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On generalized Weyl's type theorem

A. Babbah[1], M. Zohry[2]
[1,2]*(University Abdelmalek Essaadi, Faculty of Sciences, Mathematics Department,BP. 2121, Tetouan, Morocco)*

Abstract: *It is shown that if a bounded linear operator T or its adjoint T* has the single-valued extension property, then generalized Browder's theorem holds for f(T) for every f ∈ H(σ(T)). We establish the spectral theorem for the B-Weyl spectrum which generalizes [15, Theorem 2.1] and we give necessary and sufficient conditions for such operator T to obey generalized Weyl's theorem.*
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I. INTRODUCTION AND NOTATIONS

Let X denote an infinite-dimensional complex Banach space and $L(X)$ the unital (with unit the identity operator, I, on X) Banach algebra of bounded linear operators acting on X. For an operator $T \in L(X)$ write T* for its adjoint, N(T) for its null space, R(T) for its range, σ(T)for its spectrum, $\sigma_{su}(T)$ for its surjective spectrum, $\sigma_a(T)$ for its approximate point spectrum, $\alpha(T)$ for its nullity and $\beta(T)$ for its defect.

T is called an upper semi-Fredholm (resp. a lower semi-Fredholm) operator if the range R(T) of T is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$). A semi-Fredholm operator is an upper or a lower semi-Fredholm operator. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a Fredholm operator and the index of T is defined by $ind(T) = \alpha(T) - \beta(T)$.

For a T-invariant closed linear subspace Y of X, let T/Y denote the operator given by the restriction of T to Y.

For a bounded linear operator T and for each integer n, define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into itself. If for some integer n the range $R(T^n)$ is closed and $T_n = T/R(T^n)$ is a Fredholm (resp. semi-Fredholm) operator, then T is called a B-Fredholm (resp. semi-B-Fredholm) operator. In this case, from [3, Proposition 2.1] T_m is a Fredholm operator and $ind(T_m) = ind(T_n)$ for each $m \succ n$. This permits to define the index of a B-Fredholm operator T as the index of the Fredholm operator T_n where, n is any integer such that $R(T^n)$ is closed and T_n is a Fredholm operator. It is shown (see [2, Theorem 3.2]) that if S and T are two commuting B-Fredholm operators then the product ST is a B-Fredholm operator and $ind(ST) = ind(S) + ind(T)$. Let $BF(X)$ be the class of all B-Fredholm operators and $\rho_{BF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \in BF(X)\}$ be the B-Fredholm resolvent of T and let $\sigma_{BF}(T) = \mathbb{C} \setminus \rho_{BF}(T)$ be the B-Fredholm spectrum of T. The class $BF(X)$ has been studied by M. Berkani (see [3, Theorem 2.7]) where it was shown that an operator $T \in L(X)$ is a B-Fredholm operator if and only if $T = S_0 \oplus S_1$ where S_0 is a Fredholm operator and S_1 is a nilpotent one. He also proved that $\sigma_{BF}(T)$ is a closed subset of $\mathbb{C}$ contained in the spectrum $\sigma(T)$ and showed that the spectral mapping theorem holds for $\sigma_{BF}(T)$, that is, $f(\sigma_{BF}(T)) = \sigma_{BF}(f(T))$ for any complex-valued analytic function on a neighborhood of $\sigma(T)$ (see [3, Theorem 3.4]). From [21] we recall that for $T \in L(X)$, the ascent $a(T)$ and the descent $d(T)$ are given by

$$a(T) = \inf\{n \geq 0 : N(T^n) = N(T^{n+1})\}$$

And

$$d(T) = \inf\{n \geq 0 : R(T^n) = R(T^{n+1})\}$$

respectively, where the infinum over the emptyset is taken to be ∞. If $a(T)$ and $d(T)$ are both finite the $a(T)=d(T)=p$, $X=N(T^p)\oplus R(T^p)$ and $R(T^p)$ is closed.

An operator T ∈ L(X) is called semi-regular if R(T) is a closed space and $N(T)\subseteq R(T^n)$ for every n ∈ N. The semi-regular resolvent set is defined by $s\text{-}reg(T)=\{\lambda\in\mathbb{C}: T-\lambda \text{ is semi-regular}\}$ we note that $s\text{-}reg(T)=s\text{-}reg(T^*)$ is an open subset of $\mathbb{C}$. The semi-B-Fredholm resolvent set of T is given by $\rho_{SBF}(T)=\{\lambda\in\mathbb{C}: T-\lambda \text{ is semi-B-Fredholm}\}$.

We recall that an operator T ∈ L(X) has the single-valued extension property, abbreviated SVEP, if, for every open set U ⊆ C, the only analytic solution $f:U\to X$ of the equation $(T-\lambda I)f(\lambda)=0$ for all $\lambda\in U$ is the zero function on U. We will denote by $H(\sigma(T))$ the set of all complex-valued functions which are analytic on an open set containing $\sigma(T)$. As a consequence of [9, Théorème2.7], we obtain the following result.

Proposition 1 Let T ∈ L(X).

(i) If T has the SVEP then $s\text{-}reg(T)=\rho_a(T)$.

(ii) If T^* has the SVEP then $s\text{-}reg(T)=\rho_{su}(T)$.

For our investigations we need the following result.

Proposition 2 Let T ∈ L(X).

(i) If T has the SVEP then $ind(T)\leq 0$ for every $\lambda\in\rho_{SBF}(T)$.

(ii) If T^* has the SVEP then $ind(T)\geq 0$ for every $\lambda\in\rho_{SBF}(T)$.

Proof. (i) Let $\lambda\in\rho_{SBF}(T)$, then there exists an integer p such that the operator $(T-\lambda I)_{[p]}=(T-\lambda I)_{|R(T-\lambda I)^p}$ is semi-Fredholm.

From the Kato decomposition, there exists $\delta\succ 0$ such that $\{\mu: 0<|\mu-\lambda|<\delta\}\subseteq s\text{-}reg((T-\lambda I)_{[p]})$.

Since T has the single-valued extension property, Proposition 1 implies that $s\text{-}reg((T-\lambda I)_{[p]})=\rho_a((T-\lambda I)_{[p]})$. Therefore one verify that $N((T-\lambda I)_{[p]}-\mu)=0$ and so $ind(T-\mu)=ind((T-\lambda I)_{[p]}-\mu)\leq 0$, holding for $0<|\mu-\lambda|<\delta$

Thus, by the continuity of the index, $ind(T)\leq 0$.

(ii) This is included in part (i) since $ind(T^*)=-ind(T)$.

An operator $T\in L(X)$ is said to be Weyl if it is Fredholm of index zero and Browder if it is Fredholm of finite ascent and descent. The essentiel spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of T are defined by

$\sigma_e(T)=\{\lambda\in\mathbb{C}: T-\lambda \text{ is not Fredholm}\}$;

$\sigma_w(T)=\{\lambda\in\mathbb{C}: T-\lambda \text{ is not Weyl}\}$;

$\sigma_b(T)=\{\lambda\in\mathbb{C}: T-\lambda \text{ is not Browder}\}$.

It is well known that $\sigma_e(T)\subseteq\sigma_w(T)\subseteq\sigma_b(T)\subseteq\sigma(T)$.

An operator $T\in L(X)$ is called B-Weyl if it is B-Fredholm of index zero. The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined by

$\sigma_{BW}(T) = \{\lambda \in C : T - \lambda I$ is not B-Fredholm of index $0\}$.

For a subset K of C, we shall write iso(K) for its isolated points. A complex number λ_0 is said to be Riesz point of T in $L(X)$ if $\lambda_0 \in iso(\sigma(T))$ and the spectral projection corresponding to the set $\{\lambda_0\}$ has finite-dimensional range. The set of all Riesz points of T will be denoted by $\Pi_0(T)$. It is known that if $T \in L(X)$ and $\lambda \in \sigma(T)$ then $\lambda \in \Pi_0(T)$ if and only if $T - \lambda I$ is Fredholm of finite ascent and descent (see [19]). Consequently $\sigma_B(T) = \sigma(T) \setminus \Pi_0(T)$.

Let $\Pi(T)$ denote the set of all poles of the resolvent of T and $E_0(T)$ denote the set $\{\lambda \in iso(\sigma(T)) : 0 < \alpha(T - \lambda I) < \infty\}$. For a normal operator T acting on a Hilbert space H, Berkani [2, Theorem 4.5] showed that $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$ where $E(T)$ is the set of all eigenvalues of T which are isolated in $\sigma(T)$. This result gives a generalization of the classical Weyl's theorem $\sigma_W(T) = \sigma(T) \setminus E_0(T)$.

II. SVEP AND GENERALIZED WEYL'S THEOREM

The concept of Drazin invertibility plays an important role for the class of B-Fredholm operators. From [12] we recall that, for an algebra A with unit 1 we say that an element a ∈ A is Drazin invertible of degre k if there is an element b of A such that $a^k ba = a^k$, $bab = b$, $ab = ba$. The drazin spectrum of a ∈ A is defined by $\sigma_D(a) = \{\lambda \in C : a - \lambda 1$ is not Drazin invertible$\}$. In the case of $A = L(X)$, it is well known that T is Drazin invertible if and only if it has a finite ascent and descent which is also equivalent to the fact that $T = T_0 \oplus T_1$ where T_0 is an invertible operator and T_1 is a nilpotent one, see for instance [12, Proposition 6] and [7, Corollary 2.2].

Recall that $\sigma_W(T) = \bigcap\{\sigma(T + K) : K \in K(X)\}$ where $K(X)$ is the class of all compact operators acting on X.

It was proved in [2, Theorem 4.3] that for T ∈ L(X), $\sigma_{BW}(T) = \bigcap\{\sigma_D(T + F) : F \in F(X)\}$.

Let $T \in L(X)$, we will say that :

(i) T satisfies Weyl's theorem if $\sigma_W(T) = \sigma(T) \setminus E_0(T)$.

(ii) T satisfies generalized Weyl's theorem if $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$.

(iii) T satisfies Browder's theorem if $\sigma_W(T) = \sigma(T) \setminus \Pi_0(T)$.

(iv) T satisfies generalized Browder's theorem if $\sigma_{BW}(T) = \sigma(T) \setminus \Pi(T)$.

Recall from [5] that if $T \in L(X)$ satisfies generalized Weyl's theorem then it also satisfies Weyl's theorem and if T satisfies generalized Browder's theorem then it satisfies Browder's theorem.

We now turn to an another extension of the characterization of operators obeying Weyl's theorem ([1, Theorem4]).

Theorem 3 [4, Theorem 2.5] If $T \in L(X)$ then we have

(i) $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$ if and only if $E(T) = \Pi(T)$.

(ii) $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$ if and only if $\sigma_{BW}(T) = \sigma_D(T)$.

From this theorem we obtain immediately the following corollary.

Corollary 4 Let $T \in L(X)$, then T satisfies generalized Weyl's theorem if and only if $\sigma_{BW}(T) = \sigma(T) \setminus \Pi(T)$ and $E(T) = \Pi(T)$.

In [15, Theorem 2.1] it is proved that if either an operator T on an infinite dimensional separable Hilbert space or its Hilbert adjoint has the single-valued extension property, then the spectral mapping theorem holds for B-Weyl spectrum. Using a standard argument and the Riesz functional calculus, we obtain the same result for operators on infinite dimensional Banach spaces with a simple and short proof.

Proposition 5 Let $T \in L(X)$, then $\sigma_{BW}(f(T)) \subset f(\sigma_{BW}(T))$ for every $f \in H(\sigma(T))$.

Proof. Let $\lambda \in \sigma_{BW}(f(T))$, then $f(T)-\lambda I$ is not a B-Weyl's operator. As $\sigma_{BW}(f(T)) \subset \sigma(f(T)) = f(\sigma(T))$, there exists $\mu \in \sigma(T)$ such that $\lambda = f(\mu)$.
We have [illegible] where g is a non vanishing analytic function on $\sigma(T)$. So [illegible].
Since $f(T)-\lambda I$ is not a B-Weyl operator, and
[illegible],
there exists $\beta \in \{\mu, \mu_1, \cdots, \mu_n\}$ such that $T - \beta I$ is not a B-Weyl operator and since $f(\beta) = \lambda$ we get $\beta \in \sigma_{BW}(T)$.

The opposite inclusion does not hold in general. Furthermore if f is injective on $\sigma_{BW}(T)$, the last inclusion becomes an equality.

The proof of the next result is similar to that one involving $\sigma_w(T)$ (see [14, Theorem 3]).

Theorem 6 Let $T \in L(X)$, if $f \in H(\sigma(T))$ is injective on $\sigma_{BW}(T)$ then $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$.

Let $BW(X)$ be the class of $T \in L(X)$ such that $ind(T-\lambda I) \leq 0$ for all $\lambda \in \rho_{BF}(T)$ or $ind(T-\lambda I) \geq 0$ for all $\lambda \in \rho_{BF}(T)$.

We recall that hyponormals operators on a Hilbert space H lie in $BW(X)$.

The following result shows that, for operators lying in the class $BW(X)$, the spectral mapping theorem for complex polynomials implies the spectral mapping one for complex-valued analytic functions.

Theorem 7 For $T \in L(X)$ verifying the single-valued extension property, the following assertions are equivalent :
(i) $T \in BW(X)$.
(ii) $f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$ for all $f \in H(\sigma(T))$.
(iii) $p(\sigma_{BW}(T)) = \sigma_{BW}(p(T))$ for all complex polynomial p. _

Proof. *(i)=⇒(ii)* [22,Théorème 2.2.4] implies that $f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$ for all $f \in H(\sigma(T))$.
(ii)=⇒(iii) Clear.
(iii)=⇒(i) Assume that $T \notin BW(X)$. Then there are λ, μ in $\rho_{BF}(T)$ such that $ind(T-\lambda I) \succ 0$ and $ind(T-\lambda I) \prec 0$. If we consider $ind(T-\lambda I) = k$ and $-ind(T-\mu I) = l$ and the polynomial $p(t) = (t-\lambda)^l (t-\mu)^k$, then $p(T)$ is a B-Fredholm operator with $ind(p(T)) = lk - lk = 0$ thus $0 \notin \sigma_{BW}(p(T))$. Since $\lambda \in \sigma_{BW}(T)$ we get [illegible] a contradiction.

Proposition 8 If T or T^* has the single-valued extension property, then
$f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$ for any $f \in H(\sigma(T))$.

Proof. Let $f \in H(\sigma(T))$. If T or T^* has the SVEP, by Proposition 2, T lies in BW(X) and Theorem 7 concludes the proof.

Let $T \in L(X)$, the analytical core of T is the subspace, $K(T)$, defined below

$$K(T)=\{x\in X : \exists (x_n)\subset X,\ \delta>0,\ x_0=x,\ Tx_{n+1}=x_n,\ \|x_n\|\le \delta^n\|x\|\ \forall n\in\mathbb{N}\}$$

The quasi-nilpotent part of T is the subspace

$$H_0(T)=\{x\in X : \lim_{n\to\infty}\|T^n x\|^{\frac{1}{n}}=0\}$$

Both subspaces, will be of particular importance in what follows, they have been introduced and studied by Mbekhta (see [8–10]). In general neither $H_0(T)$ nor $K(T)$ is closed. The following facts are easy to verify; $T(K(T)) = K(T)$, $\ker(T^m) \subseteq H_0(T)$ for every $m \in \mathbb{N}$; if $x \in X$, then $x \in H_0(T)$ if and only if $Tx \in H_0(T)$. If T is invertible then $H_0(T) = \{0\}$.

Theorem 9 [8, Theorem 1.6] Let $T \in L(X)$, the following conditions are equivalent.
(i) λ is an isolated point of $\sigma(T)$.
(ii) $X = H_0(T-\lambda I) \oplus K(T-\lambda I)$, where $H_0(T-\lambda I) \neq \{0\}$ and the direct sum is topological.
Moreover, λ is a pole of the resolvent, $\rho(T)$, of T of order p if and only if
$H_0(T-\lambda I) = N((T-\lambda I)^p)$ and $K(T-\lambda I) = R((T-\lambda I)^p)$.

Our next goal is to show that generalized Browder's theorem is satisfied for $f(T)$ whenever T or T^* has the single-valued extension property and f in $H(\sigma(T))$. The same result was showed in [6, Theorem 1.5] for the generalized a-Browder theorem. To settle our result, we use a characterization of the pole of the resolvent in terms of ascent and descent given in [13].

Remark. It is shown in [18, Theorem 4.18] that if T verifie the single-valued extension property, then for any analytical function on an open neighbourhood of $\sigma(T)$, $f(T)$ verifie the single-valued extension property.

Theorem 10 If $T \in L(X)$ or its adjoint has the single-valued extension property, then generalized Browder's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

Proof. Assume that $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$, so $T - \lambda I$ is B-Weyl, hence B-Fredholm of index 0 and by [17, Theorem 1.82], $T - \lambda I$ is Kato type. Since T or T^* verifie SVEP, [17, Corollary 2.49] implies that $a(T-\lambda I) = d(T-\lambda I) < \infty$. Then $\lambda \in \Pi(T)$ and $\sigma(T)\setminus\Pi(T) \subseteq \sigma_{BW}(T)$. Conversely, if $\lambda \in \Pi(T)$ then λ is isolated in $\sigma(T)$ and by [4, Theorem 2.3], $T - \lambda I$ is B-Weyl, that is $\lambda \notin \sigma_{BW}(T)$ and $\sigma(T)\setminus\Pi(T) \supseteq \sigma_{BW}(T)$. Now if $f \in H(\sigma(T))$, by the last remark and the fact that $f(T^*) = [f(T)]^*$, $f(T)$ or $f(T^*)$ verifie SVEP and consequently we obtain $\sigma_{BW}(f(T)) = \sigma(f(T)) \setminus \Pi(f(T))$.

From this theorem we obtain immediately the following corollary.
Corollary 11 If $T \in L(X)$ or its adjoint T^* has the SVEP, then

(i) Generalized-Weyl's theorem holds for T if and only if $\Pi(T)=E(T)$.
(ii) Generalized-Weyl's theorem holds for T^* if and only if $\Pi(T^*)=E(T^*)$.

The next result rewrite some results due to C. Schmoeger [13] as follows.

Proposition 12 Let T ∈ L(X), the following conditions are equivalent
(i) $\lambda\in\Pi(T)$.
(ii) $\lambda\in E(T)$ and there exists an integer $p\geq 1$ for which $H_0(T-\lambda I)=N((T-\lambda I)^p)$.
(iii) $\lambda\in E(T)$ and there exists an integer $p\geq 1$ for which $K(T-\lambda I)=R((T-\lambda I)^p)$.
(iv) $\lambda\in E(T)$ and $T-\lambda I$ is of finite descent.

Proof. Without loss of generality we can assume that $\lambda=0$.
(i)⇒(ii) Since 0 is a pole of the resolvent of T of order p, it is an eigenvalue of T and an isolated point of the spectrum of T. Hence $0\in E(T)$. Finally by Theorem 9 $H_0(T)=N(T)^p$.
(ii)⇒(iii) If there exists $p\geq 1$ such that $H_0(T)=N(T^p)$ and $0\in E(T)$ from[8, Théorème 1.6] we have $X=H_0(T)\oplus K(T)$. Then one obtain $R(T^p)=T^p(H_0(T))\oplus T^p(K(T))$ and sinc $H_0(T)=N(T^p)$, $T(K(T))=K(T)$ follows that $K(T)=R(T^p)$.
(iii)⇒(iv) If there exists $p\geq 1$ such that $K(T)=R(T^p)$, since $T(K(T))=K(T)$ it follows that [illegible] and $d(T)\prec\infty$.
(iv)⇒(i) Suppose that $0\in E(T)$ and $d(T)\prec\infty$. Since 0 is isolated in $\sigma(T)$, by [13, Theorem 4] $X=H_0(T)\oplus K(T)$ and $H_0(T)\neq\{0\}$ is closed. Hence by [13, Theorem 2(b)] T has the SVEP at 0 and finally [13, Theorem 5] gives $0\in\Pi(T)$.

The following theorem follows immediately from Corollary 11 and Proposition 12.

Theorem 13 Let $T\in L(X)$ such that T or its adjoint T^* has the single-valued extension property then the following conditions are equivalent:
(i) Generalized Weyl's theorem holds for T.
(ii) $\forall\lambda\in E(T)$ there exists $p\geq 1$ for which $H_0(T-\lambda I)=N((T-\lambda I)^p)$.
(iii) $\forall\lambda\in E(T)$ there exists $p\geq 1$ for which $K(T-\lambda I)=R((T-\lambda I)^p)$.
(iv) $\forall\lambda\in E(T)$, $T-\lambda I$ is of finite descent.

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