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Edge Sum Number of Jahangir Graphs

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Abstract: *A graph is said to be an edge sum graph if the edges of G can be labeled with distinct positive integers such that the sum of all the labels incident on a vertex is again an edge label of G and if the sum of any collection of edges is a label of an edge in G, then they are incident on a vertex. The edge sum number* $\sigma_E(G)$ *of a graph G is the smallest number r of edges which added to G result in an edge sum graph. In this paper, we prove that* $\sigma_E(J_{3,4}) = 3.$

Keywords: *Sum graph, edge sum graph, sum number, edge sum number.*

I. Introduction

All terms not defined here can be found in Harary [4]. Throughout this paper, we consider only finite undirected graphs without loops. By a graph G(V,E) we mean a graph with vertex set V and edge set E. Jahangir graphs $J_{n,m}$ for $m \geq 3$ is a graph on $nm+1$ vertices that is a graph consisting of a cycle C_{nm} with one more vertex which is adjacent to m vertices of C_{nm} at distance n to each on C_{nm}.

Harary [5] introduced the concept of sum graph and sum number. A graph G is called a sum graph if the vertices of G can be labeled with distinct positive integers so that e = uv is an edge of G if and only if the sum of the labels u and v equals a label of some vertex w in G. If G is not a sum graph, adding a finite number of isolated vertices to it always yields a sum graph and the sum number of G is the smallest number of isolated vertices so added. T. Hao [3] proved an existence theorem for sum graphs and M. N. Elingham [2] proved that the sum number of any tree is just one. Several results on sum graphs and sum number of various graphs are [1,6,7,10]. D. S. T. Ramesh et. al. [8,9] defined edge sum graph, the edge analogue of sum graph and edge sum number which we denote by σ_E . In this paper, we prove that $\sigma_E(J_{3,4}) = 3.$

1.1 Definition Let G (V,E) be a graph. Let S be a set of positive integers. An edge labeling of G by elements of S is a bijection f: E $\rightarrow$ S. It induces a vertex labeling F of positive integers defined by $F(v) = \sum\{f(e) : e \text{ is incident on } v\}$ for every $v \in V.$ We call f an edge function of G and F an edge sum function of G induced by f.

1.2 Definition G is said to be an edge sum graph if there exists an edge function
f: E $\rightarrow$ S such that f and its corresponding edge sum function F on V satisfying the following conditions:

1. F is into S. That is, F (v) $\in$S for every v $\in$V.
2. For any collection of edges $e_1, e_2, \ldots, e_n \in E$ such that $f(e_1) + f(e_2) + \ldots + f(e_n) \in S$, then $e_1, e_2, \ldots, e_n$ are incident on a vertex.

1.3 Definition Let G(V,E) be an edge sum graph. Let $e_1, e_2, ..., e_m$ where $m > 1$ be a collection of edges incident on a vertex w (say). Let $ww_i = e_i$ for $1 \leq i \leq m.$ If there exists an edge $e = uv$ such that $f(e_1) + f(e_2) + ... + f(e_m) \in f(e)$ and if $\deg(u) \geq 2,$ then u is adjacent to w. Similarly, for v.

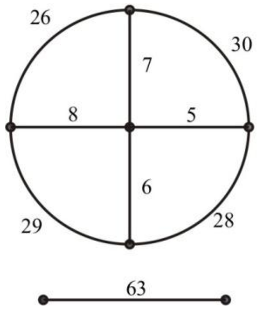
Hence, if $F(w) = f(uv)$ and $\{u, v\}$ is not a K_2 component of G then in G either $\{u, v, w\}$ form a triangle or one of $\{u, v\}$ is adjacent to w and the other is a pendent vertex.

1.4 Definition Let $\sigma_E(G) = r$. An edge function f: E → S and its corresponding edge sum function F that makes G ∪ rK_2 an edge sum graph are respectively called an **optimal edge function** and an **optimal edge sum function of G.** For a graph G with $\sigma_E(G) = r,$ there can be many optimal edge functions. Let E_1 be the edge set of G and E_2 be that of rK_2.Then, $\sigma_E(G)$ = Cardinality of the set $\{F(v) : v \in V; F(v) \notin f(E_1)\}.$ F is said to be an outer edge sum function if $F(V) \cap f(E_1) = \phi$ and an inner edge sum function if $F(V) \cap f(E_1) \neq \phi.$ The range of F has at least r elements. It has exactly r elements if and only if F is an outer edge sum function.

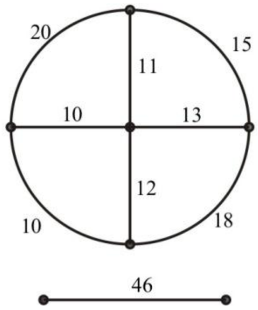
1.5 Theorem: Let f: E → S be an optimal edge function. If G has no pendent vertex and is triangle free, then F is an outer edge sum function.

Proof: Let E_1 be the edge set of G and E_2 that of rK_2. Let u ∈ V. Since F(u) ∈ S, there is an edge vw such that F(u) = f(vw). If vw∈ E_1,then $\langle u, v, w \rangle$ is K_3 or P_2 or P_1 with v or w as a pendent vertex which is a contradiction. Hence vw∈ E_2 so that F is an outer edge sum function. ■

4.2 Remark: It is easily seen that every optimal edge sum function F of a graph G is inner if G has a pendent vertex and is outer if G contains no pendent vertex and triangle free. If G has no pendent vertex but contains a triangle then F can be either inner (See Figure 1.1(a)) or outer (See Figure 1.1(b)). Here we show that $\sigma_E(W_4) = 1.$



Inner F(V) ∩ f(E_1) ≠ϕ
Figure 1.1(a)



Outer F(V) ∩ f(E_1) = ϕ.
Figure 1.1(b)

II. Edge Sum Number of Jahangir graphs

2.1 Definition: Jahangir graphs $J_{n,m}$ for $m \geq 3$ is a graph on $nm+1$ vertices that is a graph consisting of a cycle C_{nm} with one additional vertex which is adjacent to m vertices of C_{nm} at distance n to each on C_{nm}.

2.1 Theorem $\sigma_E(J_{3,4}) = 3.$

Proof: Let G = $J_{3,4}$ where $V(G) = \{v\} \cup \{v_{i,j} : 1 \leq i \leq 4; 1 \leq j \leq 3\}$ and E(G) =

$\{v_{i,j}v_{i,j+1} : 1 \leq i \leq 4; 1 \leq j \leq 2\} \cup \{v_{i,3}v_{i+1,1} : 1 \leq i \leq 3\} \cup \{v_{4,3}v_{1,1}\} \cup \{vv_{i,1} : 1 \leq i \leq 4\}$.

First let us prove that $\sigma_E(G) > 1$.

Suppose $\sigma_E(G)$ = 1.

Then there exists an optimal edge function f and its corresponding edge sum function F such that $G \cup K_2$ is an edge sum graph. Let w_1w_2 be the K_2 component of $G \cup K_2$. Since G is triangle free and has no pendent vertex; F is an outer edge sum function.

That is, $F(u) = f(w_1w_2) = a$ (say) for all $u \in V$

$F(v_{1,2}) \quad = f(v_{1,1}v_{1,2}) + f(v_{1,2}v_{1,3}) = a$

$F(v_{1,3}) \quad = f(v_{1,2}v_{1,3}) + f(v_{1,3}v_{2,1}) = a$

That is, $f(v_{1,1}v_{1,3}) = f(v_{1,3}v_{2,1})$

This is not possible as f is a bijection. Hence $\sigma_E(G) > 1.$

Suppose $\sigma_E(G) = 2$.

Then there exists an optimal edge function f and an optimal edge sum function F such that $G \cup 2K_2$ is an edge sum graph. Let w_1w_2 and w_3w_4 be the edges of the K_2 component of $G \cup 2K_2$. Let $f(w_1w_2) = z$ and $f(w_3w_4) = y$ where z = 2x.

Let $f(v_{1,1}v_{1,2}) = x - b_1 \Rightarrow f(v_{1,2}v_{1,3}) = x + b_1$

$f(v_{1,2}v_{1,3}) \quad = x + b_1 \Rightarrow f(v_{1,3}v_{2,1}) = y - x - b_1$

$f(v_{2,2}v_{2,3}) \quad = x - b_2 \Rightarrow f(v_{2,3}v_{3,1}) = x + b_2$

$f(v_{2,2}v_{2,3}) \quad = x - b_2 \Rightarrow f(v_{2,1}v_{2,2}) = y - x + b_2$

$f(v_{3,1}v_{3,2}) \quad = x - b_3 \Rightarrow f(v_{3,2}v_{3,3}) = x + b_3$

$f(v_{3,2}v_{3,3}) \quad = x + b_3 \Rightarrow f(v_{3,3}v_{4,1}) = y - x - b_3$

$f(v_{4,2}v_{4,3}) \quad = x - b_4 \Rightarrow f(v_{4,3}v_{1,1}) = x + b_4$

$f(v_{4,2}v_{4,3}) \quad = x - b_4 \Rightarrow f(v_{4,1}v_{4,2}) = y - x + b_4$

Let $f(vv_{1,1}) \quad = x_1$

$f(vv_{2,1}) \quad = x_2$

$f(vv_{3,1}) \quad = x_3$

$f(vv_{4,1}) \quad = x_4$

Suppose $F(v_{1,2}) = F(v_{2,1}) = F(v_{2,3}) = F(v_{3,2}) = F(v_{4,1}) = F(v_{4,3}) = 2x$ and

$F(v_{1,1}) = F(v_{1,3}) = F(v_{2,2}) = F(v_{3,1}) = F(v_{3,3}) = F(v_{4,2}) = y$

$F(v_{1,1}) = f(vv_{1,1}) + f(v_{4,3}v_{1,1}) + f(v_{1,1}v_{1,2}) = y$

$\Rightarrow x_1 + 2x - b_1 + b_4 = y$

$\Rightarrow x_1 = y - 2x + b_1 - b_4$

$F(v_{2,1}) = f(vv_{2,1}) + f(v_{1,3}v_{2,1}) + (v_{2,1}v_{2,2}) = 2x$

$\Rightarrow x_2 + 2y - 2x - b_1 + b_2 = 2x$

$\Rightarrow x_2 = 4x - 2y + b_1 - b_2$

$F(v_{3,1}) = f(vv_{3,1}) + f(v_{2,3}v_{3,1}) + f(v_{3,1}v_{3,2}) = y$

$\Rightarrow x_3 + 2x + b_2 - b_3 = y$

$\Rightarrow x_3 = y - 2x + b_3 - b_2$

$F(v_{4,1}) = f(vv_{4,1}) + f(v_{3,3}v_{4,1}) + f(v_{4,1}v_{4,2}) = 2x$

$\Rightarrow x_4 + 2y - 2x - b_3 + b_4 = 2x$

$\Rightarrow x_4 = 4x - 2y + b_3 - b_4$

$F(v) \quad = f(vv_{1,1}) + f(vv_{2,1}) + f(vv_{3,1}) + f(vv_{4,1})$

$= x_1 + x_2 + x_3 + x_4$

Case (i)

If F(v) = y where y < 2x

Let 2x – y = a

Therefore, a = 2x – y > 2x

Now $x_1 = b_1 - b_4 - a \Rightarrow b_1 > a + b_4$

$\Rightarrow b_1 - b_4 > a$

$x_3 = b_3 - b_2 - a \Rightarrow b_3 > a + b_2$

$\Rightarrow b_3 - b_2 > a$

$x_2 = 2a + b_1 - b_2$

$x_4 = 2a + b_3 - b_4$

$x_2 + x_4 = 4a + b_1 - b_2 + b_3 - b_4$

$= 4a + b_1 - b_4 + b_3 - b_2$

$> 4a + a + a = 6a$

$x_1 + x_2 + x_3 + x_4 > x_1 + x_3 + 6a$

$> x_1 + x_3 + 12x$ (since $a > 2x$)

$> 12x$

This is a contradiction.

Case (ii)

If F (v) = 2x where 2x < y

Let y – 2x = b

Therefore b > y > 2x

$x_1 = b + b_1 - b_4$

$x_3 = b + b_3 - b_2$

$x_2 = b_1 - b_2 - 2b$

$\Rightarrow b_1 - b_2 > 2b$

$x_4 = b_3 - b_4 - 2b$

$\Rightarrow b_3 - b_4 > 2b$

$x_1 + x_3 = 2b + b_1 - b_2 + b_3 - b_4$

$> 2b + 2b + 2b$

$= 6b > y > 2x$

This is a contradiction.

$y \geq x_1 + x_2 + x_3 + x_4 > x_1 + x_3 > 6b > y$

Hence $\sigma_E(G) > 2$. ■

The edge function given in Figure 2.1 shows that $\sigma_E(J_{3,4}) = 3$.

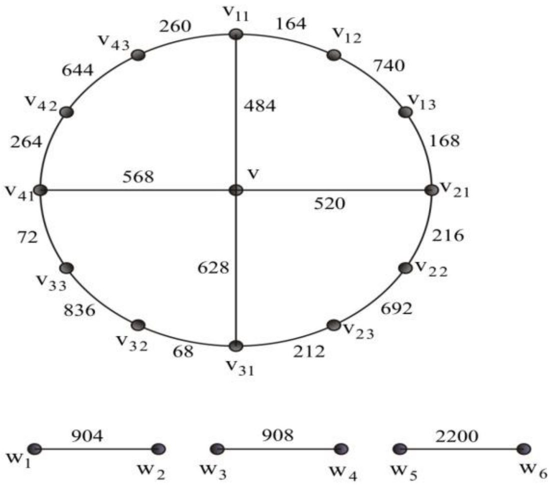


Figure 2.1

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