IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN:2319-765X. Volume 10, Issue 3 Ver. II (May-Jun. 2014), PP 01-11
www.iosrjournals.org

Introducing the Concept of Measure Manifold $(M, \Sigma_1, \mathcal{T}_1, \mu_1)$

Dr. S. C. P. Halakatti [1] and Shri H. G. Haloli[2]
[1] *Associate Professor Karnatak University Dharwad, Karnataka, India.*
[2] *Research Student, Karnatak University, Dharwad, Karnataka, India.*

Abstract: *The object of this paper is to study a measure μ on $(\mathbb{R}^n, \Sigma, \mathcal{T})$ and to introduce the concepts of measurable manifold $(M, \Sigma_1, \mathcal{T}_1)$ and measure manifold $(M, \Sigma_1, \mathcal{T}_1, \mu_1)$. Here we introduce the concept of measurable charts and measureable atlases and define measure μ_1 restricted to them respectively, we extend the study of Heine Borel property on $(M, \Sigma_1, \mathcal{T}_1, \mu_1)$, that is measure invariant.*

***Keywords**: σ-algebra, Heine Borel property, Measureable charts, Measureable Atlases, Measureable Manifold, Measure μ, Measure chart, Measure Atlas, Measure Manifold.*

I. Introduction

Let $\mathbb{R}^n$ be Euclidean space of dimension n. Generally, manifolds are defined as an n-dimensional topological manifold which is second countable, Hausdorff space that is locally Euclidean of dimension n [7],[15],[10]. On such topological manifold a differential structure was also developed to study differentiable manifolds [7],[9],[4],[15].
Let us consider some basic definitions:

Definition: 1.1 Chart

Let M be a non empty set. A pair (U, ϕ), were U is open subset of M and ϕ is a bijective map of U onto open subset of $\mathbb{R}^n$, is called an n-dimensional chart on M.

Definition: 1.2 Atlas

By an $\mathbb{R}^n$ – Atlas of a class C^k on M, we mean a collection $\mathbb{A}$ of an n-dimensional chart (U_i, ϕ_i) where $i \in N$ on M subject to following conditions,
(i) $\bigcup_{i=1}^{\infty} U_i = M$, i.e. the domain of the chart in $\mathbb{A}$ cover M.
(ii) For any Pair of charts (U_i,ϕ_i) and (U_j,ϕ_j) in $\mathbb{A}$, the sets $\phi_i(U_i \cap U_j)$ and $\phi_j(U_i \cap U_j)$ are open subsets of $\mathbb{R}^n$, and
(iii) maps, $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ and
$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$
are differentiable maps of class C^k $(k \geq 1)$. The maps $\phi_i \circ \phi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ for $i, j \in N$, are called transition maps. Since the transition map $\phi_i \circ \phi_j^{-1}$ is inverse of $\phi_j \circ \phi_i^{-1}$ and is also of class C^k. In an $\mathbb{R}^n$ – Atlas of a class C^k on M, every transition map is diffeomorphism of class C^k. An $\mathbb{R}^n$ – Atlas is said to be of class C^∞ if it is of class C^k for every positive integer k.

Definition: 1.3 Equivalence Relation:-

Let $\mathbb{A}^k(M)$ denotes the set of all $\mathbb{R}^n$ – Atlas of a class C^k on M. Two atlases $\mathbb{A}_1$ and $\mathbb{A}_2$ in $\mathbb{A}^k(M)$ are said to be **equivalent** if $\mathbb{A}_1 \cup \mathbb{A}_2$ is also in $\mathbb{A}^k(M)$.
In order that $\mathbb{A}_1 \cup \mathbb{A}_2$ be a member of ^{k} (M) for that every chart $(U_i,\phi_i) \in \mathbb{A}_1$ and every chart $(V_j,\Psi_j) \in \mathbb{A}_2$, the set $\phi_i(U_i \cap V_j)$ and $\Psi_j(U_i \cap V_j)$ be open in $\mathbb{R}^n$ and maps $\phi_i \circ \Psi_j^{-1}$ and $\Psi_j \circ \phi_i^{-1}$ be of class C^k and relation introduced is an equivalence relation on $\mathbb{A}^k(M)$ and hence partition $\mathbb{A}^k(M)$ into disjoint equivalence classes. Each of this equivalence classes is called **differentiable structure** of class C^k on M.

Definition: 1.4 Differentiable n-Manifold

A set M together with a differentiable structure of class C^k is called **differentiable n-manifold** of class C^k. $\square$

Thus a non- empty set M equipped with differentiable structure and topological structures exhibits many interesting geometrical and topological properties.
Now, in this paper, we introduce one more structure called σ- algebra which is algebraic structure on such differentiable n-manifold M, which is locally homeomorphic to an open subset of a measurable space $(\mathbb{R}^n, \Sigma, \mathcal{T})$, since M along with σ-algebra is a measurable space $(M, \Sigma_1, \mathcal{T}_1)$. We introduce a measure μ_1 on $(M, \Sigma_1, \mathcal{T}_1)$.Our aim is to study Heine Borel property on a measure space $(M, \Sigma_1, \mathcal{T}_1, \mu_1)$.

In this section, we introduce some basic definitions and theorems on algebraic structures on $\mathbb{R}^n$ and defined a measure μ on $\mathbb{R}^n$.[8],[6],[9],[10],[11],[12],[14].

Definition 1.5 σ-algebra on $\mathbb{R}^n$
A σ-algebra on a set $\mathbb{R}^n$ is a collection Σ of subsets of $\mathbb{R}^n$ such that
(i) ∅, $\mathbb{R}^n \in \Sigma$
(ii) If $V \in \Sigma$, then $V^c \in \Sigma$
(iii) If $V_i \in \Sigma$ for i ϵ N, then

$$\bigcup_{i=1}^{\infty} V_i \in \Sigma \quad , \quad \bigcap_{i=1}^{\infty} V_i \in \Sigma$$

From De-Morgan's laws, a collection Σ of subsets of $\mathbb{R}^n$ is called a σ-algebra, if it contains empty set ∅ and is closed under the operation of taking complements, countable unions and countable intersections

Definition 1.6 Measureable Space :-
The space ($\mathbb{R}^n$, Σ, $\mathcal{T}$) is called measureable topological space if the space $\mathbb{R}^n$ is a non-empty space equipped with σ-algebra, Σ. □
$\mathbb{R}^n$ is closed with respect to countable union, intersection and complements of its subsets along with finite unions and intersection property on measureable spaces ($\mathbb{R}^n$, Σ, $\mathcal{T}$).

Definition 1.7
A collection ε of an arbitrary subset of a non-empty topological place $\mathbb{R}^n$ is said to generate σ-algebra Σ(ε), if the intersection of all σ-algebra of subsets of $\mathbb{R}^n$ include ε, namely
Σ(ε) = ∩ { Σ : Σ is a σ-algebra of subsets of $\mathbb{R}^n$ and ε ⊆ Σ }, the smallest σ-algebra. □
Note that, there is at least one σ-algebra of subsets of $\mathbb{R}^n$, which includes ε and this is $\mathbb{P}(\mathbb{R}^n)$)

Definition 1.8 Borel σ-algebra
Let $\mathbb{R}^n$ be a topological space and $\mathcal{T}$-the collection of all open subsets of $\mathbb{R}^n$ be a topology on $\mathbb{R}^n$. Then σ-algebra Σ generated by $\mathcal{T}$ containing all open subsets of $\mathbb{R}^n$, is called the Borel σ-algebra of $\mathbb{R}^n$, denoted by $\mathfrak{B}_{\mathbb{R}_n}$,
i.e $\mathfrak{B}_{\mathbb{R}_n} = \Sigma\ (\mathfrak{J})$.
The elements of $\mathfrak{B}_{\mathbb{R}_n}$ are Borel sets in $\mathbb{R}^n$.

Proposition 1.9 [12]
Let Σ be a σ-algebra of subsets of $\mathbb{R}^n$ and V⊆ $\mathbb{R}^n$ be non-empty open subset of $\mathbb{R}^n$ and A ϵ Σ, if we denote
$\Sigma/_V$ = { A ∩ V : A ϵ Σ },then $\Sigma/_V$ is a restricted σ-algebra of subset of V.

Definition 1.10 Restriction of Σ on V
If Σ is σ-algebra of subsets of $\mathbb{R}^n$ and V is a non-empty open subsets of $\mathbb{R}^n$, V ⊆ $\mathbb{R}^n$ and A ϵ Σ. Then **σ-algebra restricted to V** is $\Sigma/_V$ = { A ∩ V : A ϵ Σ }.

Definition 1.11 Restriction of $\mathcal{T}$ on V:-
If $\mathcal{T}$-the collection of all open subsets of $\mathbb{R}^n$ be a topology on $\mathbb{R}^n$ and V is a non-empty open subsets of $\mathbb{R}^n$, then restriction of $\mathcal{T}$ on V is $\mathcal{T}/_V$={ V∩ G : V⊆ $\mathbb{R}^n$, Gϵ$\mathcal{T}$}
In general, if ε is any collection of subset of $\mathbb{R}^n$ and V ⊆ $\mathbb{R}^n$, we define the restriction of Σ(ε) on V is denoted by $\Sigma(\varepsilon)/_V$ and expressed as $\Sigma(\varepsilon)/_V$ = { A ∩V : A ϵ Σ(ε) }

Definition 1.12 Measureable Subspace
The space (V, $\Sigma/_V$, $\mathcal{T}/_V$) is called a **measureable subspace,** if V is non-empty open subset of ($\mathbb{R}^n$, Σ, $\mathcal{T}$) equipped with restricted σ-algebra $\Sigma/_V$.

Definition: 1.13 Push forward of a σ-algebra
If Σ is a σ-algebra of subsets of $\mathbb{R}^n$ and Σ' is a σ-algebra of subsets of $\mathbb{R}^m$, and f: $\mathbb{R}^n \to \mathbb{R}^m$ is a map then the collection {B ⊆ $\mathbb{R}^m$: f^{-1}(B) ϵ Σ} is called the **push forward of Σ** of $\mathbb{R}^n$ to Σ' to $\mathbb{R}^m$ by the function f.

Proposition : 1.14 The collection {B ⊆ $\mathbb{R}^m$: f^{-1}(B) ϵ Σ} is a σ-algebra of subsets of $\mathbb{R}^m$.

Proof: As f is map from measurable space ($\mathbb{R}^n$, Σ, $\mathcal{T}$) to measurable space ($\mathbb{R}^m$, Σ', $\mathcal{T}$'), by definition of measurable function, also inverse map $f(f^{-1}(B))=B \in \mathbb{R}^m$ for $f^{-1}(B) \in \Sigma$,the collection of such sets are sub space of $\mathbb{R}^m$, which is also σ-algebra. □

Proposition: 1.15 If ($\mathbb{R}^n$, Σ,$\mathcal{T}$) and ($\mathbb{R}^m$, Σ',$\mathcal{T}$') are two topological spaces and f: $\mathbb{R}^n \to \mathbb{R}^m$ is continuous then $f^{-1}(B)$ is a Borel set in $\mathbb{R}^n$ for every Borel set B in $\mathbb{R}^m$. □

Proof: Given that, ($\mathbb{R}^n$, Σ, $\mathcal{T}$) and ($\mathbb{R}^m$, Σ', $\mathcal{T}$') are two topological spaces. The f: $\mathbb{R}^n \to \mathbb{R}^m$ is continuous, B is Borel set in $\mathbb{R}^m$. As f is continuous, f^{-1} is also continuous. By open mapping theorem, then $f^{-1}(B)$ is a Borel set in $\mathbb{R}^n$ and is Borel set. □

Definition: 1.16 The pull back of a σ-algebra

The Σ' is a σ-algebra of subsets of $\mathbb{R}^m$ and f : $\mathbb{R}^n \to \mathbb{R}^m$ is a map, then the collection $\{f^{-1}(B) : B \in \Sigma'\}$,is called the pull back Σ by f on $\mathbb{R}^n$. □

We now consider the specific role of the measure called as Lebesgue measure not on any non-empty set $\mathbb{R}^n$ but on Real space $\mathbb{R}^n$

The main theme of Lebesgue measure on the general subsets of $\mathbb{R}^n$ is to construct the notion of abstract volume with abstract measure that reduces to the usual volume of elementary geometrical sets, such as cubes or rectangles of $\mathbb{R}^3$ and $\mathbb{R}^2$. If $\mathcal{L}(\mathbb{R}^n)$ is the collection of Lebesgue measureable sets and if $\mu : \mathcal{L}(\mathbb{R}^n) \to [0, \infty]$ is Lebesgue measure, then $\mathcal{L}(\mathbb{R}^n)$ contains all n-dimensional rectangles. The required condition on, is that, μ be countably additive.[8],[11],[14]

i.e. if $\{ A_i \in \mathcal{L}(\mathbb{R}^n) ; i \in \mathbb{N}\}$ is a countable collection of disjoint measureable sets, then their union should be measureable and

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

The countable additive requirement is an appropriate balanced condition between finite additive and uncountable additivity.

Looking at the abstract nature of $\mathbb{R}^n$, is not possible to define Lebesgue measure of all subsets of $\mathbb{R}^n$ in a geometrically reasonable way. Hausdorff (1914) showed that, for any dimension n ≥ 1, there is no countable additive measure defined on all subsets of $\mathbb{R}^n$ that is invariant under isometries and assigns measure one to the unit cube. Further, for n ≥ 3, there does not exists finitely additive measure. Banach and Tarski (1924) in their paradox showed that, there are finitely additive, isometrically invariant extensions of Lebesgue measure on $\mathbb{R}^n$ on all subsets of $\mathbb{R}^n$, but these extensions are not countablly additive.

It means, some subsets of abstract space $\mathbb{R}^n$ are too irregular to have Lebesgue measure that preserves countable additivity, in n ≥ 3 together with invariance of measure under isometries. This situation can be handled by inducing σ-algebra structure on $\mathbb{R}^n$ and $\mathcal{L}(\mathbb{R}^n)$ is a σ-algebra of Lebesgue measurable sets that includes all possible sets also it is possible to define as isometrically invariant, countablly additive outer measure on all subsets of $\mathbb{R}^n$. If $\mathbb{R}^n$ carries the topological structure along with σ-algebra such a space ($\mathbb{R}^n$, Σ, $\mathcal{T}$), is a measureable topological space, where all subsets of $\mathbb{R}^n$ are isometically invariant and have countablly addivitve outer measure.

Motivated by this approach, Lebesgue introduced some basic definitions on ($\mathbb{R}^n$, Σ, $\mathcal{T}$).

In section- 2, We introduced some basic preliminaries and re-defined some concepts which are need our work.

In section -3, the main theme of this paper, i.e. we defined new concepts like measurable charts, atlases and manifold. Also measure charts, atlases and manifold. In this section we studied the HBP on measure manifold and proved HBP on measure manifold which is measure invariant.

Lastly, the conclusion of this paper.

II. Preliminaries

We now define a measure μ that assigns a measure on each open set or Borel set that generates σ-algebra on ($\mathbb{R}^n$, Σ, $\mathcal{T}$). In order to introduce Measure Manifold (M, Σ_1, $\mathcal{T}_1$, μ_1). We define measure μ on measureable space ($\mathbb{R}^n$, Σ, $\mathcal{T}$) ([7],[8],[11],[12],[14]).

Definition 2.1 Measure μ on $\mathbb{R}^n$

A measure μ on a measureable topological space ($\mathbb{R}^n$, Σ, $\mathcal{T}$) is a function

$\mu: \Sigma \to [0,\infty]$, such that

(i) $\mu(\emptyset) = 0$

(ii) If $\{ V_n \in \Sigma, n \in \mathbb{N}\}$ is a countable disjoint collection of subsets in Σ, then

$$\mu\left(\bigcup_{i=1}^{\infty} V_i\right) = \sum_{i=1}^{\infty} \mu(V_i), \quad \forall\, V_i \in (\mathbb{R}^n, \Sigma, \mathcal{T})$$

Definition 2.2 Measure Space

A measure μ on a measureable space ($\mathbb{R}^n$, Σ, $\mathcal{T}$) is called a measure space and denoted by ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ)

Proposition 2.3

Let ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ) be a measure space and V ⊂ $\mathbb{R}^n$, A ϵ Σ(ε). If we define $^{\mu}/_{V}$: Σ →[0, ∞] by

$^{\mu}/_{V}$ (A) = μ (A ∩ V),   A ϵ Σ(ε)

Then $^{\mu}/_{V}$ is a measure on ($\mathbb{R}^n$, Σ,) with the following properties

i) $^{\mu}/_{V}$(A) = μ (A) for every A ϵ Σ, A ⊆ V

ii) $^{\mu}/_{V}$(A) = 0 for every A ϵ Σ, A ∩ V = ∅

Proof:-We have $^{\mu}/_{V}(\emptyset) = \mu\ (\emptyset\ \cap V) = \mu\ (\emptyset) = 0$.

If A_1, A_2, …ϵ Σ are pairwise disjoint, $^{\mu}/_{V}\ (\bigcup_{i=1}^{\infty} A_i) = \mu\ ((\bigcup_{i=1}^{\infty} A_i) \cap V) = \mu\ (\bigcup_{i=1}^{\infty}(A_i \cap V))$

$=\sum_{i=1}^{\infty} \mu(A_i \cap V) = \sum_{i=1}^{\infty} {}^{\mu}/_{V}(A_i)$.

Therefore, $^{\mu}/_{V}$ is a measure on ($\mathbb{R}^n$, Σ, $\mathcal{T}$) and its properties are trivial to prove. □

Definition:- 2.4 Restriction of μ on V.

Let ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ) be a measure space and V ϵ ($\mathbb{R}^n$, Σ, $\mathcal{T}$) be any non-empty open subset of $\mathbb{R}^n$, then the measure $^{\mu}/_{V}$ on ($\mathbb{R}^n$, Σ, $\mathcal{T}$) of proposition 2.3 is called **the restriction of μ on V**. □

Proposition 2.5

Let ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ) be a measure space and V ⊆ $\mathbb{R}^n$, then,

(i) $^{\Sigma}/_{V}$ = { A ∩ V : A ⊆ V ϵ $\mathbb{R}^n$ }

(ii) $^{\mu}/_{V}$: $^{\Sigma}/_{V}$ → [0, ∞] defined as, $^{\mu}/_{V}$(A) = μ (A), where A ϵ $^{\Sigma}/_{V}$,

(iii) $^{\mathcal{T}}/_{V}$={ V∩ G : V⊆ $\mathbb{R}^n$, Gϵ$\mathcal{T}$},

is called a measure on (V, $^{\Sigma}/_{V}$, $^{\mathcal{T}}/_{V}$).

The structure (V, $^{\Sigma}/_{V}$, $^{\mathcal{T}}/_{V}$, $^{\mu}/_{V}$) is called **measure subspace**.

Definition 2.6 Restriction of μ on (V, $^{\Sigma}/_{V}$, $^{\mathcal{T}}/_{V}$)

Let ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ) be a measure space and V ϵ Σ be any non-empty open subset of $\mathbb{R}^n$, then the measure $^{\mu}/_{V}$ on (V, $^{\Sigma}/_{V}$, $^{\mathcal{T}}/_{V}$).of proposition 2.5 is called **the restriction of μ on $^{\Sigma}/_{V}$.** □

Definition 2.7 The push forward of a measure

Let ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ) and ($\mathbb{R}^m$, Σ', $\mathcal{T}$', μ') are measure spaces and f: $\mathbb{R}^n$ → $\mathbb{R}^m$ be a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ and let Σ' defined by, Σ' = { B ⊆ $\mathbb{R}^m$: f^{-1}(B) ϵ Σ } be a σ-algebra on $\mathbb{R}^m$, then the push forward of Σ by f on $\mathbb{R}^m$ is defined as μ'(B) = μ(f^{-1}(B)), where B ϵ Σ'.

If μ' is a measure on ($\mathbb{R}^m$, Σ', $\mathcal{T}$', μ') it is called the **push forward of μ by f on $\mathbb{R}^m$**.

Definition: 2.8 The pull back of a measure

Let ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ) and ($\mathbb{R}^m$, Σ', $\mathcal{T}$', μ') are measure spaces and let f: $\mathbb{R}^n$ → $\mathbb{R}^m$ be a one to one onto map from $\mathbb{R}^n$ onto $\mathbb{R}^m$ and Σ – be a σ-algebra on $\mathbb{R}^n$.

i.e. Σ = { f^{-1}(B) : B ϵ Σ'}, then a pull back of Σ' by f on $\mathbb{R}^n$ is defined as μ(A) = μ'(f(A)), where A ϵ Σ

If μ is a measure on ($\mathbb{R}^n$, Σ, $\mathcal{T}$) it is called as **the pull back of μ by f on $\mathbb{R}^n$.**

Definition 2.9 Outer Lebesgue measure

The outer Lebesgue measure μ*(E) of a subset E ⊂ $\mathbb{R}^n$ or outer measure for short is

$\mu^*(E) = \inf \{ \ \sum_{j=1}^{\infty} \mu(\mathfrak{R}_j)\ ; E\ \subset\ \sum_{j=1}^{\infty} \mathfrak{R}_j\,, \mathfrak{R}_j\ \subset\ \mathfrak{R}(\mathbb{R}^n)\ \ \}$,

Where, the infimum is taken over all countable collection of rectangles $\mathfrak{R}$, whose union contains E. The map μ^*:P($\mathbb{R}^n$)→[0,∞], μ^* : E → μ^*(E) is called **outer Lebesgue measure.**

Theorem 2.10
Lebesgue outer measure μ^* has the following properties.
(i) $\mu^*(\emptyset) = 0$
(ii) if $A \subset B$ then $\mu^*(A) \leq \mu^*(B)$
(iii) if $\{ A_i \subset \mathbb{R}^n ; i \in \mathbb{N} \}$ is a countable collection of subsets of $\mathbb{R}^n$,then
$\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$ □

Let us define Carathoedory measurability,

Definition 2.11
Let μ^* be an outer measure on a set X. A subset $A \subset X$ is Caratheodory measureable with respect to μ^*, or measureable for short if $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$, for every subset $E \subset X$.

III. Construction of Measure Manifold

Let M be a topological Manifold which is second countable and Housdorff space. On Such topological Manifold a differential structure can be induced, transforming M into differentiable Manifold of dimension n. M carries Topological and differential structures smoothly .In this paper we induce the algebraic structure σ- algebra on a topological differentiable manifold that transforms M into a measurable Manifold denoted by (M, Σ_1, $\mathcal{T}_1$). The σ-algebraic structure on M admits a measure μ on a measurable Manifold transforming measurable Manifold (M, Σ_1, $\mathcal{T}_1$) into a measure Manifold denoted by (M, Σ_1, $\mathcal{T}_1$, μ_1). In this paper the first author introduces the concept of measurable charts and atlases hence measurable manifold on which a measure μ_1 has been introduced to study the measure of some topological characteristics on a measurable Manifold. The first author has introduced these conceptual framework in order to study any organic system, for example, the anatomy of a human brain, its structural and functional patterns in terms of structures of measurable Manifold and measure the behavioral patterns of human brain in term of measure Manifold.

In this paper we introduce the basics and necessary concepts and prepare a ground for evolving a mathematical model that represents any organic system in general and the structure of the brain in particular. Keeping such a larger picture in the mind the present paper is developed, where only some topological characteristics are studied on a measurable Manifold amongst many topological properties to be studied in future. In this paper we extend the Heine-Borel property on a measure manifold (M, Σ_1, $\mathcal{T}_1$, μ_1). The geometrical and algebraic structures on (M, Σ_1, $\mathcal{T}_1$, μ_1) will be studied in future work.

A countable collection of measure atlases that cover the M and satisfying the equivalence relation, induces a differentiable structure on M, converting any non-empty set M into a differentiable manifold which represents a measure space and denoted by (M, Σ_1, $\mathcal{T}_1$, μ_1).

3.1 Introducing the concepts of measurable charts and measure charts

For every $V \in (\mathbb{R}^n, \Sigma, \mathcal{T})$, there exists a homeomorphisms ϕ and ϕ^{-1} and defined ϕ: $\phi^{-1}(V) \rightarrow (\mathbb{R}^n, \Sigma, \mathcal{T})$ such that $\phi(\phi^{-1}(V)=U) = V \subset (\mathbb{R}^n, \Sigma, \mathcal{T})$,then the pair (U, ϕ) is called the **chart**.

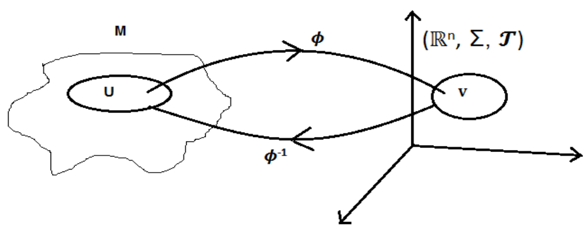

Fig.1

Definition 3.1.1 Measurable Manifold
A non-empty set M, which is modeled on measureable space ($\mathbb{R}^n$, Σ, $\mathcal{T}$) is called a **measurable manifold** denoted by (M, Σ_1, $\mathcal{T}_1$).

Definition 3.1.2 Measure Manifold
By the definition 2.1 ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ) is a measure space hence a non-empty set M modeled on ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ) is called **Measure Manifold.**

Definition 3.1.3 Measurable subspace.

If (U, Σ_1, $\mathcal{T}_1$) ⊆ (M, Σ_1, $\mathcal{T}_1$) and restriction of a σ-algebra , $\Sigma_1/_U$ on U, denoted by (U, $\Sigma_1/_U$, $\mathcal{T}_1/_U$) is called **measurable subspace.**

Definition 3.1.4 Measure subspace and Restriction of μ_1 on $\Sigma_1/_U$

Let (M, Σ_1, $\mathcal{T}_1$) be a measure space and U ϵ Σ_1 be non-empty open subset of (M, Σ_1, $\mathcal{T}_1$). The measure $\mu_1/_U$ on (U, $\Sigma_1/_U$, $\mathcal{T}_1/_U$) of proposition 2.3 is called **restriction of μ_1 on $\Sigma_1/_U$**

The pair (U, $\Sigma_1/_U$, $\mathcal{T}_1/_U$) is measureable subspace and the structure (U, $\Sigma_1/_U$, $\mathcal{T}_1/_U$,$\mu_1/_U$) is called a **measure subspace**.

Definition 3.1.5 Measurable function

Let (M, Σ_1, $\mathcal{T}_1$) and ($\mathbb{R}^n$, Σ, $\mathcal{T}$) are measurable spaces. A function ϕ: M→$\mathbb{R}^n$ is measurable if ϕ^{-1} (V) ϵ (M, Σ_1, $\mathcal{T}_1$), V ⊆ ($\mathbb{R}^n$, Σ, $\mathcal{T}$).

Note:- 3.1.6

(1) ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ) - Where Σ- collection of open subsets of V of ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ), and $\mathcal{T}$ - collections of open sets of G of $\mathbb{R}^n$,

(2) (M, Σ_1, $\mathcal{T}_1$, μ_1) - Where Σ_1- collection of open subsets of U = ϕ^{-1} (V) on M, and $\mathcal{T}_1$-collections of open sets of ϕ^{-1} (G) of M,

Definition 3.1.7 Measureable chart

Let (U, $\Sigma_1/_U$, $\mathcal{T}_1/_U$) ⊆ (M, Σ_1, $\mathcal{T}_1$, μ_1) be a non empty measurable subspace of (M, Σ_1, $\mathcal{T}_1$) if there exists a map, ϕ: (U, $\Sigma_1/_U$, $\mathcal{T}_1/_U$)→ ϕ (U, $\Sigma_1/_U$, $\mathcal{T}_1/_U$) ⊆($\mathbb{R}^n$, Σ, $\mathcal{T}$) , satisfying the following conditions,

(i) ϕ if homeomorphism

(ii) ϕ is measurable i.e ϕ^{-1} (V) ϵ (M, Σ_1, $\mathcal{T}_1$), for every Vϵ ($\mathbb{R}^n$, Σ, $\mathcal{T}$) on (U, $\Sigma_1/_U$, $\mathcal{T}_1/_U$) ⊆ (M, Σ_1, $\mathcal{T}_1$), then the structure ((U, $\Sigma_1/_U$, $\mathcal{T}_1/_U$), ϕ) is called a **measurable chart.**

Definition 3.1.8 Measurable Function

Let (M, Σ_1, $\mathcal{T}_1$, μ_1) and ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ) be measure spaces .We say that a mapping ϕ : (M, Σ_1, $\mathcal{T}_1$, μ_1) → ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ) is measurable if ϕ^{-1} (E) is measure subset of (M, Σ_1, $\mathcal{T}_1$, μ_1) for every measure subset E ⊂ ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ) .

Definition 3.1.9 Measure Preserving Map/ Invariant Measure

(i) Let (M, Σ_1, $\mathcal{T}_1$, μ_1) and ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ) be measure spaces and mapping is ϕ: (M, Σ_1, $\mathcal{T}_1$, μ_1) → ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ) measurable function .The mapping is measure preserving μ_1 (ϕ^{-1}E) = μ (E) for every measurable subset E ⊂ ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ) .When M= $\mathbb{R}^n$ and μ_1 = , we call ϕ is a transformation.

(ii) If a measurable transformation ϕ: M → M preservers a measure, then we say that ϕ is μ –invariant . If ϕ is invertible and if both ϕ and ϕ^{-1} are measurable and measure preserving , then we call ϕ and ϕ^{-1} are invertible measure preserving transformation [12].

Definition 3.1.10 Measure Chart

A measure $\mu_1/_U$ on measurable chart ((U, $\Sigma_1/_U$, $\mathcal{T}_1/_U$), ϕ) is defined in proposition 2.6 is called measure chart, denoted by ((U, $\Sigma_1/_U$, $\mathcal{T}_1/_U$ $\mu_1/_U$), ϕ) satisfying following condition,

(i) ϕ if homeomorphism

(ii) ϕ is measurable function i.e ϕ^{-1} (V)ϵ(U, Σ_1, $\mathcal{T}_1$), Vϵ ($\mathbb{R}^n$, Σ, $\mathcal{T}$) on (U, $\Sigma_1/_U$, $\mathcal{T}_1/_U$) ⊆ (M, Σ_1, $\mathcal{T}_1$)

(iii) ϕ is measure invariant.

Then, the structure ((U, $\Sigma_1/_U$, $\mathcal{T}_1/_U$, $\mu_1/_U$),ϕ) is called a measure chart □

3.2 Measurable Atlas and Measure Atlas

Now we introduce concept of the measurable atlas and measure atlas.

Let Σ_1 be a σ-algebra of charts of (M, Σ_1, $\mathcal{T}_1$, μ_1). Let Σ_1 restricted to $\mathbb{A}$ be a non-empty collection of measurable charts ((U, $\Sigma_1/_U$, $\mathcal{T}_1/_U$), ϕ). Now we consider $\boldsymbol{\sigma}$- algebra structure restricted to $\mathbb{A}$.

We say that $\mathbb{A}$~$\mathbb{B}$ if $\mathbb{A}$∪$\mathbb{B}$ ϵ (M, Σ_1, $\mathcal{T}_1$).

Definition 3.2.1 :- σ-algebra restricted on $\mathbb{A}$

$\Sigma 1/_{\mathbb{A}}$ = {$U_i \cap U_j$: for all $U_j \in \mathbb{A}$ or $\mathbb{B} \in (M, \Sigma_1, \mathcal{T}_1)$ if $\mathbb{A} \sim \mathbb{B}$ }, where $\mathbb{B}$ and $\mathbb{A}$ are measurable Atlases.

Definition 3.2.2 :- Measurable Atlas

By an $\mathbb{R}^n$ measurable atlas of class C^k on M we mean a countable collection ($\mathbb{A}$, $\Sigma_1/_{\mathbb{A}}$, $\mathcal{T}_1/_{\mathbb{A}}$) of n-dimensional measurable charts ((U_i, $\Sigma_1/_{U_i}$, $\mathcal{T}_1/_{U_i}$), ϕ_i) for all i ϵ N on (M, Σ_1, $\mathcal{T}_1$) subject to the following conditions.

($\mathbf{a_1}$) $\bigcup_{i=1}^{\infty}\left(\left(U_i, \Sigma_1/_{U_i}, \mathcal{T}_1/_{U_i}\right), \phi_i\right) = M$

i.e. the countable union of the measurable charts in ($\mathbb{A}$, $\Sigma_1/_{\mathbb{A}}$, $\mathcal{T}_1/_{\mathbb{A}}$) cover (M, Σ_1, $\mathcal{T}_1$)

($\mathbf{a_2}$) For any pair of measurable charts $\left(\left(U_i, \Sigma_1/_{U_i}, \mathcal{T}_1/_{U_i}\right), \phi_i\right)$ and $\left(\left(U_j, \Sigma_1/_{U_j}, \mathcal{T}_1/_{U_j}\right), \phi_j\right)$ in

($\mathbb{A}$,$\Sigma_1/_{\mathbb{A}}$, $\mathcal{T}_1/_{\mathbb{A}}$) , the transition maps ϕ_i o ϕ_j^{-1} and ϕ_j o ϕ_i^{-1} are

(1) differentiable maps of class C^k (K ≥ 1)

i.e. ϕ_i o ϕ_j^{-1} : ϕ_j ($U_i \cap U_j$) → ϕ_i ($U_i \cap U_j$) ⊆ ($\mathbb{R}^n$, Σ, $\mathcal{T}$)

ϕ_j o ϕ_i^{-1} : ϕ_i ($U_i \cap U_j$) → ϕ_j ($U_i \cap U_j$) ⊆ ($\mathbb{R}^n$, Σ, $\mathcal{T}$)

are differentiable maps of class C^k (K ≥ 1)

(2) Measurable

i.e. These two transition maps ϕ_i o ϕ_j^{-1} and ϕ_i o ϕ_j^{-1} are measurable functions if

(a) For any open subset K ⊆ ϕ_j ($U_i \cap U_j$) is measurable in ($\mathbb{R}^n$, Σ, $\mathcal{T}$) then (ϕ_i o ϕ_j^{-1})$^{-1}$ (K) ∈ ϕ_j ($U_i \cap U_j$) is also measurable,

(b) ϕ_j o ϕ_i^{-1} is measurable if S ⊆ ϕ_j ($U_i \cap U_j$) is measurable in ($\mathbb{R}^n$, Σ, $\mathcal{T}$) , then (ϕ_j o ϕ_i^{-1})$^{-1}$ (S) ∈ ϕ_i ($U_i \cap U_j$) is measurable.

Proposition 3.2.3

Let (M, Σ_1, $\mathcal{T}_1$, μ_1) be a measure space and $\mathbb{A} \in \Sigma_1/_{\mathbb{A}}$ be non-empty measurable Atlas, we consider $\Sigma_1/_{\mathbb{A}}$ = {$\mathbb{B} \in$(M, Σ_1, $\mathcal{T}_1$, μ_1):$\mathbb{B} \sim \mathbb{A}$} and define,

$\mu_1/_{\mathbb{A}} : \Sigma_1/_{\mathbb{A}} \rightarrow [0,\infty]$ by

$\mu_1/_{\mathbb{A}}(\mathbb{B}) = \mu_1(\mathbb{B})$, where $\mathbb{B} \in$ ($\mathbb{A}$, $\Sigma_1/_{\mathbb{A}}$, $\mathcal{T}_1/_{\mathbb{A}}$).

Then $\mu_1/_{\mathbb{A}}$ is a measure on ($\mathbb{A}$, $\Sigma_1/_{\mathbb{A}}$, $\mathcal{T}_1/_{\mathbb{A}}$).

Proof :-

We have $\mu_1/_{\mathbb{A}}(\emptyset) = \mu_1 (\emptyset \cap \mathbb{A}) = \mu_1 (\emptyset) = 0$

If $\mathbb{B}_1, \mathbb{B}_2, \ldots, \in \Sigma_1$ are pair wise disjoint Atlases,

$\mu_1/_{\mathbb{A}} (\bigcup_{i=1}^{\infty} \mathbb{B}_i) = \mu_1 ((\bigcup_{i=1}^{\infty} \mathbb{B}_i) \cap \mathbb{A}) = \mu_1(\bigcup_{i=1}^{\infty}(\mathbb{B}_i \cap \mathbb{A})) = \sum_{i=1}^{\infty} \mu_1((\mathbb{B}_i \cap \mathbb{A})$

$= \sum_{i=1}^{\infty} \mu_1/_{\mathbb{A}} (\mathbb{B}_i)$

Therefore $\mu_1/_{\mathbb{A}}$ is a measure on ($\mathbb{A}$, $\Sigma_1/_{\mathbb{A}}$, $\mathcal{T}_1/_{\mathbb{A}}$) ⊆ (M, Σ_1, $\mathcal{T}_1$, μ_1) □

Definition 3.2.4 :- Restriction of Measure μ_1 on ($\mathbb{A}$, $\Sigma_1/_{\mathbb{A}}$, $\mathcal{T}_1/_{\mathbb{A}}$).

Let (M, Σ_1, $\mathcal{T}_1$, μ_1) be a measure space and let ($\mathbb{A}$, $\Sigma_1/_{\mathbb{A}}$, $\mathcal{T}_1/_{\mathbb{A}}$) ϵ (M, Σ_1, $\mathcal{T}_1$, $_1$) be a non-empty measureable Atlas. The measure $\mu_1/_{\mathbb{A}}$ on ($\mathbb{A}$, $\Sigma_1/_{\mathbb{A}}$, $\mathcal{T}_1/_{\mathbb{A}}$) by proposition 3.2.3 is called the **restriction of measure μ_1 on ($\mathbb{A}$, $\Sigma_1/_{\mathbb{A}}$, $\mathcal{T}_1/_{\mathbb{A}}$).**

Definition 3.2.5:- Measure atlas

The structure ($\mathbb{A}$,$\Sigma_1/_{\mathbb{A}}$, $\mathcal{T}_1/_{U_n}$,$\mu_1/_{\mathbb{A}}$) is called measure Atlas if ($\mathbb{A}$,$\Sigma_1/_{\mathbb{A}}$, $\mathcal{T}_1/_{\mathbb{A}}$) is a measureable Atlas equipped with restricted measure $\mu_1/_{\mathbb{A}}$ by (proposition 3.2.3). □

Condition to be satisfied for measure atlas:-

Definition 3.2.6:- Measure Atlas :-

By an $\mathbb{R}^n$ measure atlas of class C^k on M, we mean a countable collection ($\mathbb{A}$, ${}^{\Sigma_1}/_{\mathbb{A}}$, ${}^{\mathcal{T}_1}/_{\mathbb{A}}$, ${}^{\mu_1}/_{\mathbb{A}}$) of n-dimensional measure charts ((U_i, ${}^{\Sigma_1}/_{U_i}$, ${}^{\mathcal{T}_1}/_{U_i}$, ${}^{\mu_1}/_{i}$), ϕ_i) for all i ϵ N on (M, Σ_1, $\boldsymbol{\mathcal{T}}_1$, μ_1) satisfying the following conditions:-

($\mathbf{a_1}$) $\bigcup_{i=1}^{\infty}\left(\left(U_i , {}^{\Sigma_1}/_{U_i}, {}^{\mathcal{T}_1}/_{U_i}\ {}^{\mu_1}/_{U_i}\right), \phi_i\right) = M$

i.e. the countable union of the measure charts in ($\mathbb{A}$, ${}^{\Sigma_1}/_{\mathbb{A}}$, ${}^{\mathcal{T}_1}/_{\mathbb{A}}$, ${}^{\mu_1}/_{\mathbb{A}}$) cover (M, Σ_1, $\boldsymbol{\mathcal{T}}_1$, μ_1).

($\mathbf{a_2}$) For any pair of measure charts $\left(\left(U_i , {}^{\Sigma_1}/_{U_i}, {}^{\mathcal{T}_1}/_{U_i} , {}^{\mu_1}/_{U_i}\right), \phi_i\right)$and $\left(\left(U_j , {}^{\Sigma_1}/_{U_j}, {}^{\mathcal{T}_1}/_{U_j} , {}^{\mu_1}/_{U_j}\right), \phi_j\right)$

in ($\mathbb{A}$, ${}^{\Sigma_1}/_{\mathbb{A}}$, ${}^{\mu_1}/_{\mathbb{A}}$), the transition maps ϕ_i o ϕ_j^{-1} and ϕ_j o ϕ_i^{-1} are

(1) differentiable maps of class C^k (K ≥ 1)

i.e. ϕ_i o ϕ_j^{-1} : ϕ_j ($U_i \cap U_j$) → ϕ_i ($U_i \cap U_j$) ⊆ ($\mathbb{R}^n$, Σ,$\boldsymbol{\mathcal{T}}$, μ) ,
ϕ_j o ϕ_i^{-1} : ϕ_i ($U_i \cap U_j$) → ϕ_j ($U_i \cap U_j$) ⊆ ($\mathbb{R}^n$, Σ,$\boldsymbol{\mathcal{T}}$, μ)

are differentiable maps of class C^k (K ≥ 1)

(2) Measurable .

i.e. These two transition maps ϕ_i o ϕ_j^{-1} and ϕ_j o ϕ_i^{-1} are measurable functions if

(a) For any open subset K ⊆ ϕ_i ($U_i \cap U_j$) is measurable in ($\mathbb{R}^n$, Σ,$\boldsymbol{\mathcal{T}}$, μ) then (ϕ_i o $\phi_j^{-1})^{-1}$ (K) ∈ ϕ_j ($U_i \cap U_j$) is also measurable,

(b) ϕ_j o ϕ_i^{-1} is measurable if S ⊆ ϕ_j ($U_i \cap U_j$) is measurable in ($\mathbb{R}^n$, Σ,$\boldsymbol{\mathcal{T}}$, μ) , then (ϕ_j o $\phi_i^{-1})^{-1}$ (S) ∈ ϕ_i ($U_i \cap U_j$) is measurable ,

($\mathbf{a_3}$) For any two measure atlases ($\mathbb{A}_1$, ${}^{\Sigma_1}/_{\mathbb{A}_1}$, ${}^{\mathcal{T}_1}/_{\mathbb{A}_1}$, ${}^{\mu_1}/_{\mathbb{A}_1}$) and ($\mathbb{A}_2$, ${}^{\Sigma_1}/_{\mathbb{A}_2}$, ${}^{\mathcal{T}_1}/_{\mathbb{A}_2}$, ${}^{\mu_1}/_{\mathbb{A}_2}$), we say that a mapping ,T : $\mathbb{A}_1 \rightarrow \mathbb{A}_2$ is measurable if T^{-1} (E) is measurable for every measurable subset E ⊂($\mathbb{A}_2$, ${}^{\Sigma_1}/_{\mathbb{A}_2}$, ${}^{\mathcal{T}_1}/_{\mathbb{A}_2}$, ${}^{\mu_1}/_{\mathbb{A}_2}$) and the mapping is measure preserving if $\mu_1/\mathbb{A}_1$ (T^{-1}(E)) = $\mu_1/\mathbb{A}_2$ (E),where $\mathbb{A}_1 \sim \mathbb{A}_2$ and $\mu_1/\mathbb{A}_1 = \mu_1/\mathbb{A}_2$.

Then we call T a transformation.

($\mathbf{a_4}$) If a measurable transformation T: $\mathbb{A} \rightarrow \mathbb{A}$ preserves a measure μ_1, then we say that μ_1 is T-invariant (or invariant under T). If T is invertible and if both T and T^{-1} are measurable and measure preserving then we call T an invertible measure preserving transformation. □

An $\mathbb{R}^n$, measure atlas is said to be of class C^∞ if it is of class C^k for every integer k.

Let $\mathbb{A}_m^k$(M) denotes the set of all $\mathbb{R}^n$ measure atlases of class C^k on (M, Σ_1, $\boldsymbol{\mathcal{T}}_1$, μ_1).

Now we introduce a differential structure by defining an equivalence relation in $\mathbb{A}_m^k$(M).

Definition:3.2.7 Equivalence Relation in $\mathbb{A}_m^k$(M).

Two measure atlases $\mathbb{A}_1$ and $\mathbb{A}_2$ in $\mathbb{A}_m^k$(M) are said to be **equivalent** if ($\mathbb{A}_1 \cup \mathbb{A}_2$) in $\mathbb{A}_m^k$(M).In order that $\mathbb{A}_1 \cup \mathbb{A}_2$ be a member of $\mathbb{A}_m^k$(M) we require that for every measure chart ((U_i, ${}^{\Sigma_1}/_{U_i}$, ${}^{\mathcal{T}_1}/_{V_i}$, ${}^{\mu_1}/_{U_i}$), φ_i) ∈ $\mathbb{A}_1$ and every measure chart (V_j,${}^{\Sigma_1}/_{V_j}$, ${}^{\mathcal{T}_1}/_{V_j}$, ${}^{\mu_1}/_{V_j}$), Ψ_j) ∈ $\mathbb{A}_2$ the set of $\varphi_i(U_i \cap V_j)$ and Ψ_j ($U_i \cap V_j$) be open

measurable in ($\mathbb{R}^n$, Σ, $\boldsymbol{\mathcal{T}}$, μ) and maps φ_i o $\Psi_j^{-1}$and ϕ_j o ϕ_i^{-1} be of class C^k and are measurable . The relation introduced is an equivalence relation in $\mathbb{A}_m^k$(M) and hence partitions $\mathbb{A}_m^k$(M) into disjoint equivalence classes . Each of these equivalence classes is called a **differentiable structure** of class C^k on (M, Σ_1, $\boldsymbol{\mathcal{T}}_1$, μ_1). A measure space (M, Σ_1, $\boldsymbol{\mathcal{T}}_1$, μ_1) together with a differentiable structure of class c^k is called a **differentiable measure** n-manifold of class C^k or simply a C^k –measure n-manifold .

A non empty set M equipped with differentiable structure, topological structure and algebraic structure σ-algebra is called **Measurable Manifold**. A measure μ_1 defined on (M, Σ_1, $\boldsymbol{\mathcal{T}}_1$) and the quadruple (M, Σ_1, $\boldsymbol{\mathcal{T}}_1$, μ_1) is called **Measure Manifold.**

Now we study a topological property on (M, Σ_1, $\boldsymbol{\mathcal{T}}_1$, μ_1).

3.3 Topological property on (M, Σ_1, $\mathcal{T}_1$, μ_1)

Heine-Borel property is well-defined property on Euclidean space $\mathbb{R}^n$. Now, we extend the study of this property on measure space ($\mathbb{R}^n$, Σ,$\mathcal{T}$, μ) which is Topological space also , Further this property is extended on a measure manifold (M, Σ_1, $\mathcal{T}_1$, μ_1).

The present aim of this study is to quantity (measure) the charts/measure Atlas, the union of which gives a measurable differential structure on a measure-manifold (M, Σ_1, $\mathcal{T}_1$, μ_1) and to study Heine-Borel property re-defined in-terms of measure charts and measure Atlases and examine the measure invariant properties an (M, Σ_1, $\mathcal{T}_1$, μ_1).

3.3.1 Heine-Borel property (HBP) on $\mathbb{R}^n$:-

For a subset A of the Euclidean space $\mathbb{R}^n$. A has the Heine-Borel property if every open covering of A admits a finite sub covering.

Now we extend the Heine-Borel property on measure space ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ), where elements of σ-algebra are generated by members of $\mathcal{T}$-open sets of $\mathbb{R}^n$. The elements of σ-algebra are addressed as Borel sets.

For a subset A⊆ $\mathbb{R}^n$, let (A,$\Sigma/_A$,$\mathcal{T}/_A$,$\mu/_A$) be a sub measure space of a measure space ($\mathbb{R}^n$, Σ,$\mathcal{T}$, μ). If $\mathbb{R}^n$ admits Heine-Borel property then, to prove that ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ) also admits Heine-Borel property, it is suffices to prove that every countable open measure cover has finite measure sub cover.

Definition:-3.3.2 Borel Cover

By a Borel cover viz { $\bigcup_{i=1}^{\infty} V_i$: V_i 's are Borel sets}, we mean accountable union of all Borel sets belonging to ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ).

Theorem:-3.3.3

If Heine-Borel property (HBP) holds on Euclidean space $\mathbb{R}^n$ then a measure space ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ) also admits HBP. i.e. every Borel cover for sub-measure space of ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ) has a finite Borel sub-cover.

Proof:-

Suppose $\mathbb{R}^n$ admits Heine-Borel property then every open covering of a subset A ⊂ $\mathbb{R}^n$ admits a finite sub cover.

To show that, a measure space ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ) admits Heine-Borel property, it is suffices to prove that every measure open covering of a measure subspace (A,$\Sigma/_A$,$\mathcal{T}/_A$,$\mu/_A$) of ($\mathbb{R}^n$, Σ,$\mathcal{T}$, μ) has a finite measure sub covering. Let (A,$\Sigma/_A$,$\mathcal{T}/_A$,$\mu/_A$) ⊆ ($\mathbb{R}^n$, Σ,$\mathcal{T}$, μ) is a sub measure space of ($\mathbb{R}^n$, Σ,$\mathcal{T}$, μ) and let { $\bigcup_{i=1}^{\infty}(V_i, \Sigma/_{V_i}, \mathcal{T}/_{V_i}, \mu/_{V_i})$ } be a countable measure open covering/Borel covering for (A,$\Sigma/_A$,$\mathcal{T}/_A$,$\mu/_A$)

i.e. (A,$\Sigma/_A$,$\mathcal{T}/_A$,$\mu/_A$) ⊆ $\bigcup_{i=1}^{\infty}(V_i, \Sigma/_{V_i}, \mathcal{T}/_{V_i}, \mu/_{V_i})$ …………………… (1)

Satisfying the following condition on measure,

μ (A,$\Sigma/_A$,$\mathcal{T}/_A$,$\mu/_A$) ≤ μ($\bigcup_{i=1}^{\infty}(V_i, \Sigma/_{V_i}, \mathcal{T}/_{V_i}, \mu/_{V_i})$)

= $\sum_{i=1}^{\infty} \mu(V_i, \Sigma/_{V_i}, \mathcal{T}/_{V_i}, \mu/_{V_i})$) …………………… (2)

Since A ⊂ $\mathbb{R}^n$, by Heine-Borel property on $\mathbb{R}^n$ it implies that every open covering has finite sub cover.viz { $\bigcup_{j=1}^{n} V_{i_j}$}, such that A⊂ $\bigcup_{j=1}^{n} V_{i_j}$……………………………… (3)

Since {V_i} are open sub sets in $\mathbb{R}^n$, correspondingly a {$(V_i, \Sigma/_{V_i}, \mathcal{T}/_{V_i}, \mu/_{V_i})$} is a open measure covering/Borel covering for A ⊂ ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ). HBP implies, every open cover has finite sub-cover, correspondingly, every measure subspace (A,$\Sigma/_A$,$\mathcal{T}/_A$,$\mu/_A$)⊆$\bigcup_{i=1}^{\infty}$ $(V_i, \Sigma/_{V_i}, \mathcal{T}/_{V_i}, \mu/_{V_i})$…………………..(4) which is open measure covering/Borel covering, satisfying the following condition on measure,

μ (A,$\Sigma/_A$,$\mathcal{T}/_A$,$\mu/_A$) ≤ μ($\bigcup_{i=1}^{\infty}$ $(V_i, \Sigma/_{V_i}, \mathcal{T}/_{V_i}, \mu/_{V_i})$)

=$\sum_{i=1}^{\infty} \mu(V_i, \Sigma/_{V_i}, \mathcal{T}/_{V_i}, \mu/_{V_i})$) ……………………(5)

has finite sub cover, such that,

(A,$\Sigma/_A$,$\mathcal{T}/_A$,$\mu/_A$) ⊂ $\bigcup_{j=1}^{n}(V_{i_j}, \Sigma/_{V_{i_j}}, \mathcal{T}/_{V_{i_j}}, \mu/_{V_{i_j}})$ ………………..(6)

satisfying the following condition on measure,

μ (A,$\Sigma/_A$,$\mathcal{T}/_A$,$\mu/_A$) ≤ μ($\bigcup_{j=1}^{n}(V_{i_j}, \Sigma/_{V_{i_j}}, \mathcal{T}/_{V_{i_j}}, \mu/_{V_{i_j}})$)

=$\sum_{j=1}^{n} \mu$ $(V_{i_j}, \Sigma/_{V_{i_j}}, \mathcal{T}/_{V_{i_j}}, \mu/_{V_{i_j}})$,(for finite j=1,2,…n)…………(7)

This implies that, every countable measure open cover/Borel cover has a measure sub cover/Borel sub cover. Hence HBP is true on ($\mathbb{R}^n$, Σ, $\mathcal{T}$, μ). □

Remarks:**3.3.4** (i) The significances of the extension of HBP on ($\mathbb{R}^n$, Σ, $\boldsymbol{\mathcal{T}}$, μ) is that the open subsets/Borel sets which forms a Borel cover for (A,$^{\Sigma}/_{A}$,$^{\mathcal{T}}/_{A}$,$^{\mu}/_{A}$) ⊂ ($\mathbb{R}^n$, Σ, $\boldsymbol{\mathcal{T}}$, μ) are measurable and has a measure μ.
(ii) The open cover constructed by Borel sets $\{\bigcup_{i=1}^{\infty} V_i\}$ is also measurable and have a measure μ since

$$\mu\,(\bigcup_{i=1}^{\infty} V_i) \;\le\; \sum_{i=1}^{\infty}\mu(V_i),$$

(iii) The sub cover $\{\bigcup_{j=1}^{n} V_{i_j}\}$ of $\{(\bigcup_{i=1}^{\infty} V_i)\}$ is also measurable and has a measure μ.

Theorem:3.3.5 If Heine-Borel property (HBP) holds on ($\mathbb{R}^n$, Σ,$\boldsymbol{\mathcal{T}}$, μ) then a measure space (M, Σ_1, $\boldsymbol{\mathcal{T}}_1$, μ_1) also admits HBP.
Proof :-
Suppose ($\mathbb{R}^n$, Σ, $\boldsymbol{\mathcal{T}}$, μ) admits Heine-Borel property.
To extend Heine-Borel property on (M, Σ_1, $\boldsymbol{\mathcal{T}}_1$, μ_1) it is suffices to show that for every countable union of measure chart for measure atlas $\mathbb{A}$ ⊂ M there exist a finite sub collection of measure charts. Let V_i's are measure subsets of ($\mathbb{R}^n$, Σ, $\boldsymbol{\mathcal{T}}$, μ) and let $\phi^{-1}(V_i) = U_i$ are measure subsets of (M, Σ_1, $\boldsymbol{\mathcal{T}}_1$, μ_1). Let $(\left(U_i, {}^{\Sigma_1}/_{U_i}, {}^{\mathcal{T}_1}/_{U_i}, {}^{\mu_1}/_{U_i}\right), \phi_i)$ are a measure charts.

Let, $(\mathbb{A}, {}^{\Sigma_1}/_{\mathbb{A}}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\mu_1}/_{\mathbb{A}}) \subseteq \bigcup_{i=1}^{\infty}(\left(\phi_i^{-1}(V_i), {}^{\Sigma_1}/_{\phi_i^{-1}(V_i)}, {}^{\mathcal{T}_1}/_{\phi_i^{-1}(V_i)}, {}^{\mu_1}/_{\phi_i^{-1}(V_i)}\right), \phi_i)$, for every

$V_i \in (\mathbb{R}^n, \Sigma, \boldsymbol{\mathcal{T}}, \mu)$, there exists, $\phi^{-1}(V_i) = U_i \in (M, \Sigma_1, \boldsymbol{\mathcal{T}}_1, \mu_1)$, and equation (1) implies
i.e. $(\mathbb{A}, {}^{\Sigma_1}/_{\mathbb{A}}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\mu_1}/_{\mathbb{A}}) \subseteq \bigcup_{i=1}^{\infty}(\left(U_i, {}^{\Sigma_1}/_{U_i}, {}^{\mathcal{T}_1}/_{U_i}, {}^{\mu_1}/_{U_i}\right), \phi_i)$ …………………. (8)
satisfying the following condition on measure, and equation (2) implies

$$\mu_1\,(\mathbb{A}, {}^{\Sigma_1}/_{\mathbb{A}}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\mu_1}/_{\mathbb{A}}) \le \mu_1\,(\bigcup_{i=1}^{\infty}(\left(U_i, {}^{\Sigma_1}/_{U_i}, {}^{\mathcal{T}_1}/_{U_i}, {}^{\mu_1}/_{U_i}\right), \phi_i))$$

$$\le \sum_{i=1}^{\infty}\mu_1(\left(U_i, {}^{\Sigma_1}/_{U_i}, {}^{\mathcal{T}_1}/_{U_i}, {}^{\mu_1}/_{U_i}\right), \phi_i) \ldots\ldots\ldots\ldots \quad (9)$$

has finite sub cover, such that, and equation (3) implies

$$(\mathbb{A}, {}^{\Sigma_1}/_{\mathbb{A}}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\mu_1}/_{\mathbb{A}}) \subset \bigcup_{j=1}^{n}(\left(\phi_i^{-1}(V_i)_j, {}^{\Sigma_1}/_{\phi_i^{-1}(V_i)_j}, {}^{\mathcal{T}_1}/_{\phi_i^{-1}(V_i)_j}, {}^{\mu_1}/_{\phi_i^{-1}(V_i)_j}\right), \Phi_i)$$

i.e. $(\mathbb{A}, {}^{\Sigma_1}/_{\mathbb{A}}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\mu_1}/_{\mathbb{A}}) \subset \bigcup_{j=1}^{n}(\left(U_{i_j}, {}^{\Sigma_1}/_{U_{i_j}}, {}^{\mathcal{T}_1}/_{U_{i_j}}, {}^{\mu_1}/_{U_{i_j}}\right), \Phi_i)$ …………….. (10)

satisfying the following condition on measure, and equation (4) implies

$$\mu_1\,(\mathbb{A}, {}^{\Sigma_1}/_{\mathbb{A}}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\mu_1}/_{\mathbb{A}}) \le \mu_1\,(\bigcup_{j=1}^{n}(U_{i_j}, {}^{\Sigma_1}/_{U_{i_j}}, {}^{\mathcal{T}_1}/_{U_{i_j}}, {}^{\mu_1}/_{U_{i_j}}), \phi_i)$$

$$= \sum_{j=1}^{n}\mu_1\,((U_{i_j}, {}^{\Sigma_1}/_{U_{i_j}}, {}^{\mathcal{T}_1}/_{U_{i_j}}, {}^{\mu}/_{U_{i_j}}), \phi_i), \text{(for finite j=1,2,…n)} \ldots\ldots\ldots (11)$$

This implies that, for every open countable measure chart for measure atlas $\mathbb{A}$ ⊂ M there exist a finite sub collection of measure charts. Hence HBP holds for (M, Σ_1, $\boldsymbol{\mathcal{T}}_1$, μ_1). □

Remarks:**3.3.6** (i) The significances of HBP on (M, Σ_1, $\boldsymbol{\mathcal{T}}_1$, μ_1) is that, every countable open measure cover for an atlas $(\mathbb{A}, {}^{\Sigma_1}/_{\mathbb{A}}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\mu_1}/_{\mathbb{A}}) \subseteq$ (M, Σ_1, $\boldsymbol{\mathcal{T}}_1$, μ_1) has a finite subcover which is also countable, this implies $(\mathbb{A}, {}^{\Sigma_1}/_{\mathbb{A}}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\mu_1}/_{\mathbb{A}}) \subseteq$ (M, Σ_1, $\boldsymbol{\mathcal{T}}_1$, μ_1) satisfies Lindelof property.
(ii) Since $(\mathbb{A}, {}^{\Sigma_1}/_{\mathbb{A}}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\mu_1}/_{\mathbb{A}}) \subseteq$ (M, Σ_1, $\boldsymbol{\mathcal{T}}_1$, μ_1) satisfies HBP, which implies $(\mathbb{A}, {}^{\Sigma_1}/_{\mathbb{A}}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\mu_1}/_{\mathbb{A}})$ is closed and bounded hence compact this implies, every infinite chart of $(\mathbb{A}, {}^{\Sigma_1}/_{\mathbb{A}}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\mu_1}/_{\mathbb{A}})$ has a limit point in $(\mathbb{A}, {}^{\Sigma_1}/_{\mathbb{A}}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\mu_1}/_{\mathbb{A}})$. Hence $(\mathbb{A}, {}^{\Sigma_1}/_{\mathbb{A}}, {}^{\mathcal{T}_1}/_{\mathbb{A}}, {}^{\mu_1}/_{\mathbb{A}})$ admits Bolzano-Weierstrass property (BWP).

IV. Conclusion

Any organic system is a topological manifold having finite measure. The structural and functional properties of a organic system, brain in particular, are determined by the intrinsic geometrical and topological structure of the manifold. In order to study such systems, the first author has introduced measure manifolds and extended the HBP and BWP on measure manifold. This paper is base for our future work.

References

Journal papers:

[1]. Jonathan Weitsman, "Measure on Banach Manifold, Random Surfaces, And Non Perturbative String Field Theory with Cut-Offs".arXiv:0807.2069v1[math- ph] (2008) (pp-1 - 27)

[2]. Joaquim Ortega-Cerda and Bharthi Pridhnani, "Carleson Measures and Logivinenko-Sereda Sets On Compact Manifolds".arXiv:1004.2456v1[math.CA] (2010) (pp-1 - 19)

[3]. T . Samuel , "A simple proof of vitali theorem for signed measure" The Mathematical Association of America, V-120-07, (2013) (PP-654 – 659).

Books:

[4]. Antoni A Kosinski ,Differential Manifolds (Academic Press, Har court Brace, Jovanorich Publisher Bostonsan Diego, New York, London)(1993).

[5]. C.Wayne Patty, Foundations of Topology (Second Edition, Jones and Bartlett India. PVT.LTD New Delhi, Canada, London) (2012).

[6]. Elliott H.Lieb and Michael Loss, Analysis (second Edition,Graduate studies in Mathematics V-14 American Mathematical Society , Providence, Rhode Island) (2013) .

[7]. John M .Lee , Introduction to topological Manifold (Graduate texts in Mathematics Springer Publication (USA)) (2004).

[8]. John .K. Hunter, Measure Theory (Department of Mathematics, University of California at Paris) (2011).

[9]. K . S. Amur , D.J. Shetty , C.S. Bagewadi , An Introduction to Differential Geometry (Narosa Publishing House, New Delhi) (2010).

[10]. Lawrence Conlon, Differentiable Manifolds (Second Edition Birkhauser advanced texts, Basler –Cesrbuncher –Berlin)(2001).

[11]. M.M Rao, Measure Theory and integration, (Second Edition, Revised and Expanded. Marcel Dekker .Inc. New York Basel) (2004).

[12]. M.Papadimitrakis, Notes on measure theory, (Department of Mathematics, University Of Crete Autum of 2004) (2004).

[13]. Marc . R . Roussel, Invariant Manifold (Lecture Note) (2005).

[14]. Terence Tao, An Introduction to measure theory (American Mathematical Society , Providence RI) (2012).

[15]. William M. Boothby, An Introduction to Differential Manifold and Riemannian Geometry (Academic Press, Amsterdam) (2008).