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Gaussian -Diophantine quadruples with property D (1)

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***Abstract**: A set of m Gaussian integers is called a complex Diophantine m-tuple with the property D (z) if the product of its any two distinct elements increased by z is a square of a Gaussian integer. In this paper, we present Gaussian-Diophantine quadruples with property D(1).Few examples of complex Diophantine quadruples with the property D (1) are presented.*

***Keywords:** Diophantine quadruples, Integral solutions, Gaussian integers, Pell equation*

I. Introduction

A set of positive integers $\{a_1, a_2, a_3........a_m\}$ is said to have the property D (n), $n \in Z - \{0\}$, if $a_i a_j + n$ is a perfect square for all $1 \le i \le j \le m$ and such a set is called a Diophantine m-tuples with property D(n). Many mathematicians considered the problem of the existence of a Diophantine quadruples with property D (n) for any arbitrary integer n[1] and also, for any linear polynomials in n. Further various authors considered the connections of the problem of Diaphanous, Davenport and Fibonacci numbers in [2-21].

In this paper we consider the analogous problem for Gaussian integers. Let z be a Gaussian integer and let $m \ge 2$ be an integer. A set $\{a_1, a_2, a_3......a_m\} \subset Z(i) \setminus \{0\}$ is said to have this property D (z) if the product of its any two distinct elements increased by z is a square of a Gaussian integer. If the set $\{a_1, a_2, a_3........a_m\}$ is a complex Diophantine quadruple then the same is true for the set $\{-a_1, -a_2, -a_3........-a_m\}$ .Particularly in [22], the authors have analyzed the problem of the existence of the complex Diophantine quadruples. In this paper, we present a Gaussian -Diophantine quadruple with property D (1).

MSC 2000 Mathematics subject classification: 11D99

Method of analysis:

To start with, it is seen that the pair (a, b) is Gaussian Diophantine 2-tuples with property D (1) where $a \ and \ b$ are Gaussian integers of the form

$$a = (2p^2 - 2q^2 - p) + i(4pq - q) \ and$$

$$b = (2p^2 - 2q^2 + 7p + 6) + i(4pq + 7q)$$

Let c_s be any non zero integer such that

$$a * c_s + 1 = \alpha_s^2 \quad (1)$$

$$b * c_s + 1 = \beta_s^2 \quad (2)$$

Eliminating c_s between (1) and (2) we get

$$b\alpha_s^2 - a\beta_s^2 = b - a \quad (3)$$

Substitution of the linear transformations

$$\alpha_s = X_s + aT_s \quad (4)$$

$$\beta_s = X_s + bT_s \quad (5)$$

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in (3) leads to the equation

$$X_s^2 = abT_s^2 + 1 \qquad (6)$$

where

$$ab = (4p^4 + 4q^4 - 24p^2p^2 - 36pq^2 + 12q^3 + 5p^2 - 5q^2 - 6p) + i(16p^3q - 16pq^3 + 36p^2q - 12q^3 + 10pq - 6q)$$

The general solution of (6) is given by

$$\left.\begin{aligned} X_s &= \frac{1}{2}[(X_0 + \sqrt{ab}T_0)^{s+1} + (X_0 - \sqrt{ab}T_0)^{s+1}] \\ T_s &= \frac{1}{2\sqrt{ab}}[(X_0 + \sqrt{ab}T_0)^{s+1} - (X_0 - \sqrt{ab}T_0)^{s+1}] \end{aligned}\right\} \qquad (7)$$

Taking $s = 0$ in (7) and using (4) we have

$$\alpha_0 = (4p^2 - 4q^2 + 2p - 1) + i(8pq - 2q) \qquad (8)$$

In view of (1) we have

$$c_0 = (8p^2 - 8q^2 + 12p + 4) + i(16pq + 12q) \qquad (9)$$

Observe that

$$\{(2p^2 - 2q^2 - p) + i(4pq - q),\ (2p^2 - 2q^2 + 7p + 6) + i(4pq + 7q),\ (8p^2 - 8q^2 + 12p + 4) + i(16pq + 12q)\}$$

is a Gaussian Diophantine triple with property D(1)

Again taking $s = 1$ in (7) and using (4) we obtain

$$\left.\begin{aligned} \alpha_1 = (16p^4 + 16q^4 - 88p^2q^2 + 32p^3 - 76pq^2 - 10p - 20pq^2 + 1) + \\ i(64p^3q - 64pq^3 + 96p^2q \text{-} 32q^3 - 10q) \end{aligned}\right\} \qquad (10)$$

In view of (1) we have

$$\begin{aligned} c_1 = \{&128(p^6 - 15p^4q^2 + 15p^2q^4 - q^6) + 576(p^5 - 10p^3q^2 + 5pq^4) + \\ &800(p^4 - 6p^2q^2 + q^4) + 240(p^3 - 3pq^2) - 184(p^2 - q^2) - 60p + 20\} \\ &+ i\{128(6p^5q - 20p^3q^3 + 6q^5p) + 576(5p^4q - 10p^2q^3 + q^5) + \\ &800(4p^3q - 4pq^3) + 240(3p^2q - q^3) - 368pq - 60q\} \end{aligned}$$

Hence

$$\begin{aligned} \{&(2p^2 - 2q^2 - p) + i(4pq - q),\ (2p^2 - 2q^2 + 7p + 6) + i(4pq + 7q), \\ &(8p^2 - 8q^2 + 12p + 4) + i(16pq + 12q), \\ &(128(p^6 - 15p^4q^2 + 15p^2q^4 - q^6) + 576(p^5 - 10p^3q^2 + 5pq^4) + \\ &800(p^4 - 6p^2q^2 + q^4) + 240(p^3 - 3pq^2) - 184(p^2 - q^2) - 60p + 20) \\ &+ i(128(6p^5q - 20p^3q^3 + 6q^5p) + 576(5p^4q - 10p^2q^3 + q^5) + \\ &800(4p^3q - 4pq^3) + 240(3p^2q - q^3) - 368pq - 60q)\} \end{aligned}$$

is a Gaussian Diophantine quadruple with property D(1)

The repetition of the above process leads to the generation of infinitely many Gaussian Diophantine quadruples with property D(1)

Table: Examples

(p,q)	Diophantine quadruple with property D(1)
$(0,1)$	$\{-2-i,\ 4+7i, -4+12i,\ 876+276i\}$
$(1,1)$	$\{-1+3i,\ 13+11i,\ 16+28i,\ -6024-3276i\}$
$(1,2)$	$\{-7+6i,\ 7+22i,\ -8+56i,\ 30864-36792i\}$

Note:

If $\{z_1,\ z_2,\ z_3,\ z_4\}$ is a quadruple with the property $D(z)$, then $\{\overline{z_1},\ \overline{z_2},\ \overline{z_3},\ \overline{z_4}\}$ is a quadruple with the property, $D(\overline{z})$, $z_1,\ z_2,\ z_3,\ z_4 \in Z(i)$

II. Conclusion

In this paper, we have presented a Gaussian Diophantine quadruple with property D(1).One may search for Gaussian Diophantine quadruples consisting of special numbers with suitable properties.

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