IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN:2319-765X. Volume 10, Issue 3 Ver. II (May-Jun. 2014), PP 69-77
www.iosrjournals.org

Common Coupled Fixed Point of Mappings Satisfying Rational Inequalities in Ordered Complex Valued Generalized Metric Spaces

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Abstract: *In this paper, we prove some common coupled fixed point results using rational type contractive conditions in complex valued generalized metric spaces.*
Keywords: *Common coupled fixed point, Weakly increasing map, Complex valued generalized metric space, Partially ordered set*

I. Introduction And Preliminaries

From last few years, the metric fixed point theory has become an important field of research in both pure and applied sciences. The first result in the direction was obtained by Ran and Reurings [6], in this, the authors presented some applications of their obtained results of matrix equations. In [7,8], Nieto and Lopez extended the result of Ran and Reurings [6] for non decreasing mappings and applied their result to get a unique solution for first order differential equation. While Agarwal et al. [9] and O'Regan and Petrutel [10] studied some results for a generalized contraction in ordered metric spaces. Banach's contraction Principle gives appropriate and simple condition to establish the existence and uniqueness of a solution of an operator equation $Fx = x$. Then there are many generalizations of the Banach's contraction mapping principle in the literature.

Bhaskar and Lakshmikantham [11] introduced the concept of coupled fixed point of a mapping F from $X \times X$ into X. They established some coupled fixed point results and applied their results to the study of existence and uniqueness of solution for a periodic boundary value problem. Lakshikantham and Ciric [12] introduced the concept of coupled coincidence point and proved coupled coincidence and coupled common fixed point results for a mapping F from $X \times X$ into X and g from X into X satisfying nonlinear contraction in ordered metric space. Then many results of coupled fixed point are obtained by many authors in many spaces refer as [6-9]. Recently, Azam et al. [1] introduced the concept of complex coupled metric spaces and proved some result. The idea of complex valued metric spaces can be exploited to define complex valued normed spaces and complex valued Hilbert spaces, additionally, it offers numerous research activities in mathematical analysis.

Let £ be the set of complex numbers and let $z_1, z_2 \in$ £ . Define a partial order $\le$ and £ as follows: $z_1 \le z_2$ if and oly if $\mathrm{Re}(z_1) \le \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) \le \mathrm{Im}(z_2)$.

It follows that $z_1 \le z_2$ if one of the following is satisfied:

(i) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) \le \mathrm{Im}(z_2)$

(ii) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$

(iii) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$

(iv) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$

In particular, we will write $z_1 \le z_2$ if one of (i), (ii) and (iii) is satisfied and we will write $z_1 < z_2$ if and ol y if (iii) is satisfied.

Note that

$$0 \le z_1 < z_2 \Rightarrow |z_1| < |z_2|$$
$$z_1 \le z_2,\ z_2 < z_3 \Rightarrow z_1 < z_3$$

Definition 1 ([1]). Let X be a nonempty set. Suppose that the mapping $d : X \times X \to$ £ satisfies the following:

(i) $0 \le d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(iii) $d(x, y) \le d(y, x) + d(z, y)$ for all $x, y \in X$.
Then d is called complex valued metric on X , and (X, d) is called a complex valued metric space.

Definition 2 ([18]). Let X be a nonempty set. If a mapping $d : X \times X \to £$ satisfies:

(iv) $0 \le d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(v) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(vi) $d(x, y) \le d(x,u) + d(u,v) + d(v, y)$ for all $x, y \in X$ and all distinct $u,v \in X$ each one is different from x and y .

Then d is called a complex valued generalized metric on X and (X, d) is called a complex valued generalized metric space.

The following definition is due to Altun [15]:

Definition 3 ([15]). Let $(X, ¶)$ be a partially ordered set. A pair (f, g) of self maps of X is said to be weakly increasing if $fx ¶ \ gfx$ and $gx ¶ \ fgx$ for all $x \in X$. If $f = g$, then we have $fx ¶ \ f^2 x$ for all $x \in X$ and in case, we say that f is weakly increasing map.

For coupled, we extend this as follows:
Let $(X, ¶)$ be a partially ordered set. Let $f, g : X \times X \to X$ two mapping is said to be weakly increasing if

$$f(x, y) ¶ \ g(f(x, y), f(y, x)); \ f(y, x) ¶ \ g(f(y, x), f(x, y))$$

and

$$g(x, y) ¶ \ f(g(x, y), g(y, x)); \ g(y, x) ¶ \ f(g(y, x), g(x, y)) \text{ for all } x, y \in X .$$

A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exist $0 < r \in £$ such that

$$B(x,r) = \{y \in X : d(x, y) < r\} \subseteq A .$$

A point $x \in X$ is a limit point of A whenever, for every $0 < r \in £$

$$B(x,r) \cap (A - \{x\}) \ne \phi .$$

A is called open whenever each element of A is an interior point of A . Moreover, a subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B . The family,

$$F = \{B(x,r) : x \in X, 0 < r \in £\}$$

is a sub basis for a Haudorff topology τ on X . Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in £$ with $0 < c$ there is $n_0 \in N$ such that, for all $n > n_0$, $d(x_n, x) < c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x , and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$, or $x_n \to x$ as $n \to \infty$. If for every $c \in £$ with $0 < c$ there is $n_0 \in N$, $d(x_n, x_{n+m}) < c$ then $\{x_n\}$ is called Cauchy sequence in (X, d) . If every Cauchy sequence is convergent in (X, d) , then (X, d) is called a complete complex-valued generalized space.

Lemma 1. Let (X, d) be a complex-valued generalized metric space, and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ converges to x if and only if $| d(x_n, x) | \to 0$ as $n \to \infty$.

Lemma 2. Let (X, d) be a complex-valued generalized metric space, and $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is Cauchy sequence if and only if $| d(x_n, x_{n+m}) | \to 0$ as $n \to \infty$.

Definition 4 ([11]). An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$$F(x, y) = x \text{ and } F(y, x) = y .$$

Definition 5 ([12]). (i) A coupled coincidence point of the mapping $F : X \times X \to X$ and $g : X \to X$

$$F(x, y) = g(x) \text{ and } F(y, x) = g(y) .$$

(ii) A common coupled coincidence point of the mapping $F : X \times X \to X$ and $g : X \to X$ if

$$F(x, y) = g(x) = x \text{ and } F(y, x) = g(y) = y .$$

In this paper, we extend the result of M. Abbas et al [18] fixed point to coupled fixed point in complex valued generalized metric space.

II. Main Results

Theorem 1. Let $(X,\P)$ be a partially ordered set such that there exist a complete complex valued generalized metric d on X and (S,T) a pair of weakly increasing which defined as $S,T : X \times X \to X$. Suppose that, for every comparable $x,y,u,v \in X$, we have either

$$d(S(x,y),T(u,v)) \leq \frac{a_1[d(u,S(x,y))d(x,T(u,v))^2 + d(x,T(u,v))d(u,S(x,y))^2]}{d(x,T(u,v))^2 + d(u,S(x,y))^2}$$
$$+ a_2 \frac{d(x,T(u,v))d(u,S(x,y))}{d(x,T(u,v)) + d(u,S(x,y))} + a_3\, d(x,S(x,y)) + a_4\, d(u,T(u,v)) \qquad (1)$$

in case $d(x,T(u,v)) + d(u,S(x,y)) \neq 0$, $a_i > 0$, $i = 1,2,3,4$ and $\sum_{i=1}^{4} a_i < 1$, or

$$d(S,(x,y),T(u,v)) = 0 \text{ if } d(x,T(u,v)) + d(u,S(x,y)) = 0 \qquad (2)$$

If S or T is continuous or for any nondecreasing sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \to z$ and $y_n \to z'$, we necessary have $x_n \P z$ and $y_n \P z'$ for all $n \in N$, then S and T have a common coupled fixed point.

Moreover, the set of common coupled fixed point of S and T is totally ordered if and only if S and T have one and only one common coupled fixed point.

Proof. First of all, we show that if S or T has a coupled fixed point, then it has a common coupled fixed point of S and T. Suppose (x,y) is a coupled fixed point of S, that is $S(x,y) = x$ and $S(y,x) = y$. Then we shall show that (x,y) is a coupled fixed point of T also. Let if possible (x,y) is not a coupled fixed point of T. Then from (1), we get

$$d(x,T(x,y)) = d(S(x,y),T(x,y))$$
$$\leq a_1 \frac{[d(x,S(x,y))d(x,T(x,y))^2 + d(x,T(x,y))d(x,S(x,y))^2]}{d(x,T(x,y))^2 + d(x,S(x,y))^2}$$
$$+ a_2 \frac{d(x,T(x,y))d(x,S(x,y))}{d(x,T(x,y)) + d(x,S(x,y))} + a_3\, d(x,S(x,y)) + a_4\, d(x,T(x,y))$$

which implies that $|d(x,T(x,y))| \leq a_4 |d(x,T(x,y))|$, which is a contradiction as $a_4 < 1$, so we get $d(x,T(x,y)) = 0$, this implies that $T(x,y) = x$.

Similarly, we can show that $T(y,x) = y$.

In the same way if we take (x,y) is coupled fixed point of T that it is also S.

Now let (x_0,y_0) be an arbitrary point of $X \times X$. If $S(x_0,y_0) = x_0$ and $S(y_0,x_0) = y_0$, then proof is finished. Let either $S(x_0,y_0) \neq x_0$ or $S(y_0,x_0) \neq y_0$.

Define the sequences $\{x_n\}$ and $\{y_n\}$ in X as follows:

$$x_1 = S(x_0,y_0) \P T(S(x_0,y_0),S(y_0,x_0)) = T(x_1,y_1) = x_2$$
$$y_1 = S(y_0,x_0) \P T(S(y_0,x_0),S(x_0,y_0)) = T(y_1,x_1) = y_2$$
$$x_2 = T(x_1,y_1) \P S(T(x_1,y_1),S(y_1,x_1)) = S(x_2,y_2) = x_3$$
$$y_2 = T(y_1,x_1) \P S(T(y_1,x_1),S(x_1,y_1)) = S(y_2,x_2) = y_3$$

In similar fashion, we get

$$x_1 \P x_2 \P x_3 \P \dots \P x_n \P x_{n+1} \P \dots$$

and

$$y_1 \P y_2 \P y_3 \P \dots \P y_n \P y_{n+1} \P \dots$$

Let $d(x_{2n},x_{2n+1}) > 0$ and $d(y_{2n},y_{2n+1}) > 0$ for every $n \in N$. If not, then $x_{2n} = x_{2n+1}$ and $y_{2n} = y_{2n+1}$ for some n. For all those n, $x_{2n} = x_{2n+1} = S(x_{2n},y_{2n})$ and $y_{2n} = y_{2n+1} = S(y_{2n},x_{2n})$, and proof is finished.

Now, let $d(x_{2n},x_{2n+1}) > 0$ and $d(y_{2n},y_{2n+1}) > 0$ for $n = 0,1,2,3,\dots$. As x_{2n}, x_{2n+1} and y_{2n}, y_{2n+1} are comparable, so we have

$$d(x_{2n+1},x_{2n+2}) = d(S(x_{2n},y_{2n}),T(x_{2n+1},y_{2n+1}))$$

$$\leq a_1 \frac{\begin{bmatrix} d(x_{2n+1}, S(x_{2n}, x_{2n}))d(x_{2n}, T(x_{2n+1}, y_{2n+1}))^2 \\ +d(x_{2n}, T(x_{2n+1}, y_{2n+1}))d(x_{2n+1}, S(x_{2n}, y_{2n}))^2 \end{bmatrix}}{d(x_{2n}, T(x_{2n+1}, y_{2n+1}))^2 + d(x_{2n+1}, S(x_{2n}, y_{2n}))^2}$$
$$+ a_2 \frac{[d(x_{2n}, T(x_{2n+1}, y_{2n+1}))d(x_{2n+1}, S(x_{2n}, y_{2n}))]}{d(x_{2n}, T(x_{2n+1}, y_{2n+1})) + d(x_{2n+1}, S(x_{2n}, y_{2n}))}$$
$$+ a_3\, d(x_{2n}, S(x_{2n}, y_{2n})) + a_4\, d(x_{2n+1}, T(x_{2n+1}, y_{2n+1}))$$
$$= a_3\, d(x_{2n}, x_{2n+1}) + a_4\, d(x_{2n+1}, x_{2n+2})$$

which implies that

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{a_3}{1-a_4} d(x_{2n}, x_{2n+1}) \text{ for all } n \geq 0.$$

Hence

$$d(x_{2n+1}, x_{2n+2}) \leq h\, d(x_{2n}, x_{2n+1}) \text{ for all } n \geq 0,$$

where $0 < h < \frac{a_3}{1-a_4} < 1$. Similarly,

$$d(x_{2n}, x_{2n+1}) \leq h\, d(x_{2n-1}, x_{2n}) \text{ for all } n \geq 0.$$

Hence for all $n \geq 0$, we have $d(x_{n+1}, x_{n+2}) \leq h\, d(x_n, x_{n+1})$.

Consequently,

$$d(x_{n+1}, x_{n+2}) \leq h\, d(x_n, x_{n+1}) \leq \ldots \leq h^{n+1} d(x_0, x_1) \text{ for all } n \geq 0.$$

In similar way,

$$d(y_{2n+1}, y_{2n+2}) \leq h\, d(y_{2n}, y_{2n+1}) \text{ for all } n \geq 0 \text{ and } 0 < h < 1$$

and like above

$$d(y_{2n}, y_{2n+1}) \leq h\, d(y_{2n-1}, y_{2n}) \text{ for all } n \geq 0 \text{ and } 0 < h < 1.$$

We have

$$d(y_{n+1}, y_{n+2}) \leq h\, d(y_n, y_{n+1}).$$

Consequently,

$$d(y_{n+1}, y_{n+2}) \leq h\, d(y_n, y_{n+1}) \leq \ldots \leq h^{n+1} d(y_0, y_1).$$

Now for $m > n$, we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + (x_{m-1}, x_m)$$
$$\leq h^n\, d(x_0, x_1) + h^{n+1} d(x_0, x_1) + \ldots + h^{m-1} d(x_0, x_1)$$
$$\leq \frac{h^n}{1-h} d(x_0, x_1).$$

Therefore, $|\, d(x_n, x_m)\,| \leq \frac{h^n}{1-h} |\, d(x_0, x_1)\,|$. So $|\, d(x_n, x_m)\,| \to 0$ as $n, m \to \infty$ gives that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete. So there exist a point u' in X such that $\{x_n\}$ converges to u'. Similarly, we can easily show that the sequence $\{y_n\}$ is a Cauchy sequence in X and due to completeness of X, $\{y_n\}$ converges to a point v' in X.

If S or T is continuous, then it is clear that

$$S(u', v') = u' = T(u', v')$$

and

$$S(v', u') = v' = T(v', u').$$

If neither S nor T is continuous, then by given assumption x_n ¶ u' and y_n ¶ v' for all $n \in N$. We claim that (u', v') is a coupled fixed point of S. Let if possible $d(S(u', v'), u') = z > 0$ and $d(S(v', u'), v') = z' > 0$.

From (1), we have

$$z = d(S(u', v'), u')$$
$$\leq d(u', x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + d(x_{2n+2}, S(u', v'))$$
$$= d(u', x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + d(S(u', v'), T(x_{2n+1}, y_{2n+1}))$$

$$\le d(u',x_{2n+1})+d(x_{2n+1},x_{2n+2})+a_1\frac{\begin{bmatrix} d(x_{2n+1},S(u',v'))d(u',T(x_{2n+1},y_{2n+1}))^2 \\ +d(u',T(x_{2n+1},y_{2n+1}))d(x_{2n+1},S(u',v'))^2 \end{bmatrix}}{d(u',T(x_{2n+1},y_{2n+1}))^2+d(x_{2n+1},S(u',v'))^2}$$

$$+a_2\frac{d(u',T(x_{2n+1},y_{2n+1}))d(x_{2n+1},S(u',v'))}{d(u',T(x_{2n+1},y_{2n+1}))+d(x_{2n+1},S(u',v'))}$$

$$+a_3\, d(u',S(u',v'))+a_4\, d(x_{2n+1},T(x_{2n+1},y_{2n+1}))$$

$$=d(u',x_{2n+1})+d(x_{2n+1},x_{2n+2})\ +a_1\frac{\begin{bmatrix} d(x_{2n+1},S(u',v'))d(u',x_{2n+2})^2 \\ +d(u',x_{2n+2})d(x_{2n+1},S(u',v'))^2 \end{bmatrix}}{d(u',x_{2n+2})^2+d(x_{2n+1},S(u',v'))^2}$$

$$+a_2\frac{d(u',x_{2n+2})d(x_{2n+1},S(u',v'))}{d(u',x_{2n+2})+d(x_{2n+1},S(u',v'))}\ +a_3\, d(u',S(u',v'))+a_4\, d(x_{2n+1},x_{2n+2}))$$

and so

$$|z|=|d(u',x_{2n+1})|+|d(x_{2n+1},x_{2n+2})|+a_1\frac{\begin{bmatrix} |d(x_{2n+1},S(u',v'))|\{|d(u',x_{2n+2})|\}^2 \\ +|d(u',x_{2n+2})|\{|d(x_{2n+1},S(u',v'))|\}^2 \end{bmatrix}}{|\{d(u',x_{2n+2})|\}^2+\{d(x_{2n+1},S(u',v'))\}^2}$$

$$+a_2\frac{|d(u',x_{2n+2})|\,|d(x_{2n+1},S(u',v'))|}{|d(u',x_{2n+2})+d(x_{2n+1},S(u',v'))|}\ +a_3|d(u',S(u',v'))|+|a_4\, d(x_{2n+1},x_{2n+2}))|$$

which on taking limit as $n\to\infty$, we get $|z|\le a_3|z|$, a contradiction, and so $S(u',v')=u'$. Similarly $S(v',u')=v'$.

Hence

$$S(u',v')=u'=T(u',v')$$

and

$$S(v',u')=v'=T(u',v')$$

Now suppose that the set of common coupled fixed point of S and T is totally ordered. Now we shall show that the common coupled fixed point of S and T is unique.

Let (x,y) and (u,v) are two distinct common coupled fixed point of S and T.

From (1), we obtain

$$d(S(x,y),T(u,v))\le a_1\frac{[d(u,S(x,y))d(x,T(u,v))^2+d(x,T(u,v))d(u,S(x,y))^2]}{d(x,T(u,v))^2+d(u,S(x,y))^2}$$

$$+\ a_2\frac{d(x,T(u,v))d(u,S(x,y))}{d(x,T(u,v))+d(u,S(x,y))}\ +a_3\, d(x,S(x,y))+a_4\, d(u,T(u,v))$$

$$=a_1\frac{[d(u,x)d(x,u)^2+d(x,u)d(u,x)^2]}{d(x,u)^2+d(u,x)^2}\ +a_2\frac{d(x,u)d(u,x)}{d(x,u)+d(u,x)}$$

$$+\ a_3\, d(x,x)+a_4\, d(u,u)$$

and so

$$|d(S(x,y),T(u,v))|\le\left(a_1+\frac{1}{2}a_2\right)|d(S(x,y),T(u,v))|$$

where $\left(a_1+\frac{1}{2}a_2\right)<1$.

Which is a contradiction, so $S(x,y)=T(u,v)$.

Similarly, $S(y,x)=T(v,u)$.

Hence $x=u$ and $y=v$ which proves the uniqueness.

Conversely, if S and T have only one common coupled fixed point then the set of common coupled fixed point being singleton is totally ordered.

Corollary 1. Let $(X, \P)$ be a partially ordered set such that there exist a complete complex valued generalized metric d on X and let $T : X \times X \to X$ be weakly increasing map. Suppose that, for every comparable $u, v, x, y \in X$, either

$$d(T(x,y),T(u,v)) \leq a_1 \frac{[d(u,T(x,y))d(x,T(u,v))^2 + d(x,T(u,v))d(u,T(x,y))^2]}{d(x,T(u,v))^2 + d(u,T(x,y))^2} + a_2 \frac{d(x,T(u,v))d(u,T(x,y))}{d(x,T(u,v)) + d(u,T(x,y))} + a_3\, d(x,T(x,y)) + a_4\, d(u,T(u,v)) \tag{3}$$

If $d(x,T(u,v)) + d(u,T(x,y)) \neq 0$, $a_i > 0$, $i = 1,2,3,4$ and $\sum_{i=1}^{4} a_i < 1$ or

$$d(T(x,y),T(u,v)) = 0 \text{ if } d(x,T(u,v)) + d(u,T(x,y)) = 0 \tag{4}$$

If T is continuous or for a nondecreasing sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \to z$ and $x_n \to z'$ in X, we necessary have $x_n \P z$ and $x_n \P z'$ for all $n \in N$ then T has a coupled fixed point. Moreover, the set of coupled fixed point of T is totally ordered if and only if T has one and only one coupled fixed point.

Proof. Take $S = T$ in Theorem 1.

Theorem 2. Let $(X, \P)$ be a partially ordered set such that there exist a complete complex valued generalized metric d on X and a pair (S,T) weakly increasing which defined as $S,T : X \times X \to X$. Suppose that, for every comparable $x, y, u, v \in X$, either

$$d(S(x,y),T(u,v)) \leq a \frac{[d(x,S(x,y))d(x,T(u,v)) + d(u,T(u,v))d(u,S(x,y))]}{d(x,T(u,v)) + d(u,S(x,y))} + b \frac{d(x,T(u,v))d(u,S(x,y))}{d(x,S(x,y)) + d(u,T(u,v))} \tag{5}$$

if $d(x,T(u,v)) + d(u,S(x,y)) \neq 0$ and $d(x,S(x,y)) + d(u,T(u,v)) \neq 0$, with $a + b < 1$, or

$$d(S(x,y),T(u,v)) = 0 \text{ if } d(x,T(u,v)) + d(u,S(x,y)) = 0 \text{ or } d(x,S(x,y)) + d(u,T(u,v)) = 0 \tag{6}$$

If S or T is continuous of for a nondecreasing sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \to z$ and $y_n \to z'$ in X, we necessarily have $x_n \P z$ and $y_n \P z'$ for all $n \in N$.
Then S and T have a common coupled fixed point.

Proof. First of all we shall show that if S or T has a coupled fixed point, then it has a common coupled fixed point of S and T. Suppose that (x, y) is a coupled fixed point of S. Then we shall show that (x, y) is coupled fixed point of T also. Let if possible (x, y) is not a coupled fixed point of T. Then from (5), we obtain

$$\begin{aligned}
d(x,T(x,y)) &= d(S(x,y),T(x,y)) \\
&\leq a \frac{[d(x,S(x,y))d(x,T(x,y)) + d(x,T(x,y))d(x,S(x,y))]}{d(x,T(x,y)) + d(x,S(x,y))} \\
&\quad + b \frac{d(x,T(x,y))d(x,S(x,y))}{d(x,S(x,y)) + d(x,T(x,y))} \\
&= a \frac{[d(x,x)d(x,T(x,y)) + d(x,T(x,y))d(x,x)]}{d(x,T(x,y)) + d(x,x)} \\
&\quad + b \frac{d(x,T(x,y))d(x,x)}{d(x,x) + d(x,T(x,y))} \\
&= 0
\end{aligned}$$

Hence $d(x,T(x,y)) = 0$ and this shows that $T(x,y) = x$. Similarly, we can easily, show that $T(y,x) = y$.

Hence (x, y) is a common coupled fixed point of S and T. In the same way if we take (x, y) is a coupled fixed point of T then it is also S.

Now let (x_0, y_0) be an arbitrary point of $X \times X$. If $S(x_0, y_0) = x_0$ and $S(y_0, x_0) = y_0$, then proof is hold. Let either $S(x_0, y_0) \neq x_0$ or $S(y_0, x_0) \neq y_0$.

Construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that:

$$x_1 = S(x_0, y_0) \P\ T(S(x_0, y_0), S(y_0, x_0)) = T(x_1, y_1) = x_2$$
$$y_1 = S(y_0, x_0) \le T(S(y_0, x_0), S(x_0, y_0)) = T(y_1, x_1) = y_2$$
$$x_2 = T(x_1, y_1) \P\ S(T(x_1, y_1), S(y_1, x_1)) = S(x_2, y_2) = x_3$$
$$y_2 = T(y_1, x_1) \P\ S(T(y_1, x_1), S(x_1, y_1)) = S(y_2, x_2) = y_3$$

In the same way, we obtain

$$x_1 \P\ x_2 \P\ x_3 \P\ \ldots \P\ x_n \P\ x_{n+1} \P\ \ldots$$

and

$$y_1 \P\ y_2 \P\ y_3 \P\ \ldots \P\ y_n \P\ y_{n+1} \P\ \ldots$$

Let $d(x_{2n}, x_{2n+1}) > 0$ and $d(y_{2n}, y_{2n+1}) > 0$, for every $n \in N$. If not, then $x_{2n} = x_{2n+1}$ and $y_{2n} = y_{2n+1}$ for some $n \in N$. For all those n, $x_{2n} = x_{2n+1} = S(x_{2n}, y_{2n})$ and $y_{2n} = y_{2n+1} = S(y_{2n}, y_{2n})$, and proof is hold.

Now, let $d(x_{2n}, x_{2n+1}) > 0$ and $d(y_{2n}, y_{2n+1}) > 0$ for $n = 0,1,2,3,\ldots$. As x_{2n}, x_{2n+1} and y_{2n}, y_{2n+1} are comparable, so we have

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) &= d(S(x_{2n}, y_{2n}), T(x_{2n+1}, y_{2n+1})) \\
&\le a\,\frac{\begin{bmatrix} d(x_{2n}, S(x_{2n}, y_{2n}))d(x_{2n}, T(x_{2n+1}, y_{2n+1})) \\ +d(x_{2n+1}, T(x_{2n+1}, y_{2n+1}))d(x_{2n+1}, S(x_{2n}, y_{2n})) \end{bmatrix}}{d(x_{2n}, T(x_{2n+1}, y_{2n+1})) + d(x_{2n+1}, S(x_{2n}, y_{2n}))} \\
&\quad + b\,\frac{d(x_{2n}, T(x_{2n+1}, y_{2n+1}))d(x_{2n+1}, S(x_{2n}, y_{2n}))}{d(x_{2n}, S(x_{2n}, y_{2n})) + d(x_{2n+1}, T(x_{2n+1}, y_{2n+1}))} \\
&= a\,\frac{[d(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+2})d(x_{2n+1}, x_{2n+1})]}{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})} \\
&\quad + b\,\frac{d(x_{2n}, x_{2n+2})d(x_{2n+1}, x_{2n+1})}{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})} \\
&= a\,\frac{d(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+2})}{d(x_{2n}, x_{2n+2})} \\
&= a\, d(x_{2n}, x_{2n+1}).
\end{aligned}$$

Similarly, $d(x_n, x_{2n+1}) \le ad(x_{2n-1}, x_{2n})$, for all $n \ge 0$.

Thus $d(x_{n+1}, x_{n+2}) \le ad(x_n, x_{n+1})$, for all $n \ge 0$.

Consequently,

$$d(x_{n+1}, x_{n+2}) \le a\, d(x_n, x_{n+1}) \le \ldots \le a^{n+1} d(x_0, x_1), \text{ for all } n \ge 0.$$

In similar way,

$$d(y_{n+1}, y_{n+2}) \le a\, d(y_n, y_{n+1}) \le \ldots \le a^{n+1} d(y_0, y_1), \text{ for all } n \ge 0.$$

Now for $m > n$, we have

$$\begin{aligned}
d(x_n, x_m) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + (x_{m-1}, x_m) \\
&\le a^n\, d(x_0, x_1) + a^{n+1} d(x_0, x_1) + \ldots + a^{m-1} d(x_0, x_1) \\
&\le \frac{a^n}{1-a} d(x_0, x_1).
\end{aligned}$$

So, $|\, d(x_n, x_m) \,| \le \frac{a^n}{1-a} |\, d(x_0, x_1) \,|$. So $|\, d(x_n, x_m) \,| \to 0$ as $m, n \to \infty$. It follows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete. So there exist a point u' in X such that $\{x_n\}$ converges to u'. Similarly, we can easily show that the sequence $\{y_n\}$ is a Cauchy sequence in X and due to completeness of X, $\{y_n\}$ converges to a point v' in X. If S or T is continuous, then it is clear that

$$S(u', v') = u' = T(u', v')$$

and

$$S(v', u') = v' = T(v', u').$$

If neither S nor T is continuous, then by given assumption $x_n \le u'$ and $y_n \le v'$ for all $n \in N$. We claim that (u', v') is a coupled fixed point of S. Let if possible $d(S(u', v'), u') = z > 0$ and $d(S(v', u'), v') = z' > 0$.

From (5), we have

$$\begin{aligned}
z &= d(S(u',v'),u') \\
&\le d(u',x_{2n+1}) + d(x_{2n+1},x_{2n+2}) + d(x_{2n+2},S(u',v')) \\
&= d(u',x_{2n+1}) + d(x_{2n+1},x_{2n+2}) + d(S(u',v'),T(x_{2n+1},y_{2n+1})) \\
&\le d(u',x_{2n+1}) + d(x_{2n+1},x_{2n+2}) \\
&\quad + a\frac{\begin{bmatrix} d(u',S(u',v'))d(u',T(x_{2n+1},y_{n+1})) \\ +d(x_{2n+1},T(x_{2n+1},y_{2n+1}))d(x_{2n+1},S(u',v')) \end{bmatrix}}{d(u',T(x_{2n+1},y_{2n+1})) + d(x_{2n+1},S(u',v'))} \\
&\quad + b\frac{d(u',T(x_{2n+1},y_{2n+1}))d(x_{2n+1},S(u',v'))}{d(u',S(u',u')) + d(x_{2n+1},T(x_{2n+1},y_{2n+1}))} \\
&= d(u',x_{2n+1}) + d(x_{2n+1},x_{2n+2}) + a\frac{\begin{bmatrix} d(u',S(u',v'))d(u',x_{2n+2}) \\ +d(x_{2n+1},x_{2n+2})d(x_{2n+1},S(u',v')) \end{bmatrix}}{d(u',x_{2n+2}) + d(x_{2n+1},S(u',v'))} \\
&\quad + b\frac{d(u',x_{2n+2})d(x_{2n+1},S(u',v'))}{d(u',S(u',v')) + d(x_{2n+1},x_{2n+2})}.
\end{aligned}$$

Taking modulas and $\lim n \to \infty$, on both side, we have $|z| \le 0$, a contradiction. So $S(u',v') = u'$. Similarly $S(v',u') = v'$.

Hence

$$S(u',v') = u' = T(u',v')$$

and

$$S(v',u') = v' = T(u',v')$$

Then S and T have a common coupled fixed point.

Remark. In Theorem 2, if in (5), we take $b = 0$, then point of S and T is totally ordered if and only if S and T have one and only, one common coupled fixed point.

Corollary 2. Let $(X, ¶)$ be a partially ordered set such that there exist a complete complex valued generalized metric d on X and let $T : X \times X \to X$ be weakly increasing map. Let for every comparable $x, y, u, v \in X$, either

$$\begin{aligned}
d(T(x,y),T(u,v)) &\le a\frac{[d(x,T(x,y))d(x,T(u,v)) + d(u,T(u,v))d(u,T(x,y))]}{d(x,T(u,v)) + d(u,T(x,y))} \\
&\quad + b\frac{d(x,T(u,v))d(u,T(x,y))}{d(x,T(x,y)) + d(u,T(u,v))} \qquad (8)
\end{aligned}$$

if $d(x,T(u,v)) + d(u,T(x,y)) \ne 0$ and $d(x,T(x,y)) + d(u,T(u,v)) \ne 0$ with $a + b < 1$, or

$$d(T(x,y),T(u,v)) = 0 \text{ if } d(x,T(u,v)) + d(u,T(x,y)) = 0$$

$$\text{or } d(x,T(x,y)) + d(u,T(u,v)) = 0 \qquad (9)$$

If T is continuous or for a nondecreasing sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \to z$ and $y_n \to z'$ in X necessarily have x_n ¶ z and y_n ¶ z' for all $n \in N$.

Then S and T have a common coupled fixed point.

Proof. Put $S = T$ in Theorem 2.

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