sqrt{3}}{2}}{\frac{\sqrt{3}z - 14}{2}}$$

$$tan_3(z) = \frac{z - 6\sqrt{3}}{\sqrt{3}z - 14} - - - - - - (70)$$

$When, z = 6\sqrt{3}\ , the\ value\ of\ sin_3(z)\ will\ be$

$$sin_3\left(6\sqrt{3}\right) = \frac{6\sqrt{3} - 6\sqrt{3}}{\sqrt{3}z - 14}$$

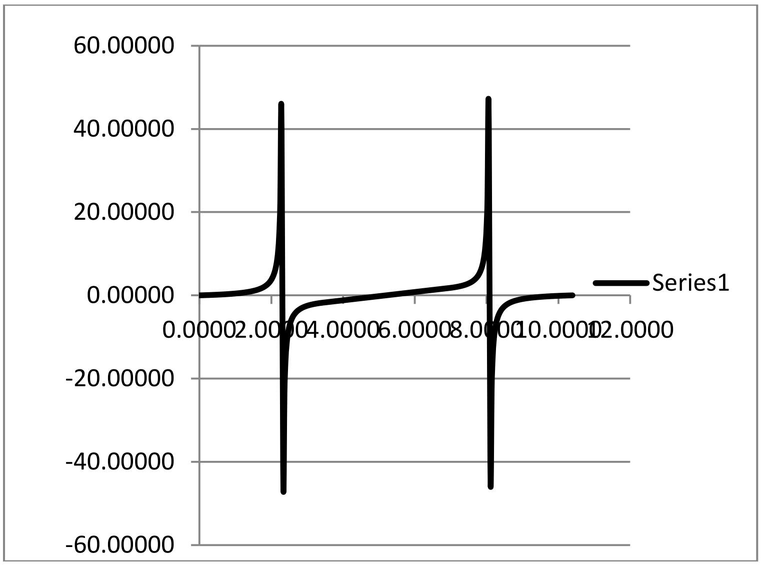
$= 0$

X. Graph of tan(z)

TABLE-3

Sl.No.	Equation	Z	tan(z)
1.	$tan_3(z) = \frac{z}{4 - \sqrt{3}z}$	0.0000	0.00000
2.		0.0289	0.00731
3.		0.0577	0.01480
4.		0.0866	0.02249
5.		0.1155	0.03039
6.		0.1443	0.03849
.	.	.	.
.	.	.	.
.	.	.	.
78	$Domain, \quad 0 \leq z \leq \frac{4}{\sqrt{3}}$	2.2517	22.51666
79		2.2805	45.61067
80		2.3094	∞
81	$tan_3(z) = \frac{z}{4 - z\sqrt{3}}$	2.3383	-46.76537
82		2.3671	-23.67136
83		2.3960	-15.97336
.	.	.	.
.	.	.	.
.	.	.	.
118		3.4064	-1.79282
119		3.4352	-1.76166
120	$Domain, \frac{4}{\sqrt{3}} \leq z \leq 2\sqrt{3}$	3.4641	-1.73205
121	$tan_3(z) = \left(z - 3\sqrt{3}\right)$	3.4930	-1.70318
122		3.5218	-1.67432
123		3.5507	-1.64545
.		.	.
.		.	.
.		.	.
178		5.1384	-0.05774
179		5.1673	-0.02887
180	$2\sqrt{3} \leq z \leq 3\sqrt{3}$	5.1962	0.00000
181	$tan_3(z) = \left(z - 3\sqrt{3}\right)$	5.2250	0.02887
182		5.2539	0.05774
183		5.2828	0.08660
.		.	.
.		.	.
.		.	.
238		6.8705	-0.51276
239		6.8993	-0.50632
240	$3\sqrt{3} \leq z \leq 4\sqrt{3}$	6.9282	-0.50000
241	$tan_3(z) = \frac{\left[\left(z - 6\sqrt{3}\right)\right]}{\sqrt{3}z - 14}$	6.9571	1.76166
242		6.9859	1.79282
243		7.0148	1.82568
.		.	.
.		.	.
.		.	.
278		8.0252	23.67136
279		8.0540	46.76537
280	$4\sqrt{3} \leq z \leq \frac{14}{\sqrt{3}}$	8.0829	∞
281	$tan_3(z) = \frac{z - 6\sqrt{3}}{\sqrt{3}z - 14}$	8.1118	-45.61067
282		8.1406	-22.51666
283		8.1695	-14.81866
.	.	.	.
.	.	.	.
.	.	.	.

Sl.No.	Equation	Z	tan(z)
358		10.3346	-0.01480
359		10.3634	-0.00731
360	$\frac{14}{\sqrt{3}} \le z \le 6\sqrt{3}$	10.3923	0.00000



XI. The basic trigonometric identities:

11.0.1. $a)\quad {sin_3}^{\mathbf{2}}z + {cos_3}^2 z = 1 - - - - - - - - (71)$

$Proof: LHS\ of\ above\ equation\ in\ the\ I\ st\ quadrant\ \ is\ \left(\frac{z}{2\sqrt{z^2-2\sqrt{3}z+4}}\right)^2 + \left(\frac{\mathbf{4-\ \sqrt{3}z}}{\mathbf{2\ \sqrt{z^2-2\ \sqrt{3}z+4}}}\right)^2$

$$= \frac{z^2}{\left(2\sqrt{z^2-2\sqrt{3}z+4}\right)^2} + \frac{\left(\mathbf{4-\ \sqrt{3}z}\right)^{\mathbf{2}}}{\left(2\sqrt{z^2-2\sqrt{3}z+4}\right)^2}$$

$$= \frac{z^2+16+3z^2-8\sqrt{3}z}{\left(2\sqrt{z^2-2\sqrt{3}z+4}\right)^2}$$

$$= \frac{4z^2-8\sqrt{3}z+16}{4z^2-8\sqrt{3}z+16}$$

$$= 1$$

This identity is true for all the domains because angles and determinant triangle are same for sin and cos functions. $And\ hypotenuse\ of\ sin\,\theta = hypotenuse\ of\ cos\,\theta\ , in\ a\ determinant\ triangle$.
$Also\ sum\ of\ square\ of\ opposite\ side\ and\ adjacent\ side\ is\ equal\ to\ square\ of\ hypotenuse$.
$consequently$
$sin^{\mathbf{2}}z + cos^2 z = 1$ throughout the triangle

11.0.2. Another method of proof

$LHS\ of\ {sin_3}^{\mathbf{2}}z + {cos_3}^2 z = 1\ can\ be\ written\ as$

$$= \frac{(Opposite\ side)^{\ 2}}{(Hypotenuse)^{\ 2}} + \frac{(Adjacent\ side)^{\ 2}}{(Hypotenuse)^{\ 2}}$$

Symbolically and generally it can also be written as

$$= \frac{\{XP(\pm a, \pm b)\}^2}{\{OP(\pm a, \pm b)\}^2} + \frac{\{O(\pm X))\}^2}{\{OP(\pm a, \pm b)\}^2}$$
$$= \frac{\{XP(\pm a, \pm b)\}^2 + \{O(\pm X))\}^2}{\{OP(\pm a, \pm b)\}^2}$$
$$= \frac{\{OP(\pm a, \pm b)\}^2}{\{OP(\pm a, \pm \mathrm{b})\}^2} ; since \{XP(\pm a, \pm b)\}^2 + \{O(\pm X))\}^2 = \{OP(\pm a, \pm b)\}^2$$
$$= 1$$

XII. Inverse Trigonometric functions w.r.t π_3^a :

12.0.0. To find the value of inverse trigonometric functions of equilateral triangle, we require to find the value of coefficient z.

To find value of z, following method is adopted.

$$We\ know\ that\ in\ the\ Ist\ quadrant\ and\ the\ domain\ 0 \leq z \leq \frac{4}{\sqrt{3}},$$
$$sin_3\, z = \frac{z}{2\sqrt{z^2 - 2\sqrt{3}z + 4}}$$

$Let\ sin_3(z) = x$

$$Then, above\ equation\ can\ be\ written\ as\ , x = \frac{z}{2\sqrt{z^2 - 2\sqrt{3}z + 4}}$$

$After\ algebraic\ manipulations\ it\ can\ be\ written\ as$

$$z^2(4x^2 - 1) - (8\sqrt{3}x^2)z + 16x^2 = 0 - - - - - - - - -(72)$$

$let\ 4x^2 - 1 = a\ ;\ -8\sqrt{3}x^2 = b\ and\ 16x^2 = c$

$This\ resebles\ the\ quadratic\ equation\ \ az^2 + bz + c = 0$

$Consequantly, the\ roots\ of\ the\ equation\ will\ be$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$Example\ 1: \ when\ \ sin_3(z) = 0.5, \qquad the\ value\ of\ z\ will\ be$

$Then, x = 0.5; hence\ , a = 4x^2 - 1 = 4 \times (0.5)^2 - 1 = 0$

$b = -8\sqrt{3}x^2 = -8\sqrt{3} \times (0.5)^2 = -3.461$

$c = 16 \times (0.5)^2 = 4$

$In\ this\ case\ resulting\ equation\ is\ not\ a\ quadratic\ equation\ because\ a = 0, hence\ z\ is\ directly$ $solvable\ as\ following$

$i.e., 3.461z = 4$

$$z = \frac{4}{3.461}$$
$$= 1.1547$$

Example 2: $sin_3(z) = 1$

$a = 4 \times 1 - 1 = 3$

$b = -8\sqrt{3} = -13.8564$

$c = 16$

$$Then, z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{13.8564 + \sqrt{(13.8564)^2 - 4 \times 3 \times 16}}{2 \times 3}$$
$$= 2.3094$$

XIII. The next regular polygon is a square. Trigonometric functions are derived as following.

13.0.0. ABCD is a unit square (a square of unit apothem) circumscribing a unit circle.

OP(a,b)- is a variable and moving radius in anti-clock wise direction from apothem OE. This OP(a,b) varies from 1 to $\sqrt{2}$ units and vice versa. Also it is the hypotenuse of the determinant triangle OEP(a,b). This triangle varies w.r.t the position of radius OP(a,b).

$Here\ the\ \angle GAP(a, b) is\ a\ conastant\ throughout\ the\ domain\ 0 \leq z \leq 2;$

$$Accordingly, opposite\ side\ GP(a, b)\ w.r.t\ the\ \Delta GAP(a, b) = AP(a, b)\ sin\left(\frac{\pi}{4}\right)$$

$GP(a,b) = \frac{z}{\sqrt{2}}$

$GP(a,b)$ *is also the opposite side of the determinant triangle* $OGP(a,b)$

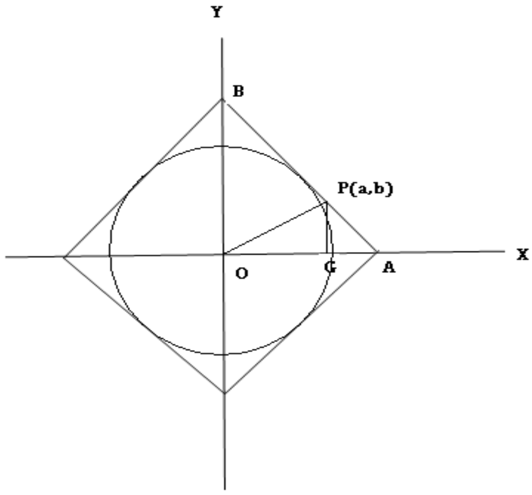
GA which is adjacent side of the $\Delta GAP(a,b) = AP(a,b) \times \cos\left(\frac{\pi}{4}\right)$

$8 = \frac{z}{\sqrt{2}}$

Then the adjacent side OG of determinant triangle is $= OA - GA$

$OG = \sqrt{2} - \frac{z}{\sqrt{2}}$; *since the square is circumscribing a unit circle.*

$= \frac{(2-z)}{\sqrt{2}}$



Fig(38)

Finally, the hypotenuse $OP(a,b)$ *of the determinant triangle* $\Delta GAP(a,b) = \sqrt{\left(\frac{z}{\sqrt{2}}\right)^2 + \left(\frac{(2-z)}{\sqrt{2}}\right)^2}$

$= \sqrt{z^2 - 2z + 2}$

Now, $\sin_4(z) = \frac{\frac{z}{\sqrt{2}}}{\sqrt{z^2 - 2z + 2}}$

$\sin_4(z) = \frac{z}{\sqrt{2}\sqrt{z^2 - 2z + 2}}$ ------(73)

For example when $z = 0$; $\sin_4(0) = \frac{0}{\sqrt{2}\sqrt{z^2 - 2z + 2}}$

$= 0$

And when $z = 2$

$\sin_4(z) = \frac{z}{\sqrt{2}\sqrt{z^2 - 2z + 2}}$

$= 1$

Next in the II nd quadrant for the domain, $2 \le z \le 4$

$\sin_4(z) = \frac{4 - z}{\sqrt{2}\sqrt{z^2 - 6z + 10}}$ ------(74)

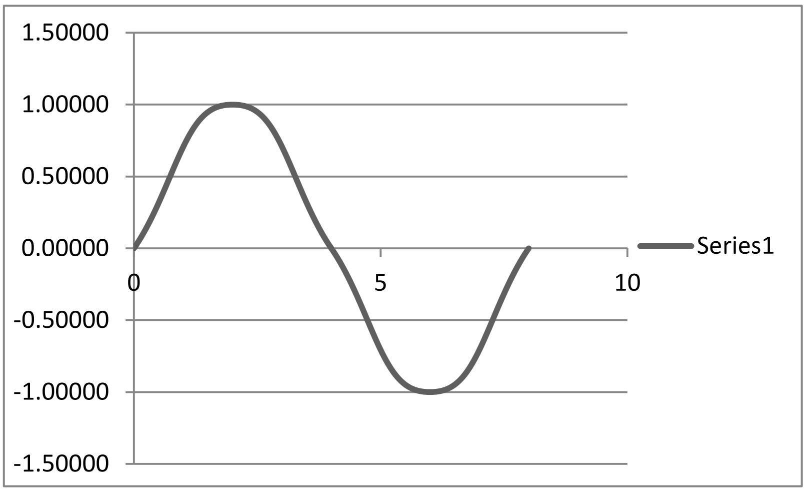
In the III rd quadrant for the domain, $4 \le z \le 6$

$\sin_4(z) = \frac{4 - z}{\sqrt{2}\sqrt{z^2 - 10z + 26}}$

In the IV th quadrant for the domain, $6 \le z \le 8$

$\sin_4(z) = \frac{z - 8}{\sqrt{2}\sqrt{z^2 - 14z + 50}}$

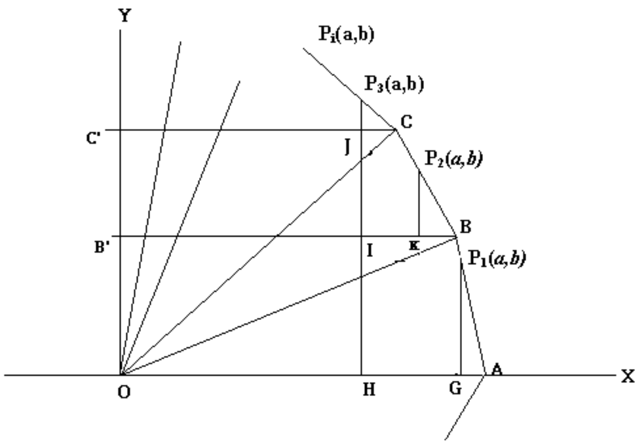
Sine graph of square



XIV. General equations to find the opposite sides of the determinant triangles of regular polygons:

Further, when the number of sides are within the reasonable limits, domains exists. When sides are sufficiently large to consider the Figure as a circle, then the domain may be considered as mono-element domain.

14.0.1. General equation for opposite side in the Ist quadrant:



Fig(39). A typical regular polygon

Let ABC… be a regular polygon of n number of sides Fig (39)

$\angle AOB, \angle BOC \cdots = \dfrac{2\pi}{n}$, *whence n is the number of sides.*

And the $\angle OAP_1(a,b), \angle B'BP_2(a,b), \cdots J'CP_n(a,b) \cdots$ *vary w.r.t the position from X axis to the end of the Ist quadrant. And the lines* $B'A, C'C \ldots J'Y \cdots$ *are parallel to* $X - axis$. And $GP_1(a,b), KP_2(a,b)$ …$JP_n(a,b)$ are parallel to Y axis.

$\angle OAB, \angle B'BC \cdots$ *are constant throughout the domain or side* .

$\angle OAB$ (angle formed with the $x-$ axis is considered as Ist angle) $= \frac{(n-2)\pi}{2n}$
; the line segment OA is on X axis.
The second $\angle B'BC = \frac{(n-2)\pi}{2n} - \angle OBB'$
$= \frac{(n-2)\pi}{2n} - \angle AOB$; angles between the parallels are equal
$= \frac{(n-2)\pi}{2n} - \frac{2\pi}{n}$
$= \frac{\pi(n-6)}{2n}$
The third angle $\angle J'CP_3(a,b) = \frac{(n-2)\pi}{2n} - 2\left(\frac{2\pi}{n}\right)$
$= \frac{(n-2)\pi}{2n} - \left(\frac{4\pi}{n}\right)$
$= \frac{(n-2)\pi}{2n} - \left(\frac{4\pi}{n}\right)$
$= \frac{n\pi - 2\pi - 8\pi}{2n}$
$= \frac{n\pi - 10\pi}{2n}$
$= \frac{\pi(n-10)}{2n}$
⋮
Equation for the angle at the i th term from initial position (in Ist quadrant) is
$\angle K'DP_i = \frac{\pi(n-2x)}{2n}$
$\angle K'DP_i = \frac{\pi(n-2(2i-1))}{2n}$ ------(75)
; where x is an ordinal odd number, ie. $x \in N = \{1,3,5,\cdots\}/$ i th side $\leq \frac{n}{4}$;
and $\frac{n}{4}$ is a natural number. And $x = 2i-1$, where i is the ordinal number of the angle or side counted from $X-$ axis.
For an equilateral triangle, $\angle OAB$ which is also equal to $\angle OAP_1(a,b)$, $x = 2i-1$
i.e, $x = (2 \times 1) - 1$
$= 1$
$\angle OAP_1(a,b) = \frac{\pi(n-2(2i-1))}{2n}$
$= \frac{\pi(3-2)}{2 \times 3}$
$= \frac{\pi}{6}$
But, its 2^{nd} angle is not in the I^{st} quadrant.
For a square , $\angle OAB$ in the I^{st} quadrant will be.
$\angle OAP_1(a,b) = \frac{\pi(n-2(2i-1))}{2n}$
$= \frac{\pi(4-2)}{2 \times 4}$
$= \frac{\pi}{4}$
Here also only one angle is in the first quadrant that has an hand parallel to X axis and the other end of the I^{st} side rest on Y axis. Whereas in the equilateral triangle side extends beyond I^{st} quadrant.
For a regular pentagon the angles will be
I^{st} $\angle OAP_1(a,b) = \frac{\pi(n-2x)}{2n}$
$= \frac{\pi(5-2)}{2 \times 5}$

$= \frac{3\pi}{10}$

$Further, for\ 2^{nd}\ \angle IBC = \frac{\pi(n-2x)}{2n}$

$= \frac{\pi(5-2\times 3)}{2\times 5}$

$= -\frac{\pi}{10}$

$Here\ the\ angle\ is\ below\ the\ parallel\ to\ X\ axis, hence\ the\ opposite\ side\ starts\ decreasing. Further\ for\ hexagon, \angle OAP_1(a,b)\ will\ vary\ as\ following$

$I^{st}\ \angle OAP_1(a,b) in\ the\ I^{st}\ quadrant\ will\ be. = \frac{\pi(n-2x)}{2n}$

$= \frac{\pi(6-2)}{2\times 6}$

$= \frac{\pi}{3}$

$2^{nd}\ \angle XYP_2(a,b)\ in\ the\ I^{st}\ quadrant\ will\ be. = \frac{\pi(n-2\times 3)}{2n}$

$= \frac{\pi(6-2\times 3)}{2\times 6}$

$= 0$

$Although\ I\ st\ quadrant\ of\ the\ hexagon\ contains\ fractional\ part\ of\ the\ second\ side, it\ is\ parallel\ to\ X-axis, hence\ the\ second\ angle\ is\ 0.$

$9.02. For\ a\ regular\ polygon\ of\ 32\ sides, each\ quadrant\ contains\ 8\ sides, then\ the\ angle\ at\ the\ 7\ th\ side\ will\ be\ \angle XYP_7(a,b) = \frac{\pi(n-2x)}{2n}$

$= \frac{\pi(32-2\times 13)}{2\times 32}; where\ x = 2i-1$

$= 0.294524$

$Further, (75)\ provides\ basis\ for\ the\ equation\ to\ find\ the\ opposite\ side.$

$accordingly, the\ longest\ opposite\ side\ of\ the\ first\ side\ is = \left(\frac{2\ \pi_n^a}{n}\right) sin\left(\frac{\pi(n-2)}{2n}\right)$

$The\ longest\ opposite\ side\ of\ the\ second\ side\ is = \left(\frac{2\ \pi_n^a}{n}\right) sin\left(\frac{\pi(n-2\times 3)}{2n}\right)$

$The\ longest\ opposite\ side\ of\ the\ second\ side\ is = \left(\frac{2\ \pi_n^a}{n}\right) sin\left(\frac{\pi(n-2\times 5)}{2n}\right)$

$\vdots$

$The\ longest\ opposite\ side\ of\ the\ i\ th\ side\ is = \left(\frac{2\ \pi_n^a}{n}\right) sin\left(\frac{\pi(n-2\times x)}{2n}\right)$

Or

$The\ longest\ opposite\ side\ of\ the\ i\ th\ side\ is = \left(\frac{2\ \pi_n^a}{n}\right) sin\left(\frac{\pi\left(n-(4i-2)\right)}{2n}\right) ----- (76)$

$Accordingly, opposite\ side\ is\ sum\ of\ the\ lengths\ from\ the\ first\ side\ to\ required\ position.$

$Magnitude\ of\ the\ Ist\ quadrant\ of\ regular\ polygons\ vary\ from\ \frac{4\pi_3^a}{9}\ to\ \frac{\pi}{2}\ or\ \frac{4}{\sqrt{3}}\ to\ \frac{\pi}{2}$

$Let, the\ arc\ of\ a\ regular\ polygon\ of\ n\ number\ of\ sides\ measured\ from\ the\ X-axis\ be\ z.$

$It\ is\ known\ that\ the\ length\ of\ each\ side\ is\ \frac{2\pi_n^a}{n}$

$If\ the\ number\ of\ sides\ with\ fractional\ side\ that\ z\ contains\ be\ denoted\ by\ (C+d)$
$where\ C\ is\ a\ whole\ number\ and\ d\ is\ a\ decimal.$

$Then, (C+d) = \frac{z}{\frac{2\pi_n^a}{n}}$

$(C+d) = \frac{nz}{2\pi_n^a}$

$Please\ note\ that\ magnitude\ of\ all\ sides\ are\ equal\ except\ the\ fractional\ part\ d\ and\ the\ corresponding\ angles\ in\ this\ context\ are\ different.$

$It\ is\ known\ that\ magnitude\ of\ each\ side\ is = \frac{2\pi_n^a}{n}$

$z = \frac{2\pi_n^a}{n}(C + d)$

$= \frac{2C\pi_n^a}{n} + \frac{2d\pi_n^a}{n}$

$let\ \frac{2d\pi_n^a}{n},\quad be\ denoted\ by\ D$

$i.e, \quad = \frac{2C\pi_n^a}{n} + D$

$For\ an\ equilateral\ triangle, magnitude\ of\ opposite\ side\ with\ in\ the\ I^{st}\ quadrant\ is$

$$GP_1(a,b) = \frac{2\pi_n^a}{n}(C + d)\left[sin\left(\frac{\pi(n-2)}{2n}\right)\right] ------(77); Fig\ (41)$$

$in\ case\ of\ an\ equilateral\ triangle\ C = 0\ because\ the\ length\ of\ the\ arc\ in\ the\ I^{st}\ quadrant\ is\ less\ than\ one\ side.$

$Hence, C + d = d$

$$Then, GP_1(a,b) = \left(\frac{2d\pi_n^a}{n}\right)\left[sin\left(\frac{\pi(n-2)}{2n}\right)\right]$$

$a) For\ example\ when\ z = \sqrt{3},; d\ will\ be = \frac{\sqrt{3}}{2\sqrt{3}}\ ; here\ 2\sqrt{3}\ is\ the\ length\ of\ the\ side\ of\ the\ considered\ triangle$

Putting these values we have

$of\ opposite\ side\ of\ of\ the\ determinant\ triangle\ will\ be,$

$$GP(a,b) = \left(\frac{2 \times \frac{1}{2} \times 3\sqrt{3}}{3}\right)\left[sin\left(\frac{\pi(3-2)}{6}\right)\right]$$

$$= \frac{\sqrt{3}}{2}$$

$b) Also, if\ D = 2.3094$

$$GP(a,b) = d\left(\frac{2\pi_n^a}{n}\right)\left[sin\left(\frac{\pi(n-2)}{2n}\right)\right]$$

$$= 2.3094\left[sin\left(\frac{\pi(n-2)}{2n}\right)\right]$$

$$= 2.3094 \times sin\left(\frac{\pi}{6}\right)$$

$$= 1.1547$$

$c) For\ square\ when\ z = 1, C\ will\ be = 0, and\ d = \frac{1}{2}, then\ we\ have$

$$GP(a,b) = d\left(\frac{2\pi_n^a}{n}\right)\left[sin\left(\frac{\pi(n-2)}{2n}\right)\right]$$

$$= \frac{1}{2}(2)\left[\frac{1}{\sqrt{2}}\right]$$

$$= \frac{1}{\sqrt{2}}$$

$d) When\ n = 32, \pi_{32}^a = 32\left[tan\left(\frac{\pi}{32}\right)\right] = 3.1517, hence\ length\ of\ each\ side\ is, \frac{2\pi_n^a}{n} = \frac{6.3035}{32} = 0.1970$

$arc\ length\ 6.4\ sides = 6.4 \times 0.1970 = 1.2608$

$(side\ in\ this\ case\ is\ the\ length\ of\ the\ side\ of\ given\ n - gon)$

$$Then\ C + d = 6 + 0.4$$

$Then, magnitude\ of\ opposite\ side\ is$

$$GP(a,b) = \frac{2\pi_n^a}{n}\left[sin\left(\frac{\pi(n-2)}{2n}\right)\right] + \cdots + \frac{2\pi_n^a}{n}\left[sin\left(\frac{\pi(n-2\times 11)}{2n}\right)\right] + D \times \left[sin\left(\frac{\pi(n-2\times 13)}{2n}\right)\right]$$

$$= \frac{2\pi_{32}^a}{32}\left[\left\{sin\left(\frac{\pi(32-2)}{64}\right)\right\} + \cdots + \left\{sin\left(\frac{\pi(32-2\times 11)}{64}\right)\right\}\right] + 0.0788 \times \left[sin\left(\frac{\pi(32-2\times 13)}{64}\right)\right]$$

$$= 0.1970\left[sin\left(\frac{15\pi}{32}\right) + \cdots + sin\left(\frac{5\pi}{32}\right)\right] + 0.0788 \times \left[sin\frac{3\pi}{32}\right]$$

$= 0.9284 + 0.0229$
$= 0.9513$
This above series of n number of sides is a convergent functional series that converges towards 1.

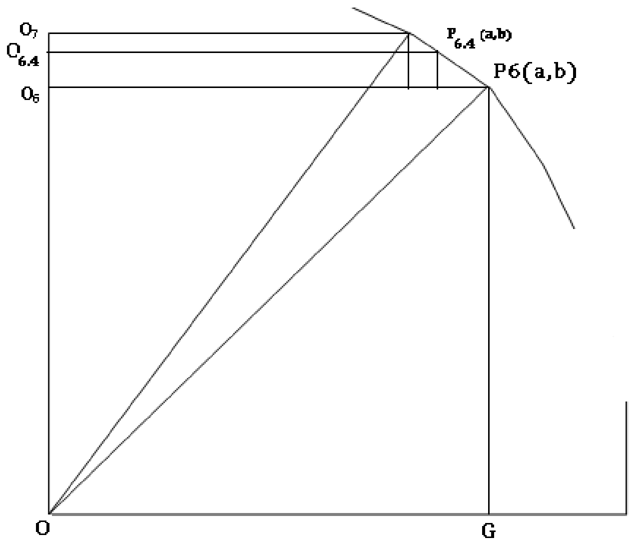
14.0.2. To check the above result geometric method is adopted as following.

Here C indicates number of sides in the given arc. Hence, at the end of 6 th side, the triangle $OP_6(a,b)O_6$ *is formable. The side* OO_6 *is the part of the required opposite side* $OO_{6.4}$ *. Some of the elements of the* $\Delta\, OP_6(a,b)O_6$ are known. Hence this triangle is chosen.

The magnitude of $OP_c(a,b)$ *is a constant* $= sec\left(\frac{\pi}{n}\right)$

Consequently, $OP_6(a,b) = sec\left(\frac{\pi}{32}\right)$

And, $\angle O_6 = \frac{\pi}{2}$



Fig(40)

Similarly, $\angle O_cP_c(a,b)O = \frac{\pi(n-2)}{2n} - \frac{\pi\big(n-2(2C-1)\big)}{2n}$

$= \frac{2C\pi}{n}$

Hence, $\quad \angle O_6P_6O(a,b) = \frac{2\times 6\times \pi}{32}$

$= \frac{3\pi}{8}$

Further, $\angle OO_cP_c(a,b) = \frac{\pi}{2}$

Then, $\angle OO_6P_6(a,b) = \frac{\pi}{2}$

applying sin law OO_i *can be found as following*

$$\frac{OO_i}{sin\left(\frac{2C\pi}{n}\right)} = \frac{OP_i(a,b)}{sin\left(\frac{\pi}{2}\right)}$$

$$OO_i = (OP_i(a,b))\left(sin\left(\frac{2C\pi}{n}\right)\right)$$

$$OO_i or\ GP_i(a,b) = \frac{\left(sin\left(\frac{2C\pi}{n}\right)\right)}{cos\left(\frac{\pi}{n}\right)} ------(78)$$

$Hence, OO_6 = \frac{sin\frac{(2\times 6\times \pi)}{32}}{cos\left(\frac{\pi}{32}\right)}$
$= 0.9284$
$Length\ of\ the\ remaining\ part\ is\ d\times \frac{2\pi_n^a}{n} = 0.4\times 0.1970$
$= 0.0788$
$Finally\ to\ find\ the\ remaining\ part\ of\ the\ opposite\ side$
$= 0.0788\times\left[sin\frac{3\pi}{32}\right]$
$= 0.0229$
$Further\ length\ of\ opposite\ side\ will\ be$
$OO_{6.4} = 0.9284 + 0.0229$
$= 0.9513$
9.04. $Now\ (79) is\ used\ for\ equating\ with\ (78) as\ following$
$If\ \ C + d = \frac{n}{4}; Following\ formula\ is\ valid\ for\ (C+d)^{th}\ term, which\ means\ if$
$the\ fraction\ d\ exists, d = 1, if\ does\ not\ exist, d = 0$

$$GP(a,b) = \frac{2\pi_n^a}{n}\left[\sin\left(\frac{\pi\big(n-2(2-1)\big)}{2n}\right) + \cdots + sin\left(\frac{\pi\big(n-2(4-1)\big)}{2n}\right) + \cdots + sin\left(\frac{\pi\big(n-2(2i_1-1)\big)}{2n}\right)\right.$$

$$\left.+d\times sin\left(\frac{\pi(n-2(2i_2-1))}{2n}\right)\right] = \frac{\left(sin\left(\frac{2C\pi}{n}\right)\right)}{cos\left(\frac{\pi}{n}\right)} + D\times sin\left(\frac{\pi\big(n-2(2i_2-1)\big)}{2n}\right) - - - - - - - - (79)$$

$$GP(a,b) = \left[\frac{2\pi_n^a}{n}\sum_{i=1}^{i=C}\left[sin\left(\frac{\pi\big(n-2(2i_1-1)\big)}{2n}\right)\right]\right] + D\times sin\left(\frac{\pi\big(n-2(2i_2-1)\big)}{2n}\right)$$

$$\left[\frac{2\pi_n^a}{n}\sum_{i=1}^{i=C}\left[sin\left(\frac{\pi\big(n-2(2C-1)\big)}{2n}\right)\right]\right] + \left(d\times\frac{2\pi_n^a}{n}\right)\times sin\left(\frac{\pi\big(n-2(2i_2-1)\big)}{2n}\right)$$

$$= \frac{sin\left(\frac{2C\pi}{n}\right)}{cos\left(\frac{\pi}{n}\right)} + D\times sin\left(\frac{\pi(n-2(2(C+1)^{th}\ term-1)}{2n}\right) - - - - - (80)$$

Case-1

$When\ n = 2y\ , where\ y\ \in\{2,3,4\cdots\}, and\ for\ C + d = \frac{n}{4}, if\ d\ exists\ d = 1, hence\ (C+1)^{th}$
$term\ exists.$

$$We\ have\ GP(a,b) = \frac{sin\left(\frac{2C\pi}{n}\right)}{cos\left(\frac{\pi}{n}\right)} + D\times sin\left(\frac{\pi(n-2(2(C+1)^{th}\ term-1)}{2n}\right) - - - -(81)$$

$We\ reqiure\ to\ show\ here\ that\ 2(2(C+1)^{th}\ term - 1) = 2y\ iff\ d\ exists\ and\ n = 2y.$
$For\ that\ following\ method\ is\ adopted.$
$We\ know\ that\ \frac{n}{4} = C + d;\ iff\ \frac{n}{4}\neq C$
$Let\ \frac{n}{4} = C + d$
$\frac{n}{4} - d = C$
$1\ is\ added\ to\ both\ sides\ of\ above\ equation, we\ have$
$\frac{n}{4} - d + 1 = (C+1)$
$Multiplied\ both\ sides\ by\ 2, and\ considered\ C+1\ as\ the\ value\ of\ the\ term$
$2\left(\frac{n}{4} - d + 1\right) = 2((C+1)^{th}\ term)$
$1\ is\ subtracted\ from\ both\ sides$
$2\left(\frac{n}{4} - d + 1\right) - 1 = 2((C+1)^{th}\ term) - 1$
$Once\ again\ both\ sides\ are\ multiplied\ by\ 2$
$2\left(2\left(\frac{n}{4} - d + 1\right) - 1\right) = 2(2((C+1)^{th}\ term) - 1)$

$Now\ RHS\ of\ the\ equation\ resembles\ LHS\ of\ \ (81)$
$When\ LHS\ of\ the\ above\ equation\ is\ brought\ to\ simple\ terms, we\ have$
$n - 4d + 2 = 2(2((C+1)^{th}\ term) - 1) ------(82)$
$Once\ again\ it\ needs\ to\ show\ that\ 4d = 2$
$Any\ even\ number\ > 5\ and\ not\ multiple\ of\ 4\ can\ be\ written\ as$
$4C + 2 = n$
$Further, n - 4C = 2 ----- (83)$
$Also, \frac{n}{4} = C + d$
$Then, n = 4C + 4d$
$And, n - 4C = 4\text{d} ----- (84)$
$Subtracting\ (84) from\ (83) we\ have$
$0 = 2 - 4d$
$4d = 2$
$or\ d = 0.5$
This value of d serves as a proof of, +Y ordinate passes through the centre of a side.
$Now\ equation\ (88) can\ be\ written\ as$
$n = 2(2((C+1)^{th}\ term) - 1)$
$i.e. 2y = 2(2((C+1)^{th}\ term) - 1)$

$$Consequently, GP(a,b) = \frac{sin\left(\frac{2C\pi}{2y}\right)}{cos\left(\frac{\pi}{2\text{y}}\right)} + D \times sin\left(\frac{\pi(2y - 2y)}{2n}\right)$$

$$GP(a,b) = \frac{sin\left(\frac{2C\pi}{2y}\right)}{cos\left(\frac{\pi}{2y}\right)}$$

$$The\ sum\ of\ the\ \ angles\ of\ sin\left(\frac{2C\pi}{2y}\right) and\ cos\left(\frac{\pi}{2y}\right)\ is$$

$$\frac{C\pi}{y} + \frac{\pi}{2y} = \frac{2C\pi + \pi}{2y}$$

$$= \frac{\pi(2C+1)}{2y}$$

$$Substituting\ \frac{n}{4}\ for\ C\ we\ have,$$

$$= \frac{\pi\left(2\left(\frac{n}{4} - d\right) + 1\right)}{2y}$$

$$= \frac{\pi\left(\left(\frac{n-4d}{2}\right) + 1\right)}{2y}$$

$$= \frac{\pi\left(\frac{n-4d+2}{2}\right)}{2y}$$

$$= \frac{\pi\left(\frac{n}{2}\right)}{2y}\ ; since\ d = 0.5\ in\ this\ case.$$

$$= \frac{y\pi}{2y}\ ; since\ n = 2y$$

$$Hence, \frac{C\pi}{y} + \frac{\pi}{2y} = \frac{\pi}{2}$$

$\therefore\ angles\ of\ sin\ and\ cos\ are\ complementary.$

$$Hence, sin\left(\frac{2C\pi}{2y}\right) = cos\left(\frac{\pi}{2y}\right)$$

$$Therefore, GP(a,b) = \frac{sin\left(\frac{2C\pi}{2y}\right)}{cos\left(\frac{\pi}{2y}\right)} = 1$$

Case-2
$When\ n\ is\ the\ multiple\ of\ 4\ , ie., n = 4y, for\ y \in N = \{1,2,3 \cdots\}$

$Then\ \ d\ will\ be = 0, hence\ the\ equation\ will\ be.$

$$GP(a,b) = \frac{sin\left(\frac{2C\pi}{n}\right)}{cos\left(\frac{\pi}{n}\right)}$$

$$GP(a,b) = \frac{sin\left(\frac{\left(\frac{n}{2}\right)\pi}{n}\right)}{cos\left(\frac{\pi}{n}\right)}\ ; since\ 2C = \frac{n}{2}$$

$$= \frac{1}{cos\left(\frac{\pi}{n}\right)}$$

$$GP(a,b) = sec\left(\frac{\pi}{n}\right)$$

$For\ example\ when\ n = 4, and\ for\ C = \frac{n}{4} = 1, then$

$$GP(a,b) = sec\left(\frac{\pi}{n}\right)$$

$$= sec\left(\frac{\pi}{4}\right)$$

$$= 1.4142$$

Further, applying limit on both sides of (86) we have

$$\lim_{n\to\infty}\left[\left\{\frac{2\pi_n^a}{n}\sum_{i=1}^{i=C,}\left[sin\left(\frac{\pi\big(n-2(2C-1)\big)}{2n}\right)\right]\right\} + D \times sin\left(\frac{\pi(n-2(2(C+1)^{th}\ term-1)}{2n}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{sin\left(\frac{2C\pi}{n}\right)}{cos\left(\frac{\pi}{n}\right)} + D \times sin\left(\frac{\pi(n-2(2(C+1)^{th}\ term-1)}{2n}\right)\right]$$

$$= \lim_{n\to\infty}\left(\frac{sin\left(\frac{2C\pi}{n}\right)}{cos\left(\frac{\pi}{n}\right)}\right) + \lim_{n\to\infty}\left\{\frac{2\pi_n^a}{n} \times d\ \times sin\left(\frac{\pi(n-2(2(C+1)^{th}\ term-1)}{2n}\right)\right\}$$

$$= \frac{\lim_{n\to\infty}\left\{sin\left(\frac{2C\pi}{n}\right)\right\}}{\lim_{n\to\infty}\left\{cos\left(\frac{\pi}{n}\right)\right\}} + \left\{\lim_{n\to\infty}\left(\frac{2\pi_n^a}{n}\right) \times d\ \times sin\left(\frac{\pi(n-2(2(C+1)^{th}\ term-1)}{2n}\right)\right\}$$

$as\ n \to \infty, The\ value\ of\ 2C \to \frac{n}{2};\ \frac{\pi}{n} \to 0\ and\ \frac{2\pi_n^a}{n}\ also\ \to 0.$

$Consequently, \lim_{n\to\infty}\left\{sin\left(\frac{2C\pi}{n}\right)\right\} = sin\left(\frac{\pi}{2}\right) = 1\ ;\ \lim_{n\to\infty}\left\{cos\left(\frac{\pi}{n}\right)\right\} = cos(0) = 1\ and \lim_{n\to\infty}\left(\frac{2\pi_n^a}{n}\right) = 0$

$Then\ \ (80)\ will\ be$

$= 1$

$This\ result\ is\ the\ value\ of\ radius\ of\ unit\ circle.$

$Similarly\ equations\ are\ derivable\ for\ other\ sides\ of\ the\ determinant\ triangles.$

XV. To find the value of arc trignometric functions like arcsin θ, arccos θ. :

$10.01. Example 1: Let\ the\ value\ of\ sin\,\theta\ in\ the\ Ist\ quadrant\ of\ a\ circle\ , be = 0.5, here\ the\ value\ of\ \theta\ is\ in\ conventional\ \pi$

$$Solution\ 1: we\ have\ OO_i = \big(OP_i(a,b)\big)\left(sin\left(\frac{2C\pi}{n}\right)\right) - - - - - (85)$$

$$\frac{OO_i}{\big(OP_i(a,b)\big)} = \left(sin\left(\frac{2C\pi}{n}\right)\right) - - - -(92)$$

$$Then, (85) will\ be\ , 0.5 = \left(sin\left(\frac{2C\pi}{n}\right)\right) - - - -(86)$$

$By\ trial\ and\ error\ method\ we\ can\ find\ that\ (87) will\ be\ true\ when\ C = 4, hence\ n = 48$

$$Consequantly, \theta = \frac{2C\pi}{n}$$

$$= \frac{2 \times 4 \times \pi}{48}$$

$= \frac{\pi}{6}$

This result can also be found as following

Solution 2: From (79) we have

$$GP(a,b) = \frac{2\pi_n^a}{n}\left[sin\left(\frac{\pi\left(n-2(2-1)\right)}{2n}\right) + \cdots + sin\left(\frac{\pi\left(n-2(4-1)\right)}{2n}\right) + \cdots + sin\left(\frac{\pi\left(n-2(2i_1-1)\right)}{2n}\right)\right.$$

$$\left. + d \times sin\left(\frac{\pi(n-2(2i_2-1))}{2n}\right)\right] = \frac{\left(sin\left(\frac{2C\pi}{n}\right)\right)}{cos\left(\frac{\pi}{n}\right)} + D \times sin\left(\frac{\pi\left(n-2(2i_2-1)\right)}{2n}\right)$$

Let $n = 32$

TABLE-4 : Showing the details of required elements to find the angle.

Ordinal number of a side	n	Number of sides in the Ist quadrant $\frac{n}{4}$	Length of each side $l = \frac{2\pi_n^a}{n}$	Length of maximum opposite side of each side $l \times sin\left(\frac{\pi\left(n-2(2i-1)\right)}{2n}\right) = m$	Length of the opposite side w.r.t each side -m	Cumulative total(opposite side) $(GP_i(a,b))$
1	2	3	4	5	6	7
1 st	4	1	2	$2 \times sin\left(\frac{\pi\{4-2(2-1)\}}{2\times 4}\right)$	$\sqrt{2}$	
1 st	8	2	0.8284	$0.8284 \times sin\left(\frac{\pi\{8-2(2-1)\}}{2\times 8}\right)$	0.7654	
2nd				$0.8284 \times sin\left(\frac{\pi\left(n-2\left((2\times 2)-1\right)\right)}{2n}\right)$	0.3170	1.0824
1st	12	3	0.5359	$0.5359 \times sin\left(\frac{\pi\{12-2(2-1)\}}{2\times 12}\right)$	0.5176	
2nd				$0.5359 \times sin\left(\frac{\pi\{12-2(2\times 2-1)\}}{2\times 12}\right)$	0.3789	0.8966
3rd				$0.5359 \times sin\left(\frac{\pi\{12-2(2\times 3-1)\}}{2\times 12}\right)$	0.1387	1.0353
1st	16	4	0.3978	$0.3978 \times sin\left(\frac{\pi\{16-2(2-1)\}}{2\times 16}\right)$	0.3902	
2nd				$0.3978 \times sin\left(\frac{\pi\{16-2(2\times 2-1)\}}{2\times 16}\right)$	0.3308	0.7210
3rd				$0.3978 \times sin\left(\frac{\pi\{16-2(2\times 3-1)\}}{2\times 16}\right)$	0.2210	0.9420
4th				$0.3978 \times sin\left(\frac{\pi\{16-2(2\times 4-1)\}}{2\times 16}\right)$	0.0776	1.0196
1st	20	5	0.3168	$0.3168 \times sin\left(\frac{\pi\{24-2(2-1)\}}{2\times 24}\right)$	0.3129	
2nd				$0.3168 \times sin\left(\frac{\pi\{24-2(2\times 2-1)\}}{2\times 24}\right)$	0.2822	0.5951
3rd				$0.3168 \times sin\left(\frac{\pi\{24-2(2\times 3-1)\}}{2\times 24}\right)$	0.2240	0.8191
4th				$0.3168 \times sin\left(\frac{\pi\{20-2(2\times 4-1)\}}{2\times 24}\right)$	0.1438	0.9629
5th				$0.3168 \times sin\left(\frac{\pi\{20-2(2\times 5-1)\}}{2\times 20}\right)$	0.0496	1.0124
1st	24	6	0.2633	$0.2633 \times sin\left(\frac{\pi\{24-2(2-1)\}}{2\times 24}\right)$	0.2611	
2nd				$0.2633 \times sin\left(\frac{\pi\{24-2(2\times 2-1)\}}{2\times 24}\right)$	0.2433	0.5043
3rd				$0.2633 \times sin\left(\frac{\pi\{24-2(2\times 3-1)\}}{2\times 24}\right)$	0.2089	0.7132
4th				$0.2633 \times sin\left(\frac{\pi\{24-2(2\times 4-1)\}}{2\times 24}\right)$	0.1603	0.8735
5th				$0.2633 \times sin\left(\frac{\pi\{24-2(2\times 5-1)\}}{2\times 24}\right)$	0.1008	0.9743

Ordinal number of a side	n	Number of sides in the Ist quadrant $\frac{n}{4}$	Length of each side $l=\frac{2\pi_n^a}{n}$	Length of maximum opposite side of each side $l\times sin\left(\frac{\pi\left(n-2(2i-1)\right)}{2n}\right)=m$	Length of the opposite side w.r.t each side -m	Cumulative total(opposite side) $(GP_i(a,b))$
6th				$0.2633\times sin\left(\frac{\pi\{24-2(2\times 6-1)\}}{2\times 24}\right)$	0.0344	1.0086
1st	28	7	0.2254	$0.2254\times sin\left(\frac{\pi\{28-2(2-1)\}}{2\times 28}\right)$	0.2239	
2nd				$0.2254\times sin\left(\frac{\pi\{28-2(2\times 2-1)\}}{2\times 28}\right)$	0.1825	0.4366
3rd				$0.2254\ \times sin\left(\frac{\pi\{28-2(2\times 3-1)\}}{2\times 28}\right)$	0.1908	0.6274
4th				$0.2254\times sin\left(\frac{\pi\{28-2(2\times 4-1)\}}{2\times 28}\right)$	0.1593	0.7868
5th				$0.2254\times sin\left(\frac{\pi\{28-2(2\times 5-1)\}}{2\times 28}\right)$	0.1199	0.9067
6th				$0.2254\times sin\left(\frac{\pi\{28-2(2\times 6-1)\}}{2\times 28}\right)$	0.0744	0.9811
7th				$0.2254\times sin\left(\frac{\pi\{28-2(2\times 6-1)\}}{2\times 28}\right)$	0.0252	1.0063
1st	32	8	0.1970	$0.1970\times sin\left(\frac{\pi\{32-2(2-1)\}}{2\times 32}\right)$	0.1961	
2nd				$0.1970\times sin\left(\frac{\pi\{32-2(2\times 2-1)\}}{2\times 32}\right)$	0.1885	0.3846
3rd			0.1970	$0.1970\ \times sin\left(\frac{\pi\{32-2(2\times 3-1)\}}{2\times 32}\right)$	0.1737	0.5583
4th				$0.1970\times sin\left(\frac{\pi\{32-2(2\times 4-1)\}}{2\times 32}\right)$	0.1523	0.7116
5th				$0.1970\times sin\left(\frac{\pi\{32-2(2\times 5-1)\}}{2\times 32}\right)$	0.1250	0.8366
6th				$0.1970\times sin\left(\frac{\pi\{32-2(2\times 6-1)\}}{2\times 32}\right)$	0.0929	0.9295
7th				$0.1970\times sin\left(\frac{\pi\{32-2(2\times 7-1)\}}{2\times 32}\right)$	0.0580	0.9875
8th				$0.1970\times sin\left(\frac{\pi\{32-2(2\times 8-1)\}}{2\times 32}\right)$	0.0193	1.0048

15.0.1. $If\ a\ polygon\ of\ non\ convenient\ number\ of\ sides\ is\ chosen\ value\ of\ \theta\ is\ approximatable\ as\ following.$

$Solution\ 2{:}\ Now$ let $a\ convenient\ regular\ polygon\ be\ chosen\ to\ find\ the\ angle\ , i.e., let\ n = 32, then\ \frac{n}{4}=8$

$$GP(a,b)=\frac{2\pi_n^a}{n}\left[sin\left(\frac{\pi(n-2)}{2n}\right)\right]+\cdots+\frac{2\pi_n^a}{n}\left[sin\left(\frac{\pi(n-2\times 11)}{2n}\right)\right]+D\times\left[sin\left(\frac{\pi(n-2\times 13)}{2n}\right)\right]$$

$To\ find\ the\ boundary$

$The\ value\ of\ \frac{2\pi_{32}^a}{32}\left[sin\left(\frac{\pi(32-2)}{2\times 32}\right)\right]+\frac{2\pi_{32}^a}{32}\left[sin\left(\frac{\pi(32-6)}{2\times 32}\right)\right]+D\times\left[sin\left(\frac{\pi(32-10)}{2\times 32}\right)\right]\ will\ be$

$$0.1970\left[sin\left(\frac{\pi(32-2)}{2\times 32}\right)\right]+0.1970\left[sin\left(\frac{\pi(32-6)}{2\times 32}\right)\right]+D\times\left[sin\left(\frac{\pi(32-10)}{2\times 32}\right)\right]$$

$$=0.1960+0.1885+D\times\left[sin\left(\frac{\pi(32-10)}{2\times 32}\right)\right]$$

$$=0.3845+D\times 0.8819$$

$Similarly, the\ expression\ for\ adjacent\ side\ is.$

$$OG=1.004839-\left[0.1970\left\{\ cos\left(\frac{\pi(32-2)}{2\times 32}\right)+cos\left(\frac{\pi(32-6)}{2\times 32}\right)\right\}+\left\{D\times cos\left(\frac{\pi(32-10)}{2\times 32}\right)\right\}\right]$$

$= 1.004839 - \left[0.1970\left\{\ cos\left(\frac{\pi(32-2)}{2\times 32}\right) + cos\left(\frac{\pi(32-6)}{2\times 32}\right)\right\} + D \times cos\left(\frac{\pi(32-10)}{2\times 32}\right)\right]$
$= 1.004839 - 0.0765 - D \times 0.4714$
$= 0.9283 - D \times 0.4714$
$Now\ the\ hypotenuse\ of\ determinant\ \Delta\ is,$
$OP(a,b) = (0.3845 + D \times 0.8819)^2 + (0.9283 - D \times 0.4714)^2$
$[2(0.3845 + D \times 0.8819)]^2 = (0.3845 + D \times 0.8819)^2 + (0.9283 - D \times 0.4714)^2$
$When\ above\ relation\ is\ brought\ to\ simple\ terms\ we\ have\ a\ quadratic\ equation\ that\ can\ be\ solved\ for\ D.$
$i.e, 2.1110D^2 + 2.5157\ D - 0.4182 = 0$
$Here\ the\ leading\ coefficient\ a = 2.1110, middle\ coefficient\ b = 2.5157\ and\ the\ constant\ term\ c = -0.4182$
$Now\ solving\ for\ D\ using\ the\ formula\ D_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2b}\ we\ have$
$D = 0.1312$
$Consequently\ d = \frac{0.1312}{0.197}$
$= 0.6660$
$Now\ to\ find\ the\ angle\ subtended\ by\ two\ sides\ and\ the\ fractional\ side\ is\ found\ as\ following$
$Angle\ subtended\ by\ each\ side\ at\ the\ center\ is = \frac{2\pi}{32}$
$= 0.1963$
$So, the\ angle\ subtended\ by\ two\ such\ sides = 2 \times 0.1963$
$= 0.3927$
$Here, segments\ \ of\ the\ side\ \ do\ not\ subtend\ the\ angle\ at\ the\ centre\ proportinately\ . Hence\ an\ approximation\ is\ done\ at\ this\ stage.$
$i.e., the\ angle\ subtended\ by\ the\ fractional\ part\ of\ the\ side\ is \approx 0.6660 \times 0.1963$
$i.e., Opposite\ side\ intersect\ the\ perimeter\ at\ C + d\ number\ of\ sides\ which\ is = 2 + 0.6660$
$= 2.666\ sides$
$$\frac{C+d}{\frac{n}{4}} = \frac{2.666}{8}$$
$Here\ \lim_{n\to\infty}(C+d) = C$
$Now\ let\ n = 400000, a\ relatively\ large\ number, then\ C\ will\ be$
$$\frac{C}{100000} = \frac{2.666}{8}$$
$Then\ C = 33325$
$Then, \theta = C \times \frac{2\pi}{n}$
$= 33325 \times \frac{2\pi}{400000}$
$= 0.5235$
$which\ is\ very\ nearly\ equal\ to\ \frac{\pi}{6} = 0.5236$
$Example2: Find\ the\ angle\ when\ sin\theta = \frac{1}{\sqrt{2}}\ (in\ the\ Ist\ quadrant)$
$Solution: It\ is\ known\ that,\ \ OO_i = \left(OP_i(a,b)\right)\left(sin\left(\frac{2C\pi}{n}\right)\right)$
$i.e, in\ the\ above\ case\ ,\qquad \frac{OO_i}{\left(OP_i(a,b)\right)} = \left(sin\left(\frac{2C\pi}{n}\right)\right)$
$$\frac{1}{\sqrt{2}} = \left(sin\left(\frac{2C\pi}{n}\right)\right)$$
$Above\ equation\ is\ true\ when\ C = 1\ and\ n = 4$
$Consequantly, \theta = \frac{2\pi}{4}$
This method is applicable for values that can be found by trial and error method.
$15.0.2. Finally, it\ is\ the\ right\ position\ to\ study\ the\ \ values\ of\ sum\ and\ differances\ of\ trigonometric\ functions.\ cos(x+y) = cos\ x\ cos\ y - sinx\ siny\ \ ------(87);$

$this\ identity\ is\ not\ true\ for\ equilateral\ triangle$
$Now, let\ we\ consider\ a\ regular\ polygon\ of\ 32\ sides. Here\ values\ of\ x\ and\ y\ are\ considerd\ only\ at\ vertices.$

$Here\ in\ the\ first\ quadrant\ values\ of\ x\ and\ y\ are\ any\ one\ \ of\ the\ multiples\ of\ \frac{2\pi_{32}^{a}}{32} = 0.1970$

$Let\ x = \frac{2\pi_{32}^{a}}{32}\ and\ y = \frac{2\pi_{32}^{a}}{16}$

$Then,\ cos(x+y) = cos\,x\,cos\,y - sinx\,siny$

$$cos\left(\frac{2\pi_{32}^{a}}{32} + \frac{2\pi_{32}^{a}}{16}\right) = cos\left(\frac{2\pi_{32}^{a}}{32}\right)cos\left(\frac{2\pi_{32}^{a}}{16}\right) - sin\left(\frac{2\pi_{32}^{a}}{32}\right)sin\left(\frac{2\pi_{32}^{a}}{16}\right)$$

$$cos\left(\frac{3\pi_{32}^{a}}{16}\right) = cos\left(\frac{2\pi_{32}^{a}}{32}\right)cos\left(\frac{2\pi_{32}^{a}}{16}\right) - sin\left(\frac{2\pi_{32}^{a}}{32}\right)sin\left(\frac{2\pi_{32}^{a}}{16}\right)$$

$\frac{3\pi_{32}^{a}}{16}\ is\ equal\ to\ arc\ length\ of\ 3\ sides$

$$cos\left(\frac{3\pi_{32}^{a}}{16}\right) = \frac{\frac{1}{cos\left(\frac{\pi}{32}\right)} - 0.197\left\{cos\left(\frac{15\pi}{32}\right) + cos\left(\frac{13\pi}{32}\right) + cos\left(\frac{11\pi}{32}\right)\right\}}{\frac{1}{cos\left(\frac{\pi}{32}\right)}}$$

$$= 1 - 0.197 \times cos\left(\frac{\pi}{32}\right)\left\{cos\left(\frac{15\pi}{32}\right) + cos\left(\frac{13\pi}{32}\right) + cos\left(\frac{11\pi}{32}\right)\right\}$$

$$= 0.8315$$

$Similarly, cos\left(\frac{2\pi_{32}^{a}}{32}\right) = 0.9808\,; cos\left(\frac{2\pi_{32}^{a}}{16}\right) = 0.9239; sin\left(\frac{2\pi_{32}^{a}}{32}\right) = 0.1951; sin\left(\frac{2\pi_{32}^{a}}{16}\right) = 0.3827$

$Substituting\ these\ values\ in\ \ (\ )\ we\ have$

$$0.8315 = (0.9808 \times 0.9239) - (0.1951 \times 0.3827)$$
$$= 0.8314$$
$$0.8315 \approx 0.8314$$

$This\ result\ shows\ that\ the\ equation\ \ cos(x+y) = cos\,x\,cos\,y - sinx\,siny\ \ \ is\ valied\ \ only$ $to\ the\ \ values\ at\ the\ vertices\ of\ the\ regular\ polygons\ from\ reasonable\ number\ of\ sides.$
$As\ n\ \to \infty, the\ resulting\ figure\ will\ be\ a\ circle. then, each\ poin$t on the circumferance of a circle is a side as well as a vertex. Consequently, (87)is true for all points on the circumferance of a circle.

XVI. Conclusion

Any knowledge that exhibits good relation and link between the subject and matter will be interesting, easy to learn and to remember, hence they can be easily applicable in required fields. With this vision an effort is made to bring the link between the equations of regular polygons in this article