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Dual Results of Opial's Inequality

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Abstract: *In this paper, we used Jensen's inequality for the case of convex functions, firstly to obtain a Calvert's generalization and second, to obtain a Maron's generalization of Opial's inequality. The main tool was adaptation of Jensen's inequality for convex functions.*
Keywords: *Integral inequalities, Maron's inequality and Calvert's generalization of Opial's inequality Jensen's inequality, convex functions.*

I. Introduction:

Opial [5] established the following interesting integral inequality:
Let $(x, y) \in C'[0,b]$ be such $x(0) = x(b) = 0$ and $x(t) \succ 0$ in $(0,b)$, then

$$\int_a^b |x(t)x'(t)|dt \le \frac{b}{4}\int_a^b (x'(t))^2\, dt \quad (1)$$

Where $\frac{b}{4}$ in the best possible constant.

In 1967 Maroni [5] obtained a generalized Opial's inequality by using Holder inequality with indices μ and υ. The result obtained is the following:

Theorem 1:

Let $p(t)$ be positive and continuous on $[\tau,\alpha]$ with $\int_\alpha^\tau p^{1-\mu}(t)dt \prec \infty$, where $\mu \succ 1, x(t)$ be absolutely function on $[\tau,\alpha]$ and $x(0)=0$. The following inequality holds.

$$\int_\alpha^\tau |x(t)x'(t)|dt \le \frac{1}{2}\left(\int_\alpha^\tau p^{1-\mu}(t)dt\right)^{\frac{2}{\mu}}\left(\int_\alpha^\tau p(t)|x'(t)|^{\upsilon}\, dt\right)^{\frac{2}{\upsilon}} \quad (2)$$

Where $\frac{1}{\mu}+\frac{1}{\upsilon}=1$. equality holds in (2) in and only if $\int_\alpha^\tau p^{1-\mu}(s)ds$

Calvert [2] also established the following result:

Theorem 2: [2] assume that

(i) $x(t)$ is absolutely continuous in $[\alpha,\tau]$ and $x(\alpha)=0$

(ii) $f(t)$ is continuous, complex-valued, defined in the range of $x(t)$ and for all real for t of the form $t(s) = \int_\alpha^s |x'(u)|du : f(|t|)$ for all t and $f(t)$ is real $t \succ 0$ and is increasing there,

(iii) $p(t)$ is positive, continuous and $\int_\alpha^\tau p^{1-\mu}(t)dt \prec \infty$, where $\frac{1}{\mu}+\frac{1}{\upsilon}=1$. then the following inequality holds.

$$\int_\alpha^\tau | f(t)x'(t)|dt \le F\left(\int_\alpha^\tau p^{1-\mu}(t)dt\right)^{\frac{2}{\mu}}\left(\int_\alpha^\tau p(t)|x'(t)|^{\upsilon}\, dt\right)^{\frac{2}{\upsilon}} \quad (3)$$

Where $F(t)=\int_{0}^{t} f(t)ds, t \succ 0.$ Equality holds in (3) if and only if $x(t)=\int_{\alpha}^{t} p^{1-\mu}(s)ds.$

The aim of this paper is to generalize Maroni and Calvert results using Jensen's inequality.

II. Some Adaptation of Jensen's inequalities:

Let φ be continuous and convex function and let $h(s,t)$ be a non negative function and λ be non decreasing function. Let $-\infty \le \xi(t) \le \eta(t) \prec \infty$ and suppose φ has a continuous inverse φ^{-1} (which is necessarily concave). Then,

$$\varphi^{-1}\left(\left|\frac{\int_{\xi(t)}^{\eta(t)} h(s,t)d\lambda(s)}{\int_{\xi(t)}^{\eta(t)} d\lambda(s)}\right|\right) \le \left(\frac{\int_{\xi(t)}^{\eta(t)} (\varphi)^{-1}(|h(s,t)|)d\lambda(s)}{\int_{\xi(t)}^{\eta(t)} d\lambda(s)}\right) \quad (4)$$

With the inequality reserved if φ is concave. The inequality (4) above is known as Jensen's inequality for convex function. Setting $\varphi(u)=u', \xi(t)=t$ and $\eta(t)=0$ in (4), then we obtain.

$$(f(t))^{\zeta} = f\left(\left|\frac{\int_{0}^{t} h(s,t)d\lambda(s)}{\int_{0}^{t} d\lambda(s)}\right|\right)^{\frac{1}{\zeta}} \le \left(\frac{\int_{0}^{t} (|h(s,t)|)^{\frac{1}{l}} d\lambda(s)}{\int_{0}^{t} d\lambda(s)}\right) \quad (5)$$

III. Main Result:

Before stating our main result in this section, we shall need the following useful Lemma:

Lemma 1:

Let $x(t), \lambda(t)$ and $f(u)$ be absolutely continuous and non decreasing functions on $[a,b]$ for $0 \le a \le b \le \infty$ with $f(t) \succ 0.$ Let l, k, o, ρ and ζ be real numbers such that $\zeta \ge 0, o \ge 0$ and also let R(t) be non negative and measurable function on $[a,b]$ such that

$$|x'(t)| \times f\left(\left|\int_{0}^{t} x'(t)R(t)d\lambda(t)\right|\right) \le \lambda(t)^{l-\zeta} y(t)^{\zeta} \times R(t)^{-1} \lambda'(t)^{-1} y'(t). \quad (6)$$

Then the following inequality holds:

$$\int_{a}^{b} |x'(t) f(t)| dt \le \int_{a}^{b} f(y(t)) y'(t) dt \quad (7)$$

Proof:

Setting $h(s,t)=x'(t)R(t)$ in (5), we have

$$(f(t))^{\zeta} = f\left(\left|\frac{\int_{0}^{t} x'(t)R(t)d\lambda(t)}{\int_{0}^{t} d\lambda(t)}\right|\right)^{\frac{1}{\zeta}} \le \left(\frac{\int_{0}^{t} (|x'(t)R(t)|)^{\frac{1}{l}} d\lambda(t)}{\int_{0}^{t} d\lambda(t)}\right) \quad (8)$$

By setting $f(\lambda(t))=\lambda(t)^{l}$ in (8) yields

$$\frac{f\left(\left|\int_0^t x'(t)R(t)d\lambda(t)\right|\right)}{\lambda(t)^l} \le \frac{\left(\int_0^t f\left|x'(t)R(t)\right|\right)^{\frac{1}{l}} d\lambda(t)^\zeta}{\lambda(t)^\zeta} \quad (9)$$

Hence,

$$f\left(\left|\int_0^t x'(t)R(t)d\lambda(t)\right|\right) \le \lambda(t)^{l-\zeta}\left(\int_0^t f\left(\left|x'(t)R(t)\right|\right)^{\frac{1}{l}} d\lambda(t)\right)^{\zeta} = \lambda(t)^{l-\zeta} y(t)^\zeta \quad (10)$$

Now let

$$y(t) = \int_0^t f\left(\left|x'(t)R(t)\right|\right)^{\frac{1}{l}} \lambda'(t) \quad (11)$$

then $y'(t) = f\left(\left|x'(t)R(t)\right|\right)^{\frac{1}{l}} \lambda'(t)$ (12)

that is $y'(t)^l = f\left(\left|x'(t)R(t)\right|\right)\lambda'(t)^l$ (13)

using the fact that $f(u) = u'$ to have

$$y'(t)^l = \left|x'(t)\right| R(t)^l \lambda'(t)^l \quad (14)$$

$$\left|x'(t)\right| = R(t)^{-1}\lambda'(t)^{-1} y'(t) \quad (15)$$

Combining both (10) and (15) yields, inequality (6) and the proof is complete.

Remarks 1:

By setting $f(u) = u', R(t) = P(t)^{-\frac{1}{k-1}}, \lambda'(t) = P(t)^{\frac{1}{k-1}}, \zeta = l$ in lemma 1 yields

$$\left|x'(t)\right| \times f\left(\left|\int_0^t x'(t)P(t)^{-\frac{1}{k-1}} P(t)^{\frac{1}{k-1}}\right| dt\right) \le \lambda(t)^{l-1} y(t)^l \times P(t)^{-\frac{1}{k-1}} P(t)^{\frac{1}{k-1}} y'(t). \quad (16)$$

Integrating both sides of inequality (16) over $[a,b]$ with the respect to t, to get

$$\int_a^b \left|x'(t)\right| \times f\left(\int_0^t \left|x'(t)\right| dt\right) \le \int_a^b y(t)' y'(t)dt \quad (17)$$

That is

$$\int_a^b \left|x'(t)\right| \times \left(\int_0^t \left|x'(t)\right| dt\right)^l \le \int_a^b y(t)^l y'(t)dt = F(y(b)) - F(y(a)). \quad (18)$$

If $y(a) = 0,$ then inequality (18) becomes

$$\int_a^b \left|x'(t)\right| \times \left(\int_0^t \left|x'(t)\right| dt\right)^l \le \int_a^b y(t)^l y'(t)dt = F(y(b)). \quad (19)$$

By using Holders inequality with o and ρ we obtain

$$y(b) = \int_a^b \left|x'(t)\right| dt = \int_a^b R^{-\frac{1}{o}}(t) R^{\frac{1}{\rho}} \left|x'(t)\right|(t)dt \le \left(\int_a^b R^{1-o}(t)dt\right)^{\frac{1}{o}} \left(\int_a^b R(t)\left|x'(t)\right|^\rho dt\right)^{\frac{1}{\rho}} \quad (20)$$

Combing inequality (19) and (20) to obtain inequality (3) if in inequality (20) $\mu = o$ and $\upsilon = \rho$ which is our desired result.

Furthermore, we need the following Lemma to obtain a generalization of Maroni.

Lemma 2:

Let $x(t), \lambda(t), f(u), R(t), l, k, o$ and ρ be as in Lemma 1 such that

$$|x'(t)| \times f\left(\left|\int_0^t x'(t)R(t)\right| d\lambda(t)\right) \le \lambda(t)^{\frac{1-\zeta}{l}} y(t)^{\frac{\zeta}{l}} \times R(t)^{-1} \lambda'(t)^{-1} y'(t). \quad (21)$$

Then, the following inequality holds:

$$\int_a^b |x'(t) f(t)| dt \le \int_a^b y(t) y'(t) dt = \frac{1}{2}\left(\int_a^b |y(t)| dt\right)^2 \quad (22)$$

Proof:
The proof is similar to the proof of lemma 1.
Since $f(u) = u^l$, inequality (10) becomes

$$\left(\left|\int_0^t x'(t)R(t)d\lambda(t)\right|\right)^l \le \lambda(t)^{l-\zeta} y(t)^{\zeta} \quad (23)$$

$$\left(\left|\int_0^t x'(t)R(t)d\lambda(t)\right|\right) \le \lambda(t)^{\frac{l-\zeta}{l}} y(t)^{-} \quad (24)$$

Combining (15) and (24) to obtain the inequality (21)
This completes the proof of the Lemma.
Consider all conditions of remark 1

$$|x'(t)| \times \left(\left|\int_0^t x'(t)P(t)^{-\frac{1}{k-1}} P(t)^{\frac{1}{k-1}}\right| dt\right) \le \lambda(t)^{\frac{l-l}{l}} y(t)^{\frac{l}{l}} \times P(t)^{-\frac{1}{k-1}} P(t)^{\frac{1}{k-1}} y'(t). \quad (25)$$

$$|x'(t)|\left(\int_0^t |x'(t)| dt\right) \le y(t) y'(t) \quad (26)$$

Putting $\int_0^t |x'(t)| dt = x(t)$ and integrate both side of inequality (26) over $[a,b]$ with the respect to t obtain

$$\int_a^b |x'(t)x(t)| dt \le \int_a^b y(t) y'(t) dt = \frac{1}{2}\left(\int_a^b |x(t)| dt\right)^2 \quad (27)$$

$$= \frac{1}{2}\left(\int_0^t |x'(t)| P(t)^{-\frac{1}{o}} P(t)^{\frac{1}{o}} dt\right)^2 \quad (28)$$

$$= \frac{1}{2}\left(\int_a^b P(t)^{1-\rho} dt\right)^{\frac{2}{\rho}} \left(\int_a^b |x'(t)| P(t)^{\frac{1}{o}} dt\right)^{\frac{2}{o}} \quad (29)$$

This is the generalization of inequality (2).

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