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Characterizations for New Classes of Analytic Functions Defined By Using Salagean Operator

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Abstract : *In this paper, we study certain subclasses* $U_{m,n}(\beta, A, B, \rho)$ *and* $U^*_{m,n}(\beta, A, B, \rho)$ *of analytic functions in the unit disk. The results presented include coefficient estimates and several subordination properties for functions belonging to these subclasses. Our results extend some earlier works.*
Keywords: *subordinating factor sequence, univalent, convex, analytic, convolution (or Hadamard product)*

I. Introduction

Let **A** denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (1.1)$$

that are analytic and univalent in the open unit disk $U = \{z \in \mathbf{C}; |z| < 1\}$. Let $g(z) \in \mathbf{A}$ be given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \qquad (1.2)$$

Furthermore, let

$$\Phi(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k, \lambda \geq 0$$

$$\Phi(z) = z + \sum_{k=2}^{\infty} \mu_k z^k, \mu \geq 0 \qquad (1.3)$$

Which are analytic and normalized by the conditions that $f(0) = f'(0) - 1 = 0$

For $f(z) \in \mathbf{A}$, Salagean [1] introduced the following differential operator,

$$D^0 f(z) = f(z), D' f(z) = zf'(z), \ldots, D^n f(z) = D(D^{n-1} f(z))(n \in \mathbf{N} = \{1, 2, \ldots\}).$$

We note that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \ \ (n \in \mathbf{N}_0 = \mathbf{N} \cup \{0\})$$

Definition 1 (Hardamard Product or Convolution)

Given two functions f and g in the class **A,** where $f(z)$ and $g(z)$ are given by (1.1) and (1.2) respectively, the Hardamard product (or Convolution) of f and g is defined (as usual) by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z) \qquad (1.4)$$

Definition 2 (Subordination Principle)

For two functions f and g, analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U, and write $f(z) \prec g(z)$ if there exists a Schwarz function $\omega(z)$ which (by definition) is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$ $(z \in U)$. Indeed it is known that

$$f(z) \prec g(z) \Rightarrow f(0) = g(0) \, and \, f(u) \subset (g(u))$$

Furthermore, if the function g is univalent in U, then we have the following equivalence [8, p. 4]:

Definition 3 [3]

Let $U_{m,n}(\beta, A, B)$ denote the subclasses of **A** consisting of functions $f(z)$ of the form (1.1) and satisfy the following subordination:

$$\frac{D^m f(z)}{D^n f(z)} - B\left|\frac{D^m f(z)}{D^m f(z)} - 1\right| \prec \frac{1+Az}{1+Bz} \qquad (1.5)$$

$$(-1 \le B < A \le 1; \beta \ge 0; m \in \mathbf{N}_0, m > n;\ z \in u)$$

Remark 1

By specializing the parameters A, B, β, m and n, certain subclasses studied by various authors are obtained. For instance,

(i) $U_{m,n}(\beta, 1-2\alpha, -1) = N_{m,n}(\alpha, \beta)$ *(see Eker and Owa [4])*

(ii) $U_{m+1,n}(\beta, 1-2\alpha, -1) = S(n,\ \alpha,\ \beta)$ *(see Rosy and Mumgundaramorthy [5], Asurf [6])*

(iii) $U_{1,0}(\beta, 1-2\alpha, -1) = U\,S(\alpha, \beta)$ *(see Shaw et al [7])*

(iv) $U_{2,1}(\beta, 1-2\alpha, -1) = UK(\alpha, \beta)$ *(see Shaw and Kulkarni[8])*

(v) $U_{1,0}(0, A, B) = S^*(A, B)$, $U_{2,1}(0, A, B) = K(A, B)$ *(see Jarowski [9] and Padmanashon and [10])*

Therefore, in the view of (1.3), definitions 1 and 3, we now give the following definitions:

Definition 4

Let $U_{m,n}(\beta, A, B, \rho)$ denote the subclasses of **A** consisting of functions $f(z)$ of the form (1.1) and satisfying the following condition;

$$\left|\frac{D^m(f*\Phi)^{\rho}(z)}{D^n(f*\psi)^{\rho}(z)}\right| - B\left|\frac{D^m(f*\Phi)^{\rho}(z)}{D^n(f*\psi)^{\rho}(z)} - 1\right| \prec \frac{1+Az}{1+Bz} \qquad (1.6)$$

$$(-1 \le B < A \le 1;\ \beta \ge 0; m, \rho \in \mathbf{N}; n \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}), m > n;\ z \in U$$

Also, we note that for $\lambda_k \ge \mu_k \ge 0$, when $\lambda_k = \mu_k = 1$ then, $\Phi(z) = \psi(z) = \dfrac{z}{1-z} \in K$ such that our **Definition 4** will be equivalent to **Definition 3**. This is because

$$(f*\Phi)(z) = (f*\psi)(z) = f$$

Definition 5 (Subordinating Factor Sequence)

A sequence $\{c_k\}_{k=0}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever $f(z)$ of the form (1.1) is analytic, univalent and convex in U, we have the subordination given by;

$$\sum_{k=1}^{\infty} a_k c_k z^k \prec f(z) \qquad (a_1 = 1, z \in U) \qquad (1.7)$$

II. Main Results

Unless otherwise stated, we shall in the sequence assume that $-1 \le B < A \le 1,\ \beta \ge 0,\ m, \rho \in \mathbf{N},\ n \in \mathbf{N}_0, m > n,\ a_k(\rho) \ge 0, \lambda_k(\rho) \ge 0, \mu_k(\rho) \ge 0;$ where $a_k(\rho), \lambda_k(\rho), \mu_k(\rho)$ are coefficients of a_k, λ_k, μ_k all depending on ρ and $z \in U$.

We now prove the following theorem which gives a sufficient condition for functions belonging to the class $U_{m,n}(\beta, A, B, \rho)$.

Theorem 1

A function $f(z)$ of the form (1.1) is in the class $U_{m,n}(\beta, A, B, \rho)$

if; $\sum_{k=2}^{\infty}\left\{\left[1+\beta(1+|B|)\right]\left[(\rho+k-1)^m \lambda_k(\rho)-(\rho+k-1)^n \mu_k(\rho)\right]+\left|B(\rho+k-1)^m \lambda_k(\rho)-A(\rho+k-1)^n \mu_k(\rho)\right|\right\}\left|a_k(\rho)\right| \le \left(A\rho^n - B\rho^m\right)-\left[1+\beta(1+|B|)\right]\left(\rho^n-\rho^m\right)$

(2.1)

Proof

It suffices to show that

$$\left|\frac{P(z)-1}{A-Bp(z)}\right|<1$$

Where

$$P(z)=\frac{D^m(f*\Phi)^\rho(z)}{D^n(f*\psi)^\rho(z)}-B\left|\frac{D^m(f*\Phi)^\rho(z)}{D^n(f*\psi)^\rho(z)}-1\right|$$

We have

$$\left|\frac{P(z)-1}{A-Bp(z)}\right|=\left|\frac{D^m(f*\Phi)^\rho(z)-\beta e^{i\theta}\left|D^m(f*\Phi)^\rho(z)-D^n(f*\psi)^\rho(z)\right|-D^n(f*\psi)^\rho(z)}{AD^m(f*\Phi)^\rho(z)-B\left[D^m(f*\Phi)^\rho(z)-\beta e^{i\theta}\left|D^m(f*\Phi)^\rho(z)-D^n(f*\psi)^\rho(z)\right|\right]}\right|$$

$$=\left|\frac{\left(\rho^n-\rho^m\right)z^\rho+\sum_{k=2}^{\infty}\left[(\rho+k-1)^m\lambda_k(\rho)-(\rho+k-1)^n\mu_k(\rho)\right]a_k(\rho)z^{\rho+k-1}+\beta e^{i\theta}\left|\left(\rho^n-\rho^m\right)z^\rho+\sum_{k=2}^{\infty}\left[(\rho+k-1)^m\lambda_k(\rho)-(\rho+k-1)^n\mu_k(\rho)\right]a_k(\rho)z^{\rho+k-1}\right|}{\left(A\rho^n-B\rho^m\right)z^\rho-\sum_{k=2}^{\infty}\left[B(\rho+k-1)^m\lambda_k(\rho)-A(\rho+k-1)^n\mu_k(\rho)\right]a_k(\rho)z^{\rho+k-1}-B\beta e^{i\theta}\left|\left(\rho^n-\rho^m\right)z^\rho+\sum_{k=2}^{\infty}\left[(\rho+k-1)^m\lambda_k(\rho)-(\rho+k-1)^n\mu_k(\rho)\right]a_k(\rho)z^{\rho+k-1}\right|}\right|$$

$$\le\left|\frac{\left(\rho^n-\rho^m\right)|z|^\rho+\sum_{k=2}^{\infty}\left[(\rho+k-1)^m\lambda_k(\rho)-(\rho+k-1)^n\mu_k(\rho)\right]\left|a_k(\rho)\right||z|^{\rho+k-1}+\beta e^{i\theta}\left|\left(\rho^n-\rho^m\right)|z|^\rho+\sum_{k=2}^{\infty}\left[(\rho+k-1)^m\lambda_k(\rho)-(\rho+k-1)^n\mu_k(\rho)\right]\left|a_k(\rho)\right||z|^{\rho+k-1}\right|}{\left(A\rho^n-B\rho^m\right)|z|^\rho+\sum_{k=2}^{\infty}B\left[(\rho+k-1)^m\lambda_k(\rho)-A(\rho+k-1)^n\mu_k(\rho)\right]\left|a_k(\rho)\right||z|^{\rho+k-1}-|B|\beta e^{i\theta}\left|\left(\rho^n-\rho^m\right)|z|^\rho-|B|\beta\sum_{k=2}^{\infty}\left[(\rho+k-1)^m\lambda_k(\rho)-(\rho+k-1)^n\mu_k(\rho)\right]\left|a_k(\rho)\right||z|^{\rho+k-1}\right|}\right|$$

$$\le\left|\frac{\left(\rho^n-\rho^m\right)+\sum_{k=2}^{\infty}\left[(\rho+k-1)^m\lambda_k(\rho)-(\rho+k-1)^n\mu_k(\rho)\right]\left|a_k(\rho)\right|+\beta\left(\rho^n-\rho^m\right)+\sum_{k=2}^{\infty}\left[(\rho+k-1)^m\lambda_k(\rho)-(\rho+k-1)^n\mu_k(\rho)\right]\left|a_k(\rho)\right|}{\left(A\rho^n-B\rho^m\right)|z|^\rho+\sum_{k=2}^{\infty}B\left[(\rho+k-1)^m\lambda_k(\rho)-A(\rho+k-1)^n\mu_k(\rho)\right]\left|a_k(\rho)\right|-|B|\beta\left|\left(\rho^n-\rho^m\right)|z|^\rho-|B|\beta\sum_{k=2}^{\infty}\left[(\rho+k-1)^m\lambda_k(\rho)-(\rho+k-1)^n\mu_k(\rho)\right]\left|a_k(\rho)\right|\right|}\right|$$

This last expression is bounded above by 1 if

$$\left|\frac{\left(\rho^n-\rho^m\right)+\sum_{k=2}^{\infty}\left[(\rho+k-1)^m\lambda_k(\rho)-(\rho+k-1)^n\mu_k(\rho)\right]\left|a_k(\rho)\right|+\beta\left(\rho^n-\rho^m\right)+\sum_{k=2}^{\infty}\left[(\rho+k-1)^m\lambda_k(\rho)-(\rho+k-1)^n\mu_k(\rho)\right]\left|a_k(\rho)\right|}{\left(A\rho^n-B\rho^m\right)|z|^\rho+\sum_{k=2}^{\infty}B\left[(\rho+k-1)^m\lambda_k(\rho)-A(\rho+k-1)^n\mu_k(\rho)\right]\left|a_k(\rho)\right|-|B|\beta\left|\left(\rho^n-\rho^m\right)|z|^\rho-|B|\beta\sum_{k=2}^{\infty}\left[(\rho+k-1)^m\lambda_k(\rho)-(\rho k-1)^n\mu_k(\rho)\right]\left|a_k(\rho)\right|\right|}\right|\le 1$$

i.e. that

$$\sum_{k=2}^{\infty}\left\{\left[1+\beta(1+|B|)\right]\left[(\rho+k-1)^m \lambda_k(\rho)-(\rho+k-1)^n \mu_k(\rho)\right]+\left|B(\rho+k-1)^m \lambda_k(\rho)-A(\rho+k-1)^n \mu_k(\rho)\right|\right\}\left|a_k(\rho)\right| \le \left(A\rho^n - B\rho^m\right)-\left[1+\beta(1+|B|)\right]\left(\rho^n-\rho^m\right)$$

and hence, the proof of Theorem 1 is obtained.

By taking $\rho=1$ in theorem 1 when $\lambda_k(\rho)=\mu_k(\rho)=1$ and $a_k(\rho)$ is the coefficient a_k depending on ρ, we obtain the following

Corollary 1

A function $f(z)$ of the form (1.1) is in the class of $U_{m,n}(\beta, A, B, \rho)$ if

$$\sum_{k=2}^{\infty}\left[\left(1+(1+|B|)\right)\left(k^m-k^n\right)+\left|Bk^m-Ak^n\right|\right]\left|a_k\right| \le A-B$$

This means that

$$U_{m,n}(\beta, A, B, 1) = U_{m,n}(\beta, A, B) = \left\{ f \in A : \frac{D^m f(z)}{D^n f(z)} - B\left|\frac{D^m f(z)}{D^n f(z)} - 1\right| \prec \frac{1+Az}{1+Bz} \right\}$$

Remark 2

(i) The result in corollary 1 which is the correct result obtained by Li and Tang [3, theorem 1], is due to M.K. Aouf et al [11]

(ii)Putting $A = 1 - 2\alpha$, $(0 \le \alpha < 1)$, $B = -1$, $m = n+1$ $(n \in \mathbf{N}_0)$ and $\rho = 1$ we obtain the result due to Rosy and Murugusudaramworthy [3, theorem 2]

Let $U^*_{m,n}(\beta, A, B, \rho)$ denote the class of $f(z \in \mathbf{A})$ whose coefficients satisfy the condition (2.1).

We note that $U^*_{m,n}(\beta, A, B, \rho) \subseteq U_{m,n}(\beta, A, B, \rho)$.

By employing the technique used earlier by [12] and Srivastava [13], we now state and prove our next result; which is a subordination result for the class $U^*_{m,n}(\beta, A, B, \rho)$. However, we first give the following lemma which is required for the proof of our next theorem.

Lemma 1 [14]

The sequence $\{c_k\}_{k=0}^{\infty}$ is a subordinating factor sequence if and only if

$$\operatorname{Re}\left\{1 + 2\sum_{k=1}^{\infty} c_k z^k\right\} > 0 \qquad (z \in U) \qquad (2.2)$$

Theorem 2

Let $f(z) \in U^*_{m,n}(\beta, A, B, \rho)$. Then

$$\frac{\Omega(2)}{2\{(A\rho^n - B\rho^m) - [1 + \beta(1+|B|)](\rho^m - \rho^n) + \Omega(2)\}}(f * h) \prec h(z) \qquad (z \in U) \qquad (2.3)$$

for every function $h \in k$ and

$$\operatorname{Re}\{f(z)\} > -\frac{(A\rho^n - B\rho^m) - [1 + \beta(1+|B|)](\rho^m - \rho^n) + \Omega(2)}{\Omega(2)}$$

The constant factor $\frac{\Omega(2)}{2\{(A\rho^n - B\rho^m) - [1 + \beta(1+|B|)](\rho^m - \rho^n) + \Omega(2)\}}$ in subordination result (2.3) cannot be replaced by a larger one.

Proof

Let $f(z) \in U^*_{m,n}(\beta, A, B, \rho)$ and let $h(z) = z + \sum_{k=2}^{\infty} c_k z^k \in$. Then we have

$$\frac{\Omega(2)}{2\{(A\rho^n - B\rho^m) - [1 + \beta(1+|B|)](\rho^m - \rho^n) + \Omega(2)\}}(f * h)(z)$$

Thus by definition 5, the subordination result (2.3) will hold if the sequence

$$\left\{\frac{\Omega(2)}{2\{(A\rho^n - B\rho^m) - [1 + \beta(1+|\beta|)](\rho^m - \rho^n) + \Omega(2)\}}\right\}_{k=1}^{\infty}$$

Is a subordinating factor sequence with $a_1 = 1$. In the view of lemma 1, this is equivalent to the following inequality;

$$\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{\Omega(2)}{(A\rho^n-B\rho^m)-[1+\beta(1+|B|)](\rho^m-\rho^n)+\Omega(2)}a_k z^k\right\}>0\ (z\in\mathbf{U}) \qquad (2.6)$$

Now, since

$$\Omega(k)=\left\{[1+\beta(1+|B|)]\left[(\rho+k-1)^m\lambda_k(\rho)-(\rho+k-1)^n\mu_k(\rho)\right]+\left|B(\rho+k-1)^m\lambda_k(\rho)-A(\rho+k-1)^n\mu_k(\rho)\right|\right\}$$

is an increasing function of $k,\ (k\geq 2)$, we have

$$\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{\Omega(2)}{(A\rho^n-B\rho^m)-[1+\beta(1+|B|)](\rho^m-\rho^n)+\Omega(2)}a_k z^k\right\}=\operatorname{Re}\left\{1+\frac{\Omega(2)}{(A\rho^n-B\rho^m)-[1+\beta(1+|B|)](\rho^m-\rho^n)+\Omega(2)}z+\frac{1}{(A\rho^n-B\rho^m)-[1+\beta(1+|B|)](\rho^m-\rho^n)+\Omega(2)}\sum_{k=2}^{\infty}\Omega(2)a_k z^k\right\}$$

$$\geq 1-\frac{\Omega(2)}{(A\rho^n-B\rho^m)-[1+\beta(1+|B|)](\rho^m-\rho^n)+\Omega(2)}r-\frac{1}{(A\rho^n-B\rho^m)-[1+\beta(1+|B|)](\rho^m-\rho^n)+\Omega(2)}\sum_{k=2}^{\infty}\Omega(k)|a_k|r^k$$

$$>1-\frac{\Omega(2)}{(A\rho^n-B\rho^m)-[1+\beta(1+|B|)](\rho^m-\rho^n)+\Omega(2)}r-\frac{(A\rho^n-B\rho^m)-[1+\beta(1+|B|)](\rho^m-\rho^n)}{(A\rho^n-B\rho^m)-[1+\beta(1+|B|)](\rho^m-\rho^n)+\Omega(2)}r$$

$$=1-r>0 \qquad (|z|=r<1)$$

where we have also made use of the assertion (2.1) of theorem 1. Thus (2.6) holds this proves the inequality (2.3). The inequality (2.4) follows from (2.3) by taking the convex function

$$h(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k$$

To prove the constant

$$\frac{\Omega(2)}{2\{(A\rho^n-B\rho^m)-[1+\beta(1+|B|)](\rho^m-\rho^n)+\Omega(2)\}}$$

We consider the function

$f_0(z)\in U^*_{m,n}(\beta,A,B,\rho)$ given by

$$f_0(z)=z-\frac{(A\rho^n-B\rho^m)-[1+\beta(1+|B|)](\rho^m-\rho^n)}{\left\{[1+\beta(1+|B|)]\left[(\rho+1)^n\lambda_2(\rho)-(\rho+1)^n\mu_2(\rho)+\right]+\left|[B(\rho+1)^m\lambda_2(\rho)-A(\rho+1)^n\mu_2(\rho)]\right|\right\}}z^2 \qquad (2.7)$$

Thus from (2.3)

$$\frac{\Omega(2)}{2\{(A\rho^n-B\rho^m)-[1+\beta(1+|B|)](\rho^m-\rho^n)+\Omega(2)\}}f_0(z)\prec\frac{z}{1-z} \qquad (z\in\mathbf{U}) \qquad (2.8)$$

Moreover, it can easily be verified for functions $f_0(z)$ given by (2.7) that

$$\min_{|z|\leq r}\left\{\operatorname{Re}\frac{\Omega(2)}{2\{(A\rho^n-B\rho^m)-[1+\beta(1+|B|)](\rho^m-\rho^n)+\Omega(2)\}}f_0(z)\right\}=-\frac{1}{2} \qquad (2.9)$$

This show that the constant

$\frac{\Omega(2)}{2\{(A\rho^n-B\rho^m)-[1+\beta(1+|B|)](\rho^m-\rho^n)+\Omega(2)\}}$ is the best possible.

This completes the proof of theorem 2.

Remark 3: When $\lambda_2(\rho)=\mu_2(\rho)=1 \quad (\lambda_k(\rho)\geq\mu_k(\rho)\geq 0)$

(i)Taking $A=1-2\alpha\ \ (0\leq\alpha<1),\ B=-1\ \ and\ \ \rho=1$ in theorem 2, we correct the result obtained by Srivastava and Eker [15, Theorem 1]

(ii) Taking $A=1-2\alpha\ \ (0\leq\alpha<1),\ B=-1,\ m=n+1\ \ and\ \ \rho=1$ in theorem 3, we obtain the result obtained by Aouf et al. [16, Corollary 4]

(iii) Taking $A = 1-2\alpha \quad (0 \le \alpha < 1), \; B = -1, \; m = 1, n = 0 \; and \quad \rho = 1$ in theorem 2, we obtain the result obtained by Frasin [17, Corollary 2.2]
(iv) Taking $A = 1-2\alpha \quad (0 \le \alpha < 1), \; B = -1, \; m = 2, \; and \; n = \rho = 1$ in theorem 3 we obtain the result obtained by Frasin [17, Corollary 2.5]
(v) Taking $\beta = \alpha \ge 0, \; B = -1, \; \rho = 1 \; and \; \lambda_2(\rho) = \mu_2(\rho) = 1$ in theorem 3, we obtain the result obtained by Oyekan and Opoola [18, Theorem 2.1]

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