IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN:2319-765X. Volume 10, Issue 2 Ver. VII (Mar-Apr. 2014), PP 59-64
www.iosrjournals.org

Coupled Coincidence Fixed Point Theorems in S-Metric Spaces

Hans Raj[1], Nawneet Hooda[2]
[1]*(Department of Mathematics, DCRUST, Murthal, Sonepat, India)*
[2]*(Department of Mathematics, DCRUST, Murthal, Sonepat, India)*

Abstract : *In this paper we prove some coupled fixed point theorems in S-metric spaces.*
MSC: *47H10, 54H25*
Keywords: *Coupled coincidence point, coupled fixed pont, mixed g-monotone property, S-metric space*

I. Introduction

Metric spaces have very wide applications in mathematics and applied sciences. Therefore, many authors have tried to introduce the generalizations of metric spaces in many ways. In 1989, Gahler [2-3], introduced the notion of 2-metric spaces and Dhage [1] introduced the notion of D-metric spaces. They proved some results related to 2-metric and D-metric spaces. After this Mustafa and Sims [4] proved that most of the results of Dhage's D-metric spaces are not valid. So, they introduced the new concept of generalized metric space called G-metric space. Now, recently Sedghi et al. [5] have introduced the notion of S-metric spaces as the generalization of G-metric and D^*-metric spaces. They proved some fixed point results in S-metric spaces. Some results have been obtained in [5-7] by Sedghi et al. In the present paper, we prove some coupled coincidence point results in S-metric space which are the generalizations of some fixed point theorems in metric spaces [8-12].

Preliminaries

Here we give some definitions which are throughout used in this paper.

Definition 2.1 ([5]). Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$.

(i) $S(x, y, z) \geq 0$

(ii) $S(x, y, z) = 0$ if and only if $x = y = z$

(iii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$

Then the pair (X, S) is called an S-metric space.

Definition 2.2 ([14]). Let $(X, \leq)$ be a partially ordered set equipped with a metric S such that (X, S) is a metric space. Further, equip the product space $X \times X$ with the following partial ordering:

for $(x, y), (u, v) \in X \times X$,

define $(u, v) \leq (x, y) \Leftrightarrow x \geq u, y \leq v$.

Definition 2.3 ([14]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. One says that F enjoys the mixed monotone property if (x, y) is monotonically nondecreasing in x and monotonically nonincreasing in y; that is, for any $x, y \in X$,

$$x^1, x^2 \in X,\ x^1 \leq x^2 \ \Rightarrow\ F(x^1, y) \leq F(x^2, y),$$
$$y^1, y^2 \in X,\ y^1 \leq y^2 \ \Rightarrow\ F(x, y^1) \geq F(x, y^2),$$

Definition 2.4 ([14]). An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if

$F(x, y) = x$ and $F(y, x) = y$.

Lemma 2.5 ([7]). In an S-metric space, we have $S(x, x, y) = S(y, y, x)$.

Definition 2.6 ([13]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ two mappings. The mapping F is said to have the mixed g-monotone property if F is monotone g-nondecreasing in its first argument and is monotone g-nonincreasing in its second argument, that is,

if, for all $x^1, x^2 \in X, g(x^1) \leq g(x^2)$ implies $F(x^1, y) \leq F(x^2, y)$, for any $y \in X$, and, for all $y^1, y^2 \in X$, $g(y^1) \le g(y^2)$ implies $F(x, y^1) \le F(x, y^2)$, for any $x \in X$.

Definition 2.7 ([13]). An element $(x, y): X \to X$ is called a coupled coincidence point of mappings $F: X \to X$ and $g: X \to X$ if

$$F(x, y) = g(x),\ F(y, x) = g(y).$$

Theorem 2.8 ([13]). Let $(X, \le)$ be a partially ordered set equipped with a metric d such that (X, d) is a complete metric space. Assume that there is a function $:[0,\infty) \to [0,\infty)$ with $(t) < t$ and $\lim_{r\to t^+} \phi(r) < t$ for each $t > 0$. Let $F: X \times X \to X$ and $g: X \to X$ be maps such that F has the mixed g-monotone property and

$$d(F(x, y), F(u, v)) \le \phi \frac{(d(g(x), g(u) + d(g(y), \mathrm{g}(v)))}{2}$$

for all $x, y, u, v \in X$ for which $g(x) \le g(u)$ and $g(y) \ge g(v)$.

Suppose that $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F besides

(a) F is continuous,

(b) X has the following properties:

(i) if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \le x$ for all n,

(ii) if a nonincreasing sequence $\{y_n\} \to y$, then $y \le y_n$ for all n.

(iii) if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \le x$ for all $n \ge 0$,

(iv) if a nonincreasing sequence $\{y_n\} \to x$, then $y_n \ge x$ for all $n \ge 0$.

If there exist $x^0, y^0 \in X$ such that

$$g(x^0) \le (x^0, y^0),\ (y^0) \ge (y^0, x^0),$$

then there exist $x, y \in X$ such that

$$g(x) = F(x, y),\ g(y) = F(y, x),$$

That is, F and g have a coupled coincidence point.

Theorem 2.9. Let $(X, \le)$ be a complete S-metric space. Suppose that there is a function $\phi:[0,\infty) \to [0,\infty)$ with $\phi(t) < t$ and $\lim_{r\to t^+} \phi(r) < t$ for each $t > 0$. Further, assume that $F: X \times X \to X$ and $g: X \to X$ are two maps such that F has the mixed g-monotone property satisfying the following condition:

(i) $F(X \times X) \subseteq g(X)$,

(ii) g is continuous and monotonically increasing.

(iii) (g, F) is commutating pair.

(iv) $$S(F(x, y), F(u, v), F(u, v)) \le \phi\left[\frac{1}{2}(S(g(x), g(u), g(u) + S(g(\mathrm{y}), g(v), g(\mathrm{v}))\right] \qquad (1)$$

for all $x, y, u, v \in X$, with $g(x) \le g(u)$ and $g(y) \ge g(v)$. Also suppose that either

(a) F is continuous or

(b) X has the following properties:

1. if a nondecreasing sequence $\{x_n\} \to x$, then

$$x_n \le x, \text{ for all } n \ge 0 \qquad (2)$$

2. If a nonincreasing sequence $\{x_n\} \to x$, then

$x_n \ge x$, for all $n \ge 0$.

If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \le F(x_0, y_0) \quad \text{and} \quad g(y_0) \ge F(y_0, x_0) \qquad (3)$$

Then F and g have a coupled coincidence point that is there exist $x, y \in X$ such that

$$g(x) = F(x, y) \quad \text{and} \quad g(y) = F(y, x) \qquad (4)$$

Proof. Let us suppose that $x, y \in X$, we construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n) \quad \text{and} \quad g(y_{n+1}) = F(y_n, x_n) \qquad (5)$$

Now, we shall show that for $n \geq 0$

$$g(x_n) \leq g(x_{n+1}) \quad \text{and} \quad g(y_n) \geq (y_{n+1}) \tag{6}$$

So (6) holds for $n = 0$. Assume (6) holds for some $n > 0$.
Suppose

$$\begin{aligned} g(x_{n+1}) &= F(x_n, y_n) \\ &\leq F(x_{n+1}, y_n) \\ &\leq F(x_{n+1}, y_{n+1}) \\ &= g(x_{n+2}) \end{aligned}$$

and

$$\begin{aligned} g(y_{n+1}) &= F(y_n, x_n) \\ &\geq F(y_{n+1}, x_n) \\ &\geq F(y_{n+1}, x_{n+1}) \\ &= g(y_{n+2}) \end{aligned}$$

Then by induction (6) holds for all $n \geq 0$.
Using (5) and (6), we get

$$\begin{aligned} S(g(x_m), g(x_{m+1}), g(x_{m+1})) &= S(F(x_{m-1}, y_{m-1}), F(x_m, y_m), F(x_m, y_m)) \\ &\leq \phi\left[\frac{1}{2}(S(g(x_{m-1}), g(x_m), g(x_m) + S(g(y_{m-1}), g(y_m), g(y_m))\right] \end{aligned}$$

Similarly, we can write by induction

$$S(g(y_m), g(y_{m+1}), g(y_{m+1})) \leq \phi\left[\frac{1}{2}(S(g(x_{m-1}), g(x_m), g(x_m) + S(g(y_{m-1}), g(y_m), g(y_m))\right]$$

So, by putting

$$\delta_m = S(g(x_m), g(x_{m+1}), g(x_{m+1}) + S(g(y_m), g(y_{m+1}), g(y_{m+1}))$$

We get

$$\begin{aligned} \delta_m &= S(g(x_m), g(x_{m+1}), g(x_{m+1}) + S(g(y_m), g(y_{m+1}), g(y_{m+1})) \\ &\leq \phi\left[\frac{1}{2}(S(g(x_{m-1}), g(x_m), g(x_m) + S(g(y_{m-1}), g(y_m), g(y_m))\right] \\ &= 2\phi\left(\frac{1}{2}\delta_{m-1}\right) \end{aligned} \tag{7}$$

Since $\phi(t) < t$ for $t > 0$. So, $\delta_m \leq \delta_{m-1}$ for all m so that $\{\delta_m\}$ is a nonincreasing sequence, since it is bounded below sequence, there exist some $\delta > 0$ such that

$$\lim_{m \to \infty} \delta_m = +\delta.$$

We have prove that $\delta = 0$. On the other hand suppose that $\delta > 0$. Putting limit as $m \to +\infty$ on both sides of (7) and having $\lim_{r \to t^+} \phi(r) < t$ for all $t > 0$ in mind, we have,

$$\begin{aligned} \delta = \lim_{m \to \infty} \delta_m &\leq \lim_{m \to \infty} 2\phi\left(\frac{1}{2}\delta_{m-1}\right) \\ &= 2\phi\left(\frac{1}{2}\delta\right) < 2 \cdot \frac{\delta}{2} = \delta \end{aligned}$$

Which gives us a contradiction so $\delta = 0$.

Therefore,

$$\lim_{m \to \infty} S(g(x_m), g(x_{m+1}), g(x_{m+1}) + S(g(y_m), g(y_{m+1}), g(y_{m+1})) = 0 \tag{8}$$

Now, we will show that the sequences $\{g(x_m)\}$ and $\{g(y_m)\}$ are Cauchy sequence. If possible, assume that atleast one of $\{g(x_m)\}$ and $\{g(y_m)\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and sequence of positive integers $\{l(K)\}$ and $\{m(K)\}$ such that for all positive integers K,

$$m(K) > l(K) > K.$$

$$S(g(x_{1(K)}),g(x_{m(K)-1}),g(x_{m(K)-1}))+S(g(y_{1(K)}),g(y_{m(K)-1}),g(y_{m(K)-1}))\geq\varepsilon .$$

Now,

$$\begin{aligned}\varepsilon &\leq S(g(x_{1(K)}),g(x_{m(K)}),g(x_{m(K)}))+S(g(y_{1(K)}),g(y_{m(K)}),g(y_{m(K)}))\\ &\leq S(g(x_{1(K)}),g(x_{m(K)-1}),g(x_{m(K)-1}))+S(g(y_{1(K)}),g(y_{m(K)-1}),g(y_{m(K)-1}))\\ &\quad+S(g(x_{m(K)-1}),g(x_{m(K)}),g(x_{m(K)}))+S(g(y_{m(K)-1}),g(y_{m(K)}),g(y_{m(K)}))\end{aligned}$$

That is

$$\begin{aligned}\varepsilon &\leq S(g(x_{1(K)}),g(x_{m(K)}),g(x_{m(K)}))+S(g(y_{1(K)}),g(y_{m(K)}),g(y_{m(K)}))\\ &\leq \varepsilon+S(g(x_{m(K)-1}),g(x_{m(K)}),g(x_{m(K)}))+S(g(y_{m(K)-1}),g(y_{m(K)}),g(y_{m(K)}))\end{aligned}$$

Taking $k\to\infty$ in the above inequality and using (8), we get

$$\lim_{k\to\infty}\left[S(g(x_{1(K)}),g(x_{m(K)}),g(x_{m(K)})+S(g(y_{1(K)}),g(y_{m(K)}),g(y_{m(K)}))\right]=\varepsilon \tag{9}$$

Again, we have

$$\begin{aligned}&S(g(x_{1(K)+1}),g(x_{m(K)+1}),g(x_{m(K)+1}))+S(g(y_{1(K)+1}),g(y_{m(K)+1}),g(y_{m(K)+1}))\\ &\quad\leq S(g(x_{1(K)+1}),g(x_{1(K)}),g(x_{1(K)}))+S(g(y_{1(K)+1}),g(y_{1(K)}),g(y_{1(K)}))\\ &\quad\quad+S(g(x_{1(K)}),g(x_{m(K)}),g(x_{m(K)}))+S(g(y_{1(K)}),g(y_{1(K)}),g(y_{1(K)}))\\ &\quad\quad+S(g(x_{m(K)}),g(x_{m(K)+1}),g(x_{m(K)+1}))+S(g(y_{m(K)}),g(y_{m(K)+1}),g(y_{m(K)+1}))\end{aligned}$$

$$\begin{aligned}&S(g(x_{1(K)}),g(x_{m(K)}),g(x_{m(K)}))+S(g(y_{1(K)}),g(y_{m(K)}),g(y_{m(K)}))\\ &\quad\leq S(g(x_{1(K)+1}),g(x_{1(K)}),g(x_{1(K)}))+S(g(y_{1(K)+1}),g(y_{1(K)}),g(y_{1(K)}))\\ &\quad\quad+S(g(x_{1(K)+1}),g(x_{m(K)+1}),g(x_{m(K)+1}))+S(g(y_{1(K)+1}),g(y_{1(K)+1}),g(y_{1(K)+1}))\\ &\quad\quad+S(g(x_{m(K)}),g(x_{m(K)+1}),g(x_{m(K)+1}))+S(g(y_{m(K)}),g(y_{m(K)+1}),g(y_{m(K)+1}))\end{aligned}$$

Taking $K\to\infty$ in above inequalities and using (8) and (9), we obtain,

$$\lim_{k\to\infty}\left[S(g(x_{1(K)+1}),g(x_{m(K)+1}),g(x_{m(K)+1})+S(g(y_{1(K)+1}),g(y_{m(K)+1}),g(y_{m(K)+1}))\right]=\varepsilon \tag{10}$$

Now,

$$\begin{aligned}&S(g(x_{1(K)+1}),g(x_{m(K)+1}),g(x_{m(K)+1}))+S(g(y_{1(K)+1}),g(y_{m(K)+1}),g(y_{m(K)+1}))\\ &\quad\leq S(F(x_{1(K)},y_{1(K)}),F(x_{m(K)},y_{m(K)}),F(x_{m(K)},y_{m(K)}))\\ &\quad\quad+S(F(y_{1(K)},x_{1(K)}),F(y_{m(K)},x_{m(K)}),F(x_{m(K)},y_{m(K)}))\\ &\quad\leq\phi\left[\frac{1}{2}(S(g(x_{1(K)}),g(x_{m(K)}),g(x_{m(K)})+S(g(y_{1(K)}),g(y_{m(K)}),g(y_{m(K)}))\right].\end{aligned}$$

Assuming $K\to\infty$ in the above inequality and using (9) and (10) and the property of ϕ, we get

$$\varepsilon\leq 2\phi\left(\frac{\varepsilon}{2}\right)<2\cdot\frac{\varepsilon}{2}=\varepsilon$$

Which leads to a contradiction. Therefore $\{g(x_m)\}$ and $\{g(y_m)\}$ are Cauchy sequences in (X,S). Since the metric space (X,S) is complete, therefore there exist $x,y\in X$ such that

$$\lim_{m\to\infty}g(x_m)=x \quad\text{and}\quad \lim_{m\to\infty}g(y_m)=y. \tag{11}$$

Now, g is continuation. So, by the continuity of g and (11), we can get

$$\lim_{m\to\infty}g(g(x_m))=g(x) \quad\text{and}\quad \lim_{m\to\infty}g(g(y_m))=g(y). \tag{12}$$

Using (5) and the commutativity of F and g, we have

$$\begin{aligned}g(g(x_{m+1}))&=g(F(x_m,y_m))\\ &=F(g(x_m),g(y_m))\end{aligned}$$

and

$$\begin{aligned}g(g(y_{m+1}))&=g(F(y_m,x_m))\\ &=F(g(y_m),g(x_m))\end{aligned}$$

Now, we will show that F and g have a coupled coincidence point. To, prove this, suppose (a) holds, then by (5) and (12) and the continuous of F and g, we get

$$\begin{aligned} g(x) &= \lim_{m\to\infty} g(g(x_{m+1})) \\ &= \lim_{m\to\infty} g(F(x_m, y_m)) \\ &= F\left(\lim_{m\to\infty} g(x_m), \lim_{m\to\infty} g(y_m)\right) \\ &= F(x, y) \end{aligned}$$

Similarly, we can show that

$$g(y) = F(y, x).$$

Hence, the element $(x, y) \in X \times X$ is a coupled coincidence point of the mappings F and g. Now, suppose that (6) holds. Since $\{g(x_m)\}$ and $\{g(y_m)\}$ is nondecreasing and nonincreasing respectively, and

$g(x_m) \to x$ as $m \to \infty$,

$g(y_m) \to y$ as $m \to \infty$,

we have

$g(x_m) \le x$ and $g(y_m) \ge y$.

Since g is monotonically increasing. So,

$g(g(x_m)) \le g(x)$ and $g(g(y_m)) \ge g(y)$.

Using triangle inequality together with (5), we have

$$\begin{aligned} S(g(x), F(x,y), F(x,y)) &\le S(g(g(x_{m+1})), F(x,y), F(x,y)) + S(g(g(x_{m+1})), g(x), g(x)) \\ &\le S(g(g(x_{m+1})), g(x), g(x)) \\ &\quad + \phi\left[\frac{1}{2}(S(g(g(x_{m+1}), g(x), g(x)) + (S(g(g(y_{m+1}), g(y), g(y))\right] \end{aligned}$$

Letting $m \to \infty$ in this inequality and using (12), we get $g(x) = F(x, y)$. Similarly, we can show that $g(y) = F(y, x)$.

Which shows that F and g have a coupled coincidence point.

Corollary 2.10. Let (X, S) is a complete S-metric space. Suppose that three is a function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(t) < t$ and $\lim_{r\to t^+} \phi(r) < t$ for each $t > 0$. Further, assume that $F : X \times X \to X$ is a mapping such that F has the mixed monotone property satisfying the following conditions:

$$S(F(x,y), F(u,v), F(u,v)) \le \phi\left[\frac{1}{2}(S(x,u,u), g(\mathrm{y},\mathrm{v},\mathrm{v})\right]$$

for all $x, y, u, v \in X$ with $x \le u$ and $y \ge v$.

Also, suppose that either

(a) F is continuous or

(b) X has the following properties:

(i) If a nondecreasing sequence $\{x_n\} \to x$, then $x_n \le x$ for all $n \ge 0$

(ii) If a nonincreasing sequence $\{x_n\} \to x$, then $x_n \ge x$ for all $n \ge 0$

If there exist $x_0, y_0 \in X$ such that

$x_0 \le F(x_0, y_0)$ and $y_0 \le F(y_0, x_0)$

Then F has a coupled fixed point in X, that is there exist $x, y \in X$ such that

$x = F(x, y)$ and $y = F(y, x)$

Proof. Assuming $g = I$, the identity mapping, in Theorem 2.9, we get the above Corollary 2.10.

Corollary 2.11. Let (X, S) be a complete S-metric space. Suppose that $F : X \times X \to X$ and $g : X \to X$ are two maps such that F has the mixed g-monotone property satisfying the following conditions:

(i) $F(X \times X) \subseteq g(X)$

(ii) g is continuous and monotonically increasing,

(iii) (g,F) is a commutating pair,

(iv) $S(F(x,y),F(u,v),F(u,v)) \leq \frac{k}{r}[S(g(x),g(u),g(u))+S(g(y),g(v),g(v))]$, $k\in[0,1)$

for all $x,y,u,v\in X$ with $g(x)\leq g(u)$ and $g(y)\geq g(v)$. Also, assume that either

(a) F is continuous or

(b) X has the following properties:

(i) If a nondecreasing sequence $\{x_n\}\to x$, then

$x_n \leq x$ for all $n\geq 0$

(ii) If a nonincreasing sequence $\{x_n\}\to x$, then

$x_n \geq x$ for all $n\geq 0$

If there exist $x_0, y_0\in X$ such that

$$g(x_0)\leq F(x_0,y_0) \quad \text{and} \quad g(y_0)\leq F(y_0,x_0)$$

Then F and g have a coupled fixed point in X, i.e. there exist $x,y\in X$ such that

$$g(x)=F(x,y) \quad \text{and} \quad g(y)=F(y,x)$$

Proof. Taking $\phi(t)=k\cdot t$ with $k\in[0,1)$ in Theorem 2.9, we obtain the above Corollary 2.11.

References

[1] B.C. Dhage, Generalized metric space and mapping with fixed point, Bull. Cal. Math. Soc. 84, 1992, 329-336.
[2] S. Gahlers, 2-metrische Raume and ihre topologische structure, Math.Nachr, 26, 1963, 115-148.
[3] S. Gahlers, Zur geometric 2-metrische raume, Revue Roumaine Math. Pures Appl., 11, 1966, 665-667.
[4] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, Journal of Nonlinear and Convex Analysis, 7 (2), 2006, 289-297.
[5] S. Sedghi, N. Shobe and A. Aliouche, A generalization of fixed point theorems in S-metric spaces, Mat. Vesnik, 64 (3) (2012), 258-266.
[6] S. Sedghi and N.V. Dzung, Fixed point theorems on S-metric spaces, accepted for publication in Mat. Vesnik (2012).
[7] T.V. An and N.V. Dung, Two fixed point theorems in S-metric spaces, preprint (2012).
[8] Z. Mustafa and B. Sims, Fixed point theorems for contractive mappings in complete G-metric spaces, Fixed Point Theory Appl. 2009, 10 (2009). Article ID 917175
[9] T.G. Bhaskar and V. Lakshmikantham, Fixed point theory in partially ordered metric spaces and applications, Nonlinear Anal., 65 (2006), 1379-1393.
[10] D.W. Boyd and S.W. Wong, On nonlinear contractions, Proc. Am. Math. Soc., 20, 1969, 458-464, doi:10.1090/S0002-9939-1969-0239559-9.
[11] M. Imdad et al, On n-tupled coincidence point results in metric spaces, Journal of Operators, 2013, 8 pages.
[12] Z. Mustafa, H. Obiedat and F. Awawdeh, Some fixed point theorem for mapping on complete metric spaces, Fixed Point Theory Appl., 2008, 12 (2008). Article ID 189870.
[13] V. Lakshmikantham and L. Ciric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Analysis: Theory, Methods & Applications A, 70 (12), 2009, 4341-4349.
[14] T. Gnana Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Analysis: Theory, Methods & Applications A, 65 (7), 2006, 1379-1393.