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On I_2-Cauchy Double Sequences in p-Adic Linear 2-Normed Spaces

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Abstract: *In this paper, we introduce the concept of $I_2^* -$convergence which is closely related to $I_2 -$convergence and the concepts I_2 and $I_2^* -$Cauchy double sequences in p-adic linear 2-normed space. Also we investigate the relation between these concepts in p-adic linear 2-normed spaces.*

Keywords: *2-normed space, p-adic linear 2-normed space, $I_2^* -$convergence, $I_2 -$ Cauchy double sequence, $I_2^* -$Cauchy double sequence.*

I. Introduction

The idea of $I -$convergence is based on the notion of the ideal I of subsets of N, the set of natural numbers. The notion of ideal convergence for single sequences was introduced first by P.Kostyrko et al [11, 12] as an interesting generalization of statistical convergence. F.Nuray and Ruckle [14] independently introduced the same concept as the name generalized statistical convergence.

The concept of a double sequence was initially introduced by Pringsheim [16] in the 1900s and this concept has been studied by many others. A double sequence of real (or complex) numbers is a function from $N \times N$ to F (where F= R or C) and we denote a double sequence as (x_{ij}) where the two subscripts run through the sequence of Natural numbers independent of each other. P.Das et al [4] introduced the concept of $I -$convergence of double sequences in a metric space and studied some properties. Also, Pratulananda Das and Prasanta Malik [15] defined the concept of $I -$limit points, $I -$cluster points, $I -$limit superior and $I -$limit inferior of double sequences. Balakrishna Tripathy, Binod Chandra Tripathy [3] introduced the notion of $I -$convergence and $I -$Cauchy double sequences and many others discussed the properties of $I -$convergence, $I^* -$convergence for double sequences and $I -$Cauchy double sequences (see ([5], [6], [25], for more details).

The concept of linear 2-normed spaces has been investigated by Gähler in 1965 [7] and has been developed extensively in different subjects by others [9, 10, 17]. A.Sahiner et al [20] introduced $I -$cluster points of convergent sequences in 2-normed linear spaces and Gurdal [8] investigated the relation between $I -$cluster points and ordinary limit points of sequences in 2-normed spaces. The concept of $I -$convergence for the double sequences in 2-normed spaces introduced by Saeed Sarabadan and Sorayya Talebi [19]. Saeed Sarabadan, Fatemeh Amoee Arani and Siamak Khalehoghli [18] obtained Condition for the equivalence of I and I*-convergence for double sequences in 2-normed spaces.

Mehmet Acikgoz [13] introduced a very understandable and readable connection between the concepts in *p*-adic numbers, *p*-adic analysis and linear 2-normed spaces. B.Surender Reddy [21] introduced some properties of *p*-adic linear 2-normed spaces and obtained necessary and sufficient conditions for *p*-adic 2-norms to be equivalent on *p*-adic linear 2-normed spaces. Recently B.Surender Reddy and D.Shankaraiah [22, 23, 24] introduced $I -$convergence, $I^* -$convergence of sequences, $I -$Cauchy, $I^* -$Cauchy sequences and their properties in *p*-adic linear 2-normed spaces and also introduced ideal convergence of double sequences in *p*-adic linear 2-normed spaces.

The main aim of this paper is we introduce the concept of $I_2^* -$convergence of double sequences which is closely related to $I_2 -$convergence of double sequences and the concepts $I_2 -$Cauchy double sequence and $I_2^* -$Cauchy double sequence in p-adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$. Also we investigate the relation between these concepts in p-adic linear 2-normed spaces.

II. Preliminaries

In this paper, we will use the notations; p for a prime number, Z - the ring of rational integers, Z^+ - the positive integers, Q - the field of rational numbers, R - the field of real numbers, R^+ - the positive real numbers, Z_p - the ring of *p*-adic rational integers, Q_p - the field of *p*-adic rational numbers, C - the field of complex numbers and C_p - the *p*-adic completion of the algebraic closure of Q_p.

Definition 2.1: A double sequence $x=(x_{ij})$ is said to be convergent to a number ξ in the Pringsheim's sense if for each $\varepsilon>0$ there exists a positive integer m such that $|x_{ij}-\xi|<\varepsilon$ whenever $i,j\geq m$.Then the number ξ is called the Pringsheim limit of the sequence x and we write as $P-\lim_{i,j\to\infty} x_{ij}=\xi$.

Definition 2.2: A double sequence $x=(x_{ij})$ is said to be Cauchy sequence if for each $\varepsilon>0$ there exists a positive integer n_0 such that $|x_{ij}-x_{mn}|<\varepsilon$ for every $i\geq m\geq n_0$ and $j\geq n\geq n_0$.

Definition 2.3: A double sequence $x=(x_{ij})$ is said to be bounded if there exists a real number $M>0$ such that $|x_{ij}|<M$ for each i and j.

Definition 2.4: Let $K\subset N\times N$ and $K(m,n)=\{(i,j):(i,j)\in K; i\leq m, j\leq n\}$. If the sequence $\left\{\frac{K(m,n)}{mn}\right\}$ has a limit in Pringsheim's sense then we say that K has a double natural density and it is denoted as $\lim_{m,n\to\infty}\frac{K(m,n)}{mn}=\delta_2(K)$.

Definition 2.5: A double sequence $x=(x_{ij})$ is said to be statistically convergent to a number ξ if for each $\varepsilon>0$, the set $A(\varepsilon)=\{(i,j)\in N\times N:|x_{ij}-\xi|\geq\varepsilon\}$ has double natural density zero. If $x=(x_{ij})$ is statistically convergent to ξ then we write $St-\lim_{i,j\to\infty} x_{ij}=\xi$.

Definition 2.6: Let I_2 be an ideal in $N\times N$. A double sequence $x=(x_{ij})$ is said to be I_2-convergent to L in Pringsheim's sense if for each $\varepsilon>0$, the set $\{(i,j)\in N\times N:|x_{ij}-L|\geq\varepsilon\}\in I_2$ and L is called I_2-limit of $x=(x_{ij})$ and we write $I_2-\lim_{i,j\to\infty} x_{ij}=L$.

Definition 2.7: A double sequence $x=(x_{ij})$ is said to be I_2^*-convergent to ξ if there exists a set $M=\{(i,j):i,j=1,2,3,.......\}\in F(I_2)$ $(i.e,(N\times N)-M\in I_2)$ such that $\lim_{i,j\to\infty} x_{ij}=\xi$ and ξ is called I_2^*-limit of $x=(x_{ij})$ and we write $I_2^*-\lim x_{ij}=\xi$.

Definition 2.8: A double sequence $x=(x_{ij})$ is I_2-convergent to zero in Pringsheim's sense is called I_2-null double sequence in Pringsheim's sense.

Definition 2.9: Let *X* be a linear space of dimension greater than 1 over *K*, where *K* is the real or complex numbers field. Suppose $N(\bullet,\bullet)$ be a non-negative real valued function on $X\times X$ satisfying the following conditions:

$(2-N_1)$: $N(x,y)>0$ and $N(x,y)=0$ if and only if *x* and *y* are linearly dependent vectors,

$(2-N_2)$: $N(x,y)=N(y,x)$ for all $x,y\in X$,
$(2-N_3)$: $N(\lambda x,y)=|\lambda|N(x,y)$ for all $\lambda\in K$ and $x,y\in X$,
$(2-N_4)$: $N(x+y,z)\le N(x,z)+N(y,z)$ for all $x,y,z\in X$.
Then $N(\bullet,\bullet)$ is called a 2-norm on *X* and the pair $(X,N(\bullet,\bullet))$ is called a linear 2-normed space.

Definition 2.10: Suppose a mapping $d_p: X\times X\times X\to R$ on a non-empty set X satisfying the following conditions, for all $x,y,z\in X$

D_1) For any two different elements x and y in X there is an element z in X such that $d_p(x,y,z)\ne 0$.
D_2) $d_p(x,y,z)=0$ when two of three elements are equal.
D_3) $d_p(x,y,z)=d_p(x,z,y)=d_p(y,z,x)$.
D_4) $d_p(x,y,z)\le d_p(x,y,w)+d_p(x,w,z)+d_p(w,y,z)$ for any w in X . Then d_p is called *p*-adic 2-metric on X and the pair (X, d_p) is called *p*-adic 2-metric space. If *p*-adic 2-metric also satisfies the condition $d_p(x,y,z)\le \max\{d_p(x,y,w),d_p(x,w,z),d_p(y,w,z)\}$ for $x,y,z,w\in X$, then d_p is called a *p*-adic ultra 2-metric and the pair (X,d_p) is called a *p*-adic ultra 2-metric space.

Definition 2.11: Let *X* be a linear space of dimension greater than 1 over *K*, where *K* is the real or complex numbers field. Suppose $N(\bullet,\bullet)_p$ be a non-negative real valued function on $X\times X$ satisfying the following conditions:
$(2-pN_1)$: $N(x,z)_p=0$ if and only if *x* and *z* are linearly dependent vectors.
$(2-pN_2)$: $N(xy,z)_p=N(x,z)_p.N(y,z)_p$ for all $x,y,z\in X$,
$(2-pN_3)$: $N(x+y,z)_p\le N(x,z)_p+N(y,z)_p$ for all $x,y,z\in X$,
$(2-pN_4)$: $N(\lambda x,z)_p=|\lambda|\,N(x,z)_p$ for all $\lambda\in K$ and $x,z\in X$.
Then $N(\bullet,\bullet)_p$ is called a *p*-adic 2-norm on *X* and the pair $(X,N(\bullet,\bullet)_p)$ is called *p*-adic linear 2-normed space.

For every *p*-adic linear 2-normed space $(X,N(\bullet,\bullet)_p)$ the function defined on $X\times X\times X$ by $d_p(x,y,z)=N(x-z,y-z)_p$ is a *p*-adic 2-metric. Thus every *p*-adic linear 2-normed space $(X,N(\bullet,\bullet)_p)$ will be considered to be a *p*-adic 2-metric space with this 2-metric. A double sequence (x_{ij}) of *p*-adic 2-metric space (X,d_p) converges to $x\in X$ in *p*-adic 2-metric if for every $\varepsilon>0$, there is an $l\ge 1$ such that $d_p(x_{ij},x,z)=N(x_{ij}-z,x-z)_p<\varepsilon$ for every $i,j\ge l$. For the given two double sequences of *p*-adic 2-metric space (X,d_p) which are (x_{ij}) and (y_{ij}) converges to $x,y\in X$ in the *p*-adic 2-metric space respectively, then the double sequence of sums $x_{ij}+y_{ij}$ and the product $x_{ij}y_{ij}$ converges to the sum $x+y$ and to the product xy of the limits of initial double sequences.

A double sequence (x_{ij}) of *p*-adic 2-metric space (X,d_p) is a Cauchy double sequence with respect to the *p*-adic 2-metric if for each $\varepsilon>0$, there is an $l\ge 1$ such that $d_p(x_{ij},x_{mn},z)=N(x_{ij}-z,x_{mn}-z)_p<\varepsilon$, for every $i\ge m\ge l$, $j\ge n\ge l$.

Definition 2.12: A double sequence $x=(x_{ij})$ in a $p-$adic linear 2-normed space $(X,N(\bullet,\bullet)_p)$ is said to be convergent to $l\in X$ if for each $\varepsilon>0$ there exists $m\in N$ such that $N(x_{ij}-l,z)_p<\varepsilon$ for each

$i, j \geq m$ and for each $z \in X$. If $x = (x_{ij})$ is convergent to l then we write $\lim_{i,j\to\infty} x_{ij} = l$ or $x_{ij} \xrightarrow{[N(\bullet,\bullet)_p]_X} l$.

Definition 2.13: A double sequence $x = (x_{ij})$ in a $p-$adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ is said to be bounded if for each non zero $z \in X$ and for all $i, j \in N$ there exists $M > 0$ such that $N(x_{ij}, z)_p < M$. Note that a convergent double sequence need not be bounded.

Definition 2.14: A double sequence $x = (x_{ij})$ in a $p-$adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ is said to be Cauchy double sequence if for each $\varepsilon > 0$ there exists a positive integer n_0 such that $N(x_{ij} - x_{mn}, z)_p < \varepsilon$ for every $i \geq m \geq n_0$ and $j \geq n \geq n_0$.

A *p*-adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ is called complete if every Cauchy sequence is convergent in *p*-adic linear 2-normed space. A p-adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ is called *p*-adic 2-Banach space if *p*-adic linear 2-normed space is complete.

Proposition 2.15: If a double sequence $\{x_{ij}\}$ in a *p*-adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ is convergent to $x \in X$, then $\lim_{i,j\to\infty} N(x_{ij}, z)_p = N(x, z)_p$ for each $z \in X$.

Proposition 2.16: If $\lim_{i,j\to\infty} N(x_{ij}, z)_p$ exists then we say that (x_{ij}) is a Cauchy sequence with respect to $N(\bullet,\bullet)_p$.

Proof: Let us suppose that $\lim_{i,j\to\infty} N(x_{ij}, z)_p = x$. Then we can obtain a constant M_1 such that $i, j > M_1$ $\Rightarrow N(x - x_{ij}, z)_p < \frac{\varepsilon}{2}$. If $i, j, m, n > M_1$ then $N(x - x_{ij}, z)_p < \frac{\varepsilon}{2}$ and $N(x - x_{mn}, z)_p < \frac{\varepsilon}{2}$, hence by using the triangle inequality, we have $N(x_{ij} - x_{mn}, z)_p = N(x_{ij} - x + x - x_{mn}, z)_p$ $\leq N(x_{ij} - x, z)_p + N(x - x_{mn}, z)_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

$\Rightarrow (x_{ij})$ is a Cauchy sequence with respect to $N(\bullet,\bullet)_p$.

III. Main Results

In this section, we introduce the concept of I_2^*-convergence of double sequences which is closely related to I_2-convergence of double sequences in p-adic linear 2-normed space ($X, N(\bullet,\bullet)_p$) and we introduce the concepts I_2-Cauchy double sequence and I_2^*-Cauchy double sequence in p-adic linear 2-normed space ($X, N(\bullet,\bullet)_p$). Also we investigate the relation between these concepts in p-adic linear 2-normed spaces.

A family of sets $I \subseteq 2^Y$ (power sets of Y) is said to be an ideal if $\Phi \in I$, I is additive i.e., $A, B \in I \Rightarrow A \bigcup B \in I$ and hereditary i.e., $A \in I, B \subseteq A \Rightarrow B \in I$.

A non empty family of sets $F \subset 2^Y$ is a filter on Y if and only if $\Phi \notin F$, $A \bigcap B \in F$ for each $A, B \in F$, and any subset of an element of F is in F. An ideal I is called non-trivial if $I \neq \Phi$ and $Y \notin I$. Clearly I is a non-trivial ideal if and only if $F = F(I) = \{Y - A : A \in I\}$ is a filter in

Y, called the filter associated with the ideal I. A non-trivial ideal I is called admissible if and only if $\{\{n\} : n \in Y\} \subset I$.

An admissible ideal $I \subset 2^Y$ is said to have the property (AP) if for any sequence $\{A_1, A_2, A_3, ...\}$ of mutually disjoint sets of I there is a sequence $\{B_1, B_2, B_3, ...\}$ of sets such that each symmetric difference $A_i \Delta B_i$, $i = 1,2,3....,$ is finite and $B = \bigcup_{i=1}^{\infty} B_i \in I$.

In order to distinguish between the ideals of N and $N \times N$ we shall denote the ideals of N by I and ideals of $N \times N$ by I_2. In general, there is no connection between I and I_2.

A non trivial ideal I_2 in $N \times N$ is called strongly admissible if $\{i\} \times N$ and $N \times \{i\}$ belong to I_2 for each $i \in N$. It is clear that a strongly admissible ideal is admissible also.

Let $I_0 = \{A \subset N \times N : (\exists m(A) \in N)(i, j \geq m(A) \Rightarrow (i, j) \in (N \times N) - A\}$. Then I_0 is a non trivial strongly admissible ideal and I_2 is strongly admissible if and only if $I_0 \subseteq I_2$. $I_2 \subset 2^{N \times N}$ is a non trivial ideal if and only if the class $F = F(I) = \{(N \times N) - A : A \in I_2\}$ is a filter in $N \times N$.

An admissible ideal $I_2 \subset 2^{N \times N}$ satisfies the condition (AP_2) if for each countable family of disjoint sets $\{A_1, A_2, A_3, ...\}$ belongs to I_2, there exists a countable family of sets $\{B_1, B_2, B_3, ...\}$ such that $A_j \Delta B_j$ is included in the finite union of rows and columns in $N \times N$ for each $j \in N$ and $B = \bigcup_{j=1}^{\infty} B_j \in I_2$ (hence $B_j \in I_2$ for each $j \in N$).

Definition 3.1: A double sequence $x = (x_{ij})$ in a $p-$adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ is said to be I_2-convergent to $l \in X$ if for each $\varepsilon > 0$ and non zero $z \in X$, the set $A(\varepsilon) = \{(i, j) \in N \times N : N(x_{ij} - l, z)_p \geq \varepsilon\} \in I_2$ and l is called the I_2-limit of the sequence $x = (x_{ij})$.
If $x = (x_{ij})$ is I_2-convergent to l, then we write $I_2 - \lim_{i,j \to \infty} x_{ij} = l$ or $I_2 - \lim_{i,j \to \infty} N(x_{ij} - l, z)_p = 0$ or $I_2 - \lim_{i,j \to \infty} N(x_{ij}, z)_p = N(l, z)_p$ for each non zero $z \in X$.

Now introducing the definition of I_2^*-convergence for double sequence $x = (x_{ij})$ which is closely related to I_2-convergence of double sequence $x = (x_{ij})$ in a $p-$adic linear 2-normed space as follows.

Definition 3.2: A double sequence $x = (x_{ij})$ in a $p-$adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ is said to be I_2^*-convergent to $l \in X$ if there exists a set $M \in F(I_2)$ $(i.e., (N \times N) - M \in I_2)$ such that $\lim_{i,j \to \infty} x_{ij} = l$ for $(i, j) \in M$ and l is called the I_2^*-limit of the sequence $x = (x_{ij})$.
If $x = (x_{ij})$ is I_2^*-convergent to l, then we write $I_2^* - \lim_{i,j \to \infty} x_{ij} = l$ or $I_2^* - \lim_{i,j \to \infty} N(x_{ij} - l, z)_p = 0$ or $I_2^* - \lim_{i,j \to \infty} N(x_{ij}, z)_p = N(l, z)_p$ for each non zero $z \in X$.

Lemma 3.3: Let $\{P_i\}_{i=1}^{\infty}$ be a countable collection of subsets of $N \times N$ such that $\{P_i\}_{i=1}^{\infty} \in F(I_2)$ for each i, where $F(I_2)$ is a filter associated with a strongly admissible ideal I_2 with property (AP$_2$). Then there is a set $P \subset N \times N$ such that $P \in F(I_2)$ and the set $P - P_i$ is finite for all i.

Lemma 3.4: Let $I_2 \subset 2^{N \times N}$ be a strongly admissible ideal and $(X, N(\bullet,\bullet)_p)$ be a $p-$adic linear 2-normed space. If $I_2^* - \lim_{i,j\to\infty} x_{ij} = l$, then $I_2 - \lim_{i,j\to\infty} x_{ij} = l$ for each non zero $z \in X$.

Proof: Suppose $I_2^* - \lim_{i,j\to\infty} x_{ij} = l$. Then there exists $H \in I_2$ such that for $M = (N \times N) - H \in F(I_2)$ we have

$$\lim_{i,j\to\infty} x_{ij} = l,\ (i,j) \in M, \quad \text{for each non zero } z \in X \tag{3.5}$$

Let $\varepsilon > 0$. By virtue of equation (3.5) there exists a positive integer n_1 such that $N(x_{ij} - l, z)_p < \varepsilon$ for every $(i,j) \in M$ with $i, j \geq n_1$.

Let $A = \{1,2,3,......, n_1 - 1\}$, $B = \{(i,j) \in M : N(x_{ij} - l, z)_p \geq \varepsilon\}$. Then it is clear that $B \subset (A \times N) \bigcup (N \times A)$ and therefore $B \in I_2$. Obviously the set $\{(i,j) \in N \times N : N(x_{ij} - l, z)_p \geq \varepsilon\} \subset B \bigcup H$ and therefore the set $\{(i,j) \in N \times N : N(x_{ij} - l, z)_p \geq \varepsilon\} \in I_2$ for each non zero $z \in X$. This implies that $I_2 - \lim_{i,j\to\infty} x_{ij} = l$, for each non zero $z \in X$.

Theorem 3.6: Let $I_2 \subset 2^{N \times N}$ be a strongly admissible ideal with property (AP_2) and $(X, N(\bullet,\bullet)_p)$ be a $p-$adic linear 2-normed space. Then for an arbitrary double sequence $x = (x_{ij})$ of elements of X, if $I_2 - \lim_{i,j\to\infty} x_{ij} = l$ then $I_2^* - \lim_{i,j\to\infty} x_{ij} = l$.

Proof: Suppose that $I_2 - \lim_{i,j\to\infty} x_{ij} = l$. Then for any $\varepsilon > 0$ and non zero $z \in X$, $A(\varepsilon) = \{(i,j) \in N \times N : N(x_{ij} - l, z)_p \geq \varepsilon\} \in I_2$.

Now put $A_1 = \{(i,j) \in N \times N : N(x_{ij} - l, z)_p \geq 1\}$ and $A_k = \left\{(i,j) \in N \times N : \frac{1}{k} \leq N(x_{ij} - l, z)_p < \frac{1}{k-1}\right\}$ for $k \geq 2$ and for each non zero $z \in X$. It is clear that $A_m \bigcap A_n = \phi$ for $m \neq n$ and $A_m \in I_2$ for each $m \in N$. By virtue of (AP_2) there exists a countable family of sets $\{B_1, B_2, B_3, ...\}$ such that $A_m \Delta B_m$ is a included in finite union of rows and columns in $N \times N$ for each $m \in N$ and $B = \bigcup_{m=1}^{\infty} B_m \in I_2$. Put $M = (N \times N) - B$ and to prove the theorem it is sufficient to prove that $\lim_{i,j\to\infty} x_{ij} = l$ for $(i,j) \in M$.

Let $\delta > 0$. Choose $k \in N$ such that $\frac{1}{k} < \delta$. Then we have

$$\{(i,j) \in N \times N : N(x_{ij} - l, z)_p \geq \delta\} \subseteq \bigcup_{m=1}^{k} A_m \tag{3.7}$$

Since $A_m \Delta B_m$, $m = 1,2,3,.....,k$ is a finite set, there exists $n_0 \in N$ such that $\left(\bigcup_{m=1}^{k} B_m\right) \bigcap \{(i,j) : i, j \geq n_0\} = \left(\bigcup_{m=1}^{k} A_m\right) \bigcap \{(i,j) : i, j \geq n_0\}$. If $i, j > n_0$ and $(i,j) \in M$, then $(i, j) \notin B$. This implies that $(i, j) \notin \bigcup_{m=1}^{k} B_m$ and therefore $(i, j) \notin \bigcup_{m=1}^{k} A_m$. Hence for every $i, j > n_0$ and $(i, j) \in M$, we have by equation (3.7), $N(x_{ij} - l, z)_p < \delta$, for each $z \in X$.

$\Rightarrow \lim_{i,j \to \infty} x_{ij} = l$, for $(i, j) \in M$.

Thus $I_2^* - \lim_{i,j \to \infty} x_{ij} = l$.

From the Lemma (3.4) and Theorem (3.6) we obtain the following Lemma which gives the equivalence between I_2-convergence and $I_2^* -$convergence in $p-$adic linear 2-normed spaces.

Lemma 3.8: Let $I_2 \subset 2^{N \times N}$ be a strongly admissible ideal with the property (AP_2) and $(X, N(\bullet,\bullet)_p)$ be a $p-$adic linear 2-normed space. Then a double sequence in X is I_2-convergent to l in X if and only if it is $I_2^* -$convergent to l in X.

Definition 3.9: Let $(X, N(\bullet,\bullet)_p)$ be a $p-$adic linear 2-normed space and $x \in X$. Then x is called accumulation point of X if there exists a sequence (x_k) of distinct elements of X such that $x_k \neq x$ for any k and $x_n \xrightarrow{N(\bullet,\bullet)_p} x$.

Theorem 3.10: Let $I_2 \subset 2^{N \times N}$ be a strongly admissible ideal and $(X, N(\bullet,\bullet)_p)$ be a $p-$adic linear 2-normed space. If X has at least one accumulation point for any arbitrary double sequence $x = (x_{ij})$ in X and for each $a \in X$, $I_2 - \lim_{i,j \to \infty} x_{ij} = a$ implies $I_2^* - \lim_{i,j \to \infty} x_{ij} = a$, then I_2 has the property (AP_2).

Proof: Suppose that $a \in X$ is accumulation point of X. Then there is a sequence (b_k) of distinct elements of X such that $b_k \neq a$ for any k, and $b_k \xrightarrow{N(\bullet,\bullet)_p} a$.

Put $\varepsilon_k^{(z)} = N(b_k - a, z)_p$ for $k \in N$. Let $(A_j)_{j \in N}$ be a disjoint family of non empty sets from I_2. Define a sequence (x_{mn}) for all $z \in X$ such that

(i) . $x_{mn} = b_j$ if $(m,n) \in A_j$ and

(ii) . $x_{mn} = a$ if $(m,n) \notin A_j$ for any j.

Let $\delta > 0$ be given and $z_0 \in X$. Choose $k \in N$ such that $\varepsilon_k^{(z_0)} < \delta$. Then we have $A^{z_0}(\delta) = \{(m,n) : N(x_{mn} - a, z_0)_p \geq \delta\} \subseteq A_1 \cup A_2 \cup .... \cup A_k$. Hence $A^{z_0}(\delta) \in I_2$ and so $I_2 - \lim_{m,n \to \infty} x_{mn} = a$.

By virtue of our assumption, we have $I_2^* - \lim_{m,n \to \infty} x_{mn} = a$. Hence there exists a set $B \in I_2$ such that

$$M = (N \times N) - B \in F(I_2) \text{ and } \lim_{m,n \to \infty} x_{mn} = a, (m,n) \in M \tag{3.11}$$

Let $B_j = A_j \cap B$ for $j \in N$. Then $B_j \in I_2$ for each $j \in N$ and $\bigcup_{j=1}^{\infty} B_j = B \cap \left(\bigcup_{j=1}^{\infty} A_j \right) \subseteq B$ and so $\bigcup_{j=1}^{\infty} B_j \in I_2$, fix $j \in N$.

If $A_j \bigcap M$ is not included in the finite union of rows and columns in $N \times N$, then M must contain an infinite sequence of elements (m_k, n_k) and $\lim_{m_k, n_k \to \infty} x_{m_k n_k} = b_j \neq a$ for all $k \in N$, which contradicts equation (3.11). Hence $A_j \bigcap M$ must be contained in the finite union of rows and columns in $N \times N$. Thus

$A_j \Delta B_j = (A_j - B_j) \bigcup (B_j - A_j)$

$= A_j - B_j$ since $B_j - A_j = \phi$

$= A_j - (A_j \bigcap B)$

$= A_j - B = A_j \bigcap M$ is also included in the finite union of rows and columns. Thus the ideal I_2 has the property (AP_2).

Definition 3.12: Let $I_2 \subset 2^{N \times N}$ be a strongly admissible ideal and $(X, N(\bullet,\bullet)_p)$ be a $p-$adic linear 2-normed space. A double sequence $x = (x_{ij})$ in X is said to be $I_2 -$Cauchy double sequence in X, if for each $\varepsilon > 0$ and non zero $z \in X$, there exists $m = m(\varepsilon, z), n = n(\varepsilon, z) \in N$ such that $\{(i, j) \in N \times N : N(x_{ij} - x_{mn}, z)_p \geq \varepsilon\} \in I_2$.

Definition 3.13: Let $I_2 \subset 2^{N \times N}$ be a strongly admissible ideal and $(X, N(\bullet,\bullet)_p)$ be a $p-$adic linear 2-normed space. A double sequence $x = (x_{ij})$ in X is said to be $I_2^* -$Cauchy double sequence in X, if for each $\varepsilon > 0$ and non zero $z \in X$, there exists a set $M \in F(I_2)$ $(i.e., \mathrm{H} = (N \times N) - M \in I_2)$ such that the double sequence $(x_{mn})_{(m,n) \in M}$ is a Cauchy sequence in X. i.e., $\lim_{i,j,m,n \to \infty} N(x_{ij} - x_{mn}, z)_p = 0$ for each non zero $z \in X$ and $(i, j), (m, n) \in M$.

Theorem 3.14: Let $I_2 \subset 2^{N \times N}$ be a strongly admissible ideal and $(X, N(\bullet,\bullet)_p)$ be a $p-$adic linear 2-normed space. If a double sequence $x = (x_{ij})$ is $I_2^* -$Cauchy double sequence in X, then it is $I_2 -$Cauchy double sequence in X.

Proof: Suppose $x = (x_{ij})$ is $I_2^* -$Cauchy double sequence a in $p-$adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$. Then there exists a set $M \in F(I_2)$ $(i.e., \mathrm{H} = (N \times N) - M \in I_2)$ such that $N(x_{ij} - x_{mn}, z)_p < \varepsilon$ for every $\varepsilon > 0$ and for all $(\mathrm{i, j}), (\mathrm{m, n}) \in M; \mathrm{i, j, m, n} \geq l$ and $l = l(\varepsilon) \in \mathrm{N}$.

Now $A(\varepsilon) = \{(i, j) \in N \times N : N(x_{ij} - x_{mn}, z)_p \geq \varepsilon\}$

$$\subset H \bigcup \left[M \bigcap \left((\{1,2,3,.....,l-1\} \times N) \bigcup (N \times \{1,2,3,.......,l-1\}) \right) \right] \tag{3.15}$$

Since I_2 is a strongly admissible ideal, therefore

$H \bigcup \left[M \bigcap \left((\{1,2,3,.....,l-1\} \times N) \bigcup (N \times \{1,2,3,.......,l-1\}) \right) \right] \in I_2$.

From equation (3.15) and by the definition of ideal, $A(\varepsilon) \in \mathrm{I}_2$.This shows that the double sequence $x = (x_{ij})$ is $I_2 -$Cauchy double sequence in X.

Now we will prove in the following theorem that $I_2^* -$convergence implies $I_2 -$Cauchy condition in $p -$adic linear 2-normed space.

Theorem 3.16: Let $I_2 \subset 2^{N \times N}$ be a strongly admissible ideal and $(X, N(\bullet,\bullet)_p)$ be a $p-$adic linear 2-normed space. If a double sequence (x_{ij}) in X is $I_2^* -$convergent to $x \in X$, then (x_{ij}) is $I_2 -$Cauchy double sequence in X.

Proof: Suppose a double sequence (x_{ij}) in a $p-$adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ is I_2^*-convergent to $x \in X$. Then there exists a set $M \in F(I_2)$ $(i.e., \mathrm{H} = (N \times N) - M \in I_2)$ such that $\lim_{m,n\to\infty} x_{mn} = x, (m,n) \in M$. It shows that there exists $k_0 = k_0(\varepsilon)$ such that

$$N(x_{mn} - x, z)_p < \frac{\varepsilon}{2}, \text{ for every } \varepsilon > 0, \text{ non zero } z \in X \text{ and } m,n > k_0 \qquad (3.17)$$

For $(i, j),(m,n) \in M$ and $i, j,m,n > k_0$,

$$\begin{aligned} N(x_{ij} - x_{mn}, z)_p &= N(x_{ij} - x + x - x_{mn}, z)_p \\ &= N((x_{ij} - x) - (x_{mn} - x), z)_p \\ &\le N(x_{ij} - x, z)_p + N(x_{mn} - x, z)_p \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad , \quad \text{by equation (3.17)} \\ &= \varepsilon, \text{ for each } \varepsilon > 0, \text{ non zero } z \in X \text{ and } i, j,m,n > k_0. \end{aligned}$$

Therefore $\lim_{i,j,m,n\to\infty} N(x_{ij} - x_{mn}, z)_p = 0$ which implies $(x_{ij})_{(i,j)\in M}$ is a Cauchy double sequence in X and hence $(x_{mn})_{(m,n)\in N\times N}$ is a I_2^*-Cauchy double sequence in X. By Theorem (3.14), $(x_{mn})_{(m,n)\in N\times N}$ is a I_2-Cauchy double sequence in X.

From Theorem (3.16) and Lemma (3.8) we have the following corollary which gives the relation between I_2-convergence and I_2-Cauchy double sequence in a $p-$adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$.

Corollary 3.18: Let $I_2 \subset 2^{N\times N}$ be a strongly admissible ideal with the property (AP_2) and $(X, N(\bullet,\bullet)_p)$ be a $p-$adic linear 2-normed space. If a double sequence (x_{ij}) is I_2-convergent to x in X, then (x_{ij}) is I_2-Cauchy double sequence in X.

Finally, we will give the following Theorem which states the equivalence of I_2-Cauchy double sequence and I_2^*-Cauchy double sequence in a $p-$adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$ in the case I_2 has the property (AP_2).

Theorem 3.19: Let $I_2 \subset 2^{N\times N}$ be a strongly admissible ideal with the property (AP_2) and $(X, N(\bullet,\bullet)_p)$ be a $p-$adic linear 2-normed space. Then a double sequence $x = (x_{ij})$ is I_2-Cauchy double sequence in X if and only if $x = (x_{ij})$ is I_2^*-Cauchy double sequence in X.

Proof: Suppose a double sequence $x = (x_{ij})$ is I_2^*-Cauchy double sequence in a $p-$adic linear 2-normed space $(X, N(\bullet,\bullet)_p)$.Then by Theorem (3.14) , $x = (x_{ij})$ is I_2-Cauchy double sequence in X.

Now it is sufficient to prove that, if a double sequence $x = (x_{ij})$ is I_2-Cauchy double sequence in X, then it is I_2^*-Cauchy double sequence in X .Suppose $x = (x_{ij})$ is I_2-Cauchy double sequence in X. Then there exists $m = m(\varepsilon), n = n(\varepsilon) \in N$ such that

$A(\varepsilon) = \{(m,n) : N(x_{ij} - x_{mn}, z)_p \ge \varepsilon\} \in I_2$ for each $\varepsilon > 0$ and non zero $z \in X$.

Let $P_k = \{(i,j) \in N \times N : N(x_{ij} - x_{m_k n_k}, z)_p < \frac{1}{k}\}, k \in N$ where $m_k = m(\frac{1}{k}), n_k = n(\frac{1}{k})$.

Since $H_k = (N \times N) - P_k = \{(i,j) \in N \times N : N(x_{ij} - x_{m_k n_k}, z)_p \geq \frac{1}{k}\} \in I_2$ for each $k \in N$ and non zero $z \in X$, therefore $P_k \in F(I_2)$, for each $k \in N$.

Since I_2 has the property (AP_2), then by Lemma (3.3) there exists a set $P \subset N \times N$ such that $P \in F(I_2)$ and $P - P_k$ is finite for all $k \in N$. Now we have to show that $\lim_{i,j,m,n \to \infty} N(x_{ij} - x_{mn}, z)_p = 0$, for $(i,j),(m,n) \in P$ and for each non zero $z \in X$. For this, let $\varepsilon > 0$ and $l \in N$ be such that $l > \frac{2}{\varepsilon}$. If $(i,j),(m,n) \in P$, then $P - P_l$ is finite, so there exists $\alpha = \alpha(l)$ such that $(i,j),(m,n) \in P_l$ for all $i,j,m,n > \alpha(l)$. Therefore

$N(x_{ij} - x_{m_l n_l}, z)_p < \frac{1}{l}$ and $N(x_{mn} - x_{m_l n_l}, z)_p < \frac{1}{l}$ for all $i,j,m,n > \alpha(l)$ and non zero $z \in X$.

Now
$$\begin{aligned} N(x_{ij} - x_{mn}, z)_p &= N(x_{ij} - x_{m_l n_l} + x_{m_l n_l} - x_{mn}, z)_p \\ &= N((x_{ij} - x_{m_l n_l}) - (x_{mn} - x_{m_l n_l}), z)_p \\ &\leq N(x_{ij} - x_{m_l n_l}, z)_p + N(x_{mn} - x_{m_l n_l}, z)_p \\ &< \frac{1}{l} + \frac{1}{l} = \frac{2}{l} < \varepsilon \end{aligned}$$
, for all $i,j,m,n > \alpha(l)$ and non zero $z \in X$.

Hence for each $\varepsilon > 0$ there exists $\alpha = \alpha(\varepsilon)$ such that for $i,j,m,n > \alpha(\varepsilon)$ and $(i,j),(m,n) \in P \in F(I_2)$, we have $N(x_{ij} - x_{mn}, z)_p < \varepsilon$ for each non zero $z \in X$. This shows that the sequence $x = (x_{ij})$ is $I_2^* -$Cauchy double sequence in X. Thus a double sequence $x = (x_{ij})$ is $I_2 -$Cauchy double sequence in X if and only if $x = (x_{ij})$ is $I_2^* -$Cauchy double sequence in X.

References

[1] G.Bachman, *Introduction to p-Adic Numbers and Valuation Theory* (Academic Press, 1964).

[2] G. Bachman and L. Narici, *Functional Analysis* (New York and London, Academic Press, 1966).

[3] Balakrishna Tripathy and Binod Chandra Tripathy, On I-convergent double sequences ,*Soochow Journal of Mathematics , 31, 2005 549-560,.*

[4] P.Das ,P.Kostyrko,W.Wikzynski and P.Malik,I and I* -convergence of double sequences , *Math. Slovaca, 58(5),2008,605-620*.

[5] Erdinç Dündar and Bilal Altay, On Some Properties of $I_2 -$ Convergence and $I_2 -$ Cauchy of Double Sequences, Gen. Math. Notes, Vol. 7, No.1, November 2011, pp.1-12.

[6] Erdinç Dündar and Özer Talo, $I_2 -$ Cauchy Double Sequences of Fuzzy Numbers, *Gen. Math. Notes, 16(2), 2013, 103-114*.

[7] S. Gähler, Linear 2-normerte raume, *Math. Nachr, 28, 1965, 1-45*.

[8] M. Gurdal, On ideal convergent sequences in 2-normed spaces, *Thai Journal of Mathematics, 4(1), 2006, 85-91.*

[9] M. Gurdal and Isil Acik, On I-Cauchy sequences in2-normedspaces,*Mathematical Inequalities & Applications, 11(2) ,2008,349-354.*

[10] M. Gurdal, Ahmet Sahiner, Extremal I-Limit points of Double Sequences, *Applied Mathematical E-Notes, 8,2008, 131-137*.

[11] P. Kostyrko, M. Macaj and T. Salat, I-convergence, *Real Anal. Exchange, 26(2), 2000, 669-686.*

[12] P. Kostyrko, M. Macaj, T. Salat and M. Sleziak, I-convergence and extremal I-limit points, *Math. Slovaca, 55, 2005, 443-464*.

[13] Mehmet Acikgoz, N. Aslan, N. Koskeroglu and S. Araci, p-adic approach to linear 2-normed spaces, *Mathematica Moravica, 13(2) 2009, 7-22.*

[14] F.Nuray and W.H.Ruckle, Generalized statistical convergence and convergence free spaces, *J.Math.Anal.Appl., 245,2000, 513-527.*

[15] Pratulananda Das and Prasanta Malik, On Extrimal I-limit points of double sequences, *Tatra Mt.Math.Publ. 40, 2008, 91-102*.

[16] A.Pringsheim, Zur theorie der zweifach unendlichen Zahlenfolgen, *Math. Ann., 53, 1900, 289-321*.

[17] W. Raymond, R. Freese and J. Cho, *Geometry of Linear 2-Normed Spaces* (Nova Science Publishers, 2001).

[18] Saeed Sarabadan,Fatemeh Amoee Arani and Siamak Khalehoghli, A Condition for the Equivalence of I and I*-convergence in 2-normed spaces, *Int.J.Contemp. Math. Sciences, 6(43), 2011, 2147-2159.*

[19] Saeed Sarabadan and Sorayya Talebi, On I-convergence of Double sequences in 2- normed spaces, *Int.J.Contemp.Math.Sciences,.7(14), 2012, 673-684.*

[20] A.Sahiner, M. Gurdal, S. Saltan and H. Gunawan, Ideal convergence in 2-normed spaces, *Taiwanese Journal of Mathematics, 11(5) 2007, 1477-1484.*

[21] B.Surender Reddy, Equivalence of p-adic 2-norms in p-adic linear 2-normed spaces, *International Journal of Open Problems in Computer Science and Mathematics, 3(5),(2010, 25-38.*

[22] B.Surender Reddy and D.Shankaraiah, On Ideal convergent sequences in p-adic linear 2- normed spaces, *General Mathematics Notes17(1), 2013, 88-104.*

[23] B.Surender Reddy and D.Shankaraiah, On I-Cauchy sequences in p-adic linear 2-normed spaces, *International Journal of Pure and Applied Mathematics, 89(4), 2013, 483-496.*

[24] B.Surender Reddy and D.Shankaraiah, On Ideal Convergent of Double Sequences in p-adic linear 2-normed spaces, *Global Journal of Pure and Applied Mathematics, 10(1), 2014, 7-20.*

[25] Vijay Kumar, On I and I* -Convergence of double sequences, *Mathematical Communications 12, 2007, 171-181.*