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The Best Growth and Approximation of Entire Functions of Two Complex Variables in Banach Spaces

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Abstract: *In this Paper we are studding the polynomial approximation of entire functions of two complex variables in Banach spaces; concept is depend on index-pair. The characterizations of (p,q) −order of entire functions of two complex variables have been studied in terms of approximation errors. The results can be extended to m-variables but to reduce the mechanical labour we have considered only two variables.*
Key Words: *Approximation error, order, type, entire function, index-pair.*

I. Introduction:

Let $\varphi(z_{1.}z_2) = \sum_{m_1,m_2=0}^{\infty}\{a_{m_1,m_2}(z_1^{m_1}, z_2^{m_2})\}$ be a function of the complex variables z_1 and z_2, regular for $|z_1| \le r_n \;\; n = 1,2$. If r_1 and $r_2$can be taken arbitrarily large, then $\varphi(z_{1.}z_2)$ represents an entire function of complex variables z_1 and z_2. Following Bose and Sharma [1] we define the maximum modulus of $\varphi(z_{1.}z_2)$ as

$$M(r_1, r_2) = \max_{|z_n|\le r_n}|\varphi(z_{1.}z_2)| \;, n = 1,2.$$

The order ρ of the entire function $\varphi(z_{1.}z_2)$ is defined as [1];

$$\rho = \lim_{r_1,r_{2\to\infty}} sup \frac{\log\log M(r_1,r_2)}{\log(r_1,r_2)}$$

Bose and Sharma [1], obtained the following characterizations for order of entire functions of two complex variables.

Theorem 1.1. The entire function $\varphi(z_{1.}z_2) = \sum_{m_1,m_2=0}^{\infty}\{a_{m_1,m_2}(z_1^{m_1}, z_2^{m_2})\}$ is of finite order if and only if

$$\mu = \lim_{m_1,m_{2\to\infty}} sup \frac{\log(\mathrm{m_1}^{\mathrm{m_1}},\mathrm{m_2}^{\mathrm{m_2}})}{log\left(|a_{m_1,m_2}|^{-1}\right)}$$

is finite and then the order ρ of $\varphi(z_{1.}z_2)$ is equal to μ .

Let $H_\vartheta, \vartheta > 0$ denote the space of functions $f(z_{1.}z_2)$ analytic in the unit bi-disc $U = \{z_1, z_{1,} \in: |z_1| < 1, |z_2| < 1\}$ Such that $\|\varphi\|_{T_\vartheta} = \lim_{r_1,r_2} M_\vartheta(\varphi, r_{1,}r_2) < \infty,$

Where

$M_\vartheta(\varphi, r_{1,}r_2) = \left\{\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\varphi(r_1e^{i\theta_1}, r_2e^{i\theta_2})|^\vartheta d\theta_1 d\theta_2\right\}^{1/\vartheta}$, and let $H'_{\vartheta\,,}\vartheta > 0$ denote the space of functions $f(z_{1.}z_2)$ analytic in U and satisfying the condition

$$\|\varphi\|_{T_\vartheta} = \left\{\frac{1}{\pi^2}\int_{|z_1|<1}\int_{|z_2|<1}|\varphi(z_{1.}z_2)|^\vartheta dx_1dy_1\; dx_2dy_2\right\}^{1/\vartheta} < \infty$$

Set

$\|\varphi\|_\infty = sup\{|\varphi(z_{1.}z_2)|: z_{1,}z_2 \in U\}$

H_ϑ and $H'_{\vartheta\,,}$ are Banach spaces for $\vartheta \ge 1$ In analogy with spaces of functions of one variable, we callH_ϑ and $H'_{\vartheta\,,}$ the Hardy and Bergman spaces respectively.

Following the Vakarchuk and Zhir [4] we say that the function $\varphi(z_{1.}z_2)$ analytic in U belongs to the space$B(u, v, k)$ where $0 < u < v \le \infty$ and $0 < k \le \infty$ if

$$\|\varphi\|_{u,v,k} = \left\{\int_0^1\int_0^1\{(1-r_1)(1-r_2)\}^{k\left(\frac{1}{u}-\frac{1}{v}\right)^{-1}} M_v^k(\varphi, r_1, r_2)\, dr_1dr_2\right\}^{\frac{1}{N}} < \infty$$

$$\|\varphi\|_{u,v,\infty} = sup\left\{\{(1-r_1)(1-r_2)\}^{\left(\frac{1}{u}-\frac{1}{v}\right)^{-1}} M_v(\varphi, r_1, r_2): 0 < r_1, r_2 < 1\right\} < \infty$$

The space $B(u, v, k)$ is a Banach space for $u > 0$ and $v, k \ge 1$ otherwise it is a Frechet space. Further, we have

$$H_{\vartheta} \sqsubset H'_{\vartheta\,,} = B\left(\frac{v}{2}, v, v\right), 1 \le v < \infty. \qquad (1.1)$$

Let Y is a Banach space and let $E_{m_{1},m_{2}(\varphi,Y)}$ be the best approximation of a function $\varphi(z_{1.}z_{2}) \in Y$ by elements of the space Q that consists of algebraic polynomials of degree $\le m_1 + m_2$ in two complex variables.

$$E_{m_{1},m_{2}(\varphi,Y)} = inf\{\|\varphi - q\|_Y : q \in Q\}\,.$$

Recently, Ganti and Srivastava [2] characterized the order and type in terms of the approximation errors $E_{m_{1},m_{2}(\varphi,B(u,v,k))}$ and $E_{m_{1},m_{2}(\varphi,T_{\vartheta})}$. But their results leave to study a big class of entire functions such as slow growth and fast growth. To bridge this gap in this chapter we pick up the concept of $(p,q)-$ order introduced by Juneja et al. [3] and consider it for entire functions of two variables. Roughly speaking, this concept is a modification of the classical definition of order obtained by replacing logarithms by iterated logarithms, where the degrees of iteration are determined by p and q.

To the best of our knowledge, characterizations for the $(p,q)-$order of entire functions of two complex variables in Banach spaces have not been obtained so far. We define the $(p,q)-$order of an entire function $\varphi(z_{1.}z_{2})$ by

$$(p,q) = \lim_{r_1,r_2\to\infty} sup\ \frac{log^{p}M(r_1,r_2)}{log^{q}(r_1,r_2)} \qquad (1.3)$$

Where p and q are integers such that $p \ge q \ge 1$.

Notations: We are using the following notations in this paper.

(i) $log^{[m]}x = exp^{[-m]}x = \log\left(log^{[m-1]}x\right) = \exp\left(exp^{[-(m-1)]}x\right), m = 0, \pm1. \pm2 \dots\dots$
Provided that$0 < log^{[m-1]}x < \infty\ with\ log^{[0]}x = exp^{[0]}x = x.$

(ii) $\beta^{1/k}\left[(n+1)k+1; k\left(\frac{1}{u}-\frac{1}{2}\right)\right] = \beta(n,u,2,k\ )$

$\beta^{1/k}\left[(m+1)k+1; k\left(\frac{1}{u}-\frac{1}{2}\right)\right] = \beta(m,u,2,k)$

$\beta^{1/k}\left[(n+1)k+1, k\left(\frac{1}{u}-\frac{1}{v}\right)\right] = \beta(n,u,v,k)$

$\beta^{1/k}\left[(m+1)k+1; k\left(\frac{1}{u}-\frac{1}{v}\right)\right] = \beta(m,u,v,k)$

II. Basic results:

In this section we have given some lemmas as basic results, which have been used in the sequel.

Lemma 2.1. If $\varphi(z_{1.}z_{2}) = \sum_{m_1,m_2=0}^{\infty}\left\{a_{m_1,m_2}\left(z_1^{m_1}, z_2^{m_2}\right)\right\}$ be an entire function and for a pair of integers $(p,q)\,, p \ge 2, q \ge 1\ \rho(p,q)$ be defined by (1.3) then

$$\rho(p,q) = P\left(L(p,q)\right)$$

Where

$$L(p,q) = \lim_{m_1+m_2\to\infty} sup\, \frac{log^{[p-1]}\{(m_1+m_2)a_{m_1,m_2}\}}{log^{[q-1]}\left\{\frac{1}{m_1+m_2} log\left|a_{m_1,m_2}\right|^{-1}\right\}} \qquad (2.1)$$

4.

$$P\left(L(p,q)\right) = \begin{cases} L(p,q) & if\ p > q \\ 1 + L(p,q) & if\ p = q = 2 \\ max\left(1 + L(p,q)\right) & if 3 \le p = q < \infty \\ \infty & if\ \ p = q = \infty \end{cases}$$

And

$$a_{m_1,m_2} = \begin{cases} (\mathrm{m_1}^{\mathrm{m_1}}, \mathrm{m_2}^{\mathrm{m_2}})^{\frac{1}{m_1+m_2}} & ;\ m_{1,}m_{2,} \ge 1\ for\ (p,q) = (2,1) \\ 1 & ;\ m_{1,}m_{2,} \ge 1\ for\ 2 \le q \le p < \infty \\ 0 & ;\ at\ least\ one\ m_{1,}m_{2,} = 0 \end{cases}$$

III. Main Results: In this section we prove our main results.

Theorem3.1: If $\varphi(z_{1.}z_2) = \sum_{m_1,m_2=0}^{\infty}\{a_{m_1,m_2}(z_1^{m_1}, z_2^{m_2})\}$ be an entire function and for a pair of integers (p,q) , $p \geq 2$, $q \geq 1$, be defined by (1.3) then

$$\rho(p,q) = P\left(L'(p,q)\right) \text{ where}$$

$$L'(p,q) = \lim_{m_1+m_2\to\infty} sup \frac{log^{[p-1]}\{(m_1+m_2)a_{m_1,m_2}\}}{log^{[q-1]}\left\{\frac{1}{m_1+m_2}log\left[E_{m_1,m_2}(\varphi,\beta(u,v,k))\right]^{-1}\right\}} \qquad (3.1)$$

Proof. We prove the above result in two steps, first we consider the space $\beta(u,v,k)$, $v = 2,0 < v < 2$ and $k \geq 1$ .Let $\varphi(z_{1.}z_2) \in \beta(u,v,k)$ be of (p,q) order $\rho(p,q)$ From (2.1), for any $\varepsilon > 0$ there exists a natural number $m_0 = m_0(\varepsilon)$ such that

$$|a_{m_1,m_2}| \leq \left[exp^{[q-1]}\left\{log^{[p-2]}\{(m_1+m_2)a_{m_1,m_2}\}\right\}^{\frac{1}{L'(p,q)+\varepsilon}}\right]^{-(m_1+m_2)} \quad , m_{1,}m_2 > m_0 \quad (3.2)$$

Now first we show that $\rho(p,q) \geq P\left(L'(p,q)\right)$.If $L'(p,q) = \infty$ then $\rho(p,q) = \infty$ or else $\varphi(z_{1.}z_2)$ is not an entire function. If $L'(p,q) = 0, \rho(p,q) \geq P\left(L'(p,q)\right)$ since $\rho(p,q)$ is nonnegative then we have $0 < \varepsilon < L'(p,q) < \infty$.

We denote the partial sum of the Taylor series of a function $\varphi(z_{1.}z_2)$ by

$$H_{m_1,m_2}(\varphi, z_{1.}z_2) = \sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2} a_{i_1,i_2}\, z_1^{i_1}, z_2^{i_2}$$

and

$$E_{m_1,m_2}(\varphi,\beta(u,2,k)) = \left\|\varphi - H_{m_1,m_2}(\varphi)\right\|_{u.2,k}$$

$$= \left\{\int_0^1\int_0^1\{(1-r_1)(1-r_2)\}^{k\left(\frac{1}{u}-\frac{1}{2}\right)-1}\left(\sum_{i_1}\sum_{i_2} r_1^{2i_1}\, r_2^{2i_2}|a_{i_1,i_2}|^2\right)^{\frac{k}{2}} dr_1 dr_2\right\}^{\frac{1}{k}} \quad , \quad (3.3)$$

Where

$\sum_{i_1}\sum_{i_2} r_1^{2i_1}\, r_2^{2i_2}|a_{i_1,i_2}|^2 = R_1 + R_2 + \sum_{i_1=m_1+1}^{\infty}\sum_{i_2=m_2+1}^{\infty} r_1^{2i_1} r_2^{2i_2}|a_{i_1,i_2}|^2$,

$R_1 = \sum_{i_1=0}^{m_1}\sum_{i_2=m_2+1}^{\infty} r_1^{2i_1} r_2^{2i_2}|a_{i_1,i_2}|^2$ and $R_2 = \sum_{i_1=m_1+1}^{\infty}\sum_{i_2=0}^{m_2} r_1^{2i_1} r_2^{2i_2}|a_{i_1,i_2}|^2$

Since R_1, R_2 are bounded and $r_1, r_2 < 1$,therefore (3.3) becomes

$$E_{m_1,m_2}(\varphi,\beta(u,2,k)) \leq C\left[\int_0^1\left\{(1-r)^{k\left(\frac{1}{u}-\frac{1}{2}\right)-1}\right\} r^{(s+1)k} dr\right]\left\{\sum_{i_1=m_1+1}^{\infty}\sum_{i_2=m_2+1}^{\infty}|a_{i_1,i_2}|^2\right\}^{\frac{1}{2}}$$

Where

$$\left[\int_0^1\left\{(1-r)^{k\left(\frac{1}{u}-\frac{1}{2}\right)-1}\right\} r^{(s+1)k} dr\right]$$

$$= \left[\int_0^1\left\{(1-r_1)^{k\left(\frac{1}{u}-\frac{1}{2}\right)-1}\right\} r_1^{(m_1+1)k} dr_1\right] \times \left[\int_0^1\left\{(1-r_2)^{k\left(\frac{1}{u}-\frac{1}{2}\right)-1}\right\} r_2^{(m_2+1)k} dr_2\right]$$

Therefore

$$E_{m_1,m_2}(\varphi,\beta(u,2,k)) \leq C'\beta(m_{1,}u,2,k)\beta(m_{2,}u,2,k)\left\{\sum_{i_1=m_1+1}^{\infty}\sum_{i_2=m_2+1}^{\infty}|a_{i_1,i_2}|^2\right\}^{\frac{1}{2}} \qquad (3.4)$$

where C' is a constant and $\beta(a,b)(a,b) > 0$ denotes the beta function.

In view of (3.2), we have

$$\sum_{i_1=m_1+1}^{\infty}\sum_{i_2=m_2+1}^{\infty}|a_{i_1,i_2}|^2 \leq \sum_{i_1=m_1+1}^{\infty}\sum_{i_2=m_2+1}^{\infty}\left\{exp^{[q-1]}\{log^{[p-2]}(i_1+i_2)\}^{\frac{1}{L'+\varepsilon}}\right\}^{-2(i_1+i_2)}$$

$$= O(1)\left[exp^{[q-1]}\{log^{[p-2]}(m_1+1+m_2+1)\}^{\frac{1}{L'(p,q)+\varepsilon}}\right]^{-2(m_1+1+m_2+1)}$$

Using the above inequality in (3.4) we get

$$E_{m_1,m_2}(\varphi,\beta(u,2,k)) \le C'' \beta(m_{1,}u,2,k)\beta(m_{2,}u,2,k)$$

$$\times \left[exp^{[q-1]}\{log^{[p-2]}(m_1+m_2+2)\}^{\frac{1}{L'(p,q)+\varepsilon}}\right]^{-(m_1+m_2+2)} \quad (3.5)$$

The result for has been obtained by Ganti and Srivastava [2].

Now consider for $(p\,,q) = (2\,,2)$

$$E_{m_1,m_2}(\varphi,\beta(u,2,k)) \le C'' \beta(m_{1,}u,2,k)\beta(m_{2,}u,2,k) \times$$

$$\left[exp\{(m_1+1+m_2+1)\}^{\frac{1}{L'(2,2)+\varepsilon}}\right]^{-(m_1+1+m_2+1)}$$

Or

$$log\, E_{m_1,m_2}(\varphi,\beta(u,2,k)) - log\,\beta(m_{1,}u,2,k) - \log\beta(m_{2,}u,2,k)$$

$$\le log\left[exp\{(m_1+1+m_2+1)\}^{\frac{1}{L'(2,2)+\varepsilon}}\right]^{-(m_1+1+m_2+1)}$$

7.

Or

$$log\, E_{m_1,m_2}(\varphi,\beta(u,2,k)) - log\,\beta(m_{1,}u,2,k) - \log\beta(m_{2,}u,2,k)$$

$$\le -(m_1+m_2+2)log\left\{exp\{(m_1+m_2+2)\}^{\frac{1}{L'(2,2)+\varepsilon}}\right\}$$

Or

$$\frac{1}{log\, E_{m_1,m_2}(\varphi,\beta(u,2,k)) - log\,\beta(m_{1,}u,2,k) - \log\beta(m_{2,}u,2,k)} \ge \frac{1}{-(m_1+m_2+2)log\left\{exp\{(m_1+m_2+2)\}^{\frac{1}{L'(2,2)+\varepsilon}}\right\}}$$

Or

$$L'(2,2)+\varepsilon \ge \frac{log(m_1+m_2)}{log\left[-\frac{1}{m_1+m_2}\{log\, E_{m_1,m_2}(\varphi,\beta(u,2,k)) - log\,\beta(m_{1,}u,2,k) - \log\beta(m_{2,}u,2,k)\}\right]}$$

Since

$$\left\{\beta\left[(n+1)k+1; k\left(\frac{1}{u}-\frac{1}{2}\right)\right]\right\}^{1/(n+1)} \cong 1. \qquad (3.6)$$

Now proceeding to limits, we get

$$L'(2,2) \ge \lim_{m_1+m_2\to\infty} sup \frac{log(m_1+m_2)}{\log\log\left[log\, E_{m_1,m_2}(\varphi,\beta(u,2,k))\right]^{-\frac{1}{(m_1+m_2)}}}$$

Or

$$P\left(L'(2,2)\right) - 1 \ge \lim_{m_1+m_2\to\infty} sup \frac{log(m_1+m_2)}{\log\log\left[log\, E_{m_1,m_2}(\varphi,\beta(u,2,k))\right]^{-\frac{1}{(m_1+m_2)}}}$$

Or

$$P\left(L'(2,2)\right) - 1 \ge L'(2,2)$$

8.

Or

$$\rho(2\,,2) \ge P\left(L'(2,2)\right) \ge L'(2,2)+1 \qquad (3.7)$$

Now for $(p\,,q) \ne (2,1)$ *and* $(2,2)$ we have from (3.5) that

$$E_{m_1,m_2}(\varphi,\beta(u,2,k)) \le C'' \beta(m_{1,}u,2,k)\beta(m_{2,}u,2,k)$$

$$\times \left[exp^{[q-1]}\{log^{[p-2]}(m_1+m_2+2)\}^{\frac{1}{L'(p,q)+\varepsilon}}\right]^{-(m_1+m_2+2)}$$

Or

$$log\, E_{m_1,m_2}(\varphi,\beta(u,2,k)) - log\,\beta(m_{1,}u,2,k) - \log\beta(m_{2,}u,2,k)$$

$$\le -(m_1+m_2+2)log\left[exp^{[q-1]}\{log^{[p-2]}(m_1+m_2+2)\}^{\frac{1}{L'(p,q)+\varepsilon}}\right]$$

Taking (3.6) into account, we get

$$\left[E_{m_1,m_2}(\varphi,\beta(u,2,k))\right]^{\frac{-1}{(m_1+m_2)}} \geq exp^{[q-1]}\left\{log^{[p-2]}(m_1+m_2)\right\}^{\frac{1}{L'(p,q)+\varepsilon}}$$

Or

$$log^{[q-1]}\left[E_{m_1,m_2}(\varphi,\beta(u,2,k))\right]^{\frac{-1}{(m_1+m_2)}} \geq \left\{log^{[p-2]}(m_1+m_2)\right\}^{\frac{1}{L'(p,q)+\varepsilon}}$$

Or

$$L'(p,q)+\varepsilon \geq \frac{log^{[p-1]}(m_1+m_2)}{log^{[q]}\left[E_{m_1,m_2}(\varphi,\beta(u,2,k))\right]^{\frac{-1}{(m_1+m_2)}}}$$

Proceeding to limits, we obtain

$$L'(p,q) \geq \lim_{m_1+m_2\to\infty} sup \frac{log^{[p-1]}(m_1+m_2)}{log^{[q]}\left[E_{m_1,m_2}(\varphi,\beta(u,2,k))\right]^{\frac{-1}{(m_1+m_2)}}} \qquad (3.8)$$

9.

Combining other results for $(p\,,q)=(2{,}1)\ and\ (2{,}2)$ with (3.8) we get

$$\rho(p\,,q) \geq P\left(L'(p,q)\right) \qquad (3.9)$$

To prove reverse inequality consider (eq.2.4 [2]) which gives

$$\left|a_{m_1+1\,,m_2+1}\right|\beta(m_1,u,2,k)\beta(m_2,u,2,k) \leq E_{m_1,m_2}(\varphi,\beta(u,2,k))$$

Or

$$log\left|a_{m_1+1\,,m_2+1}\right| + log\beta(m_1,u,2,k) + log\beta(m_2,u,2,k) \leq logE_{m_1,m_2}(\varphi,\beta(u,2,k))$$

Again (3.6) taking into account in above inequality, we obtain

$$\lim_{m_1+m_2\to\infty} sup \frac{log^{[p-1]}(m_1+m_2)}{log^{[q]}\left[E_{m_1,m_2}(\varphi,\beta(u,2,k))\right]^{\frac{-1}{(m_1+m_2)}}} \geq \lim_{m_1+m_2\to\infty} sup \frac{log^{[p-1]}(m_1+m_2)}{log^{[q]}\left[a_{m_1,m_2}\right]^{\frac{-1}{(m_1+m_2)}}}$$

Now using Lemma (2.1), since $\rho(p\,,q) \geq 1\ for\ p=q$ the inequality for $p=q\geq 3$ gives

$$\rho(p\,,q) \leq max\left(1, L'(p,q)\right)$$

and for $p>q$ it gives

$$\rho(p\,,q) \leq L'(p,q) \qquad (3.10)$$

Hence combining above results we get $\rho(p\,,q) \leq P\left(L'(p,q)\right)$

This is the proof of first step.

Now we consider the space $\beta(u,v,k) \neq 2$ we have

$$E_{m_1,m_2}(\varphi,\beta(u,v,k)) \leq \left\|\varphi - H_{m_1,m_2}(\varphi)\right\|_{u,v,k}$$

$$=\left\{\int_0^1\int_0^1 \{(1-r_1)(1-r_2)\}^{k\left(\frac{1}{u}-\frac{1}{v}\right)-1}\left(\sum_{i_1}\sum_{i_2} r_1^{2i_1} r_2^{2i_2}\left|a_{i_1,i_2}\right|^2\right)^{\frac{k}{v}} dr_1 dr_2\right\}^{\frac{1}{k}} \qquad (3.11)$$

Where

$\sum_{i_1}\sum_{i_2} r_1^{2i_1} r_2^{2i_2}\left|a_{i_1,i_2}\right|^v = R_1 + R_2 + \sum_{i_1=m_1+1}^{\infty}\sum_{i_2=m_2+1}^{\infty} r_1^{2i_1} r_2^{2i_2}\left|a_{i_1,i_2}\right|^v,$

$R_1 = \sum_{i_1=0}^{m_1}\sum_{i_2=m_2+1}^{\infty} r_1^{2i_1} r_2^{2i_2}\left|a_{i_1,i_2}\right|^v$ and $R_2 = \sum_{i_1=m_1+1}^{\infty}\sum_{i_2=0}^{m_2} r_1^{2i_1} r_2^{2i_2}\left|a_{i_1,i_2}\right|^v$

Since R_1, R_2 are bounded and $r_1, r_2 < 1$,therefore (3.11) becomes

$$E_{m_1,m_2}(\varphi,\beta(u,v,k)) \leq C''\left[\int_0^1\left\{(1-r)^{k\left(\frac{1}{u}-\frac{1}{v}\right)-1}\right\} r^{(s+1)k} dr\right]\left\{\sum_{i_1=m_1+1}^{\infty}\sum_{i_2=m_2+1}^{\infty}\left|a_{i_1,i_2}\right|^v\right\}^{\frac{1}{v}}$$

Where

$$\left\{\int_0^1 \left\{(1-r)^{k\left(\frac{1}{u}-\frac{1}{v}\right)-1}\right\} r^{(s+1)k} dr\right\}$$
$$= \left\{\int_0^1 \left\{(1-r_1)^{k\left(\frac{1}{u}-\frac{1}{v}\right)-1}\right\} {r_1}^{(m_1+1)k} dr_1\right\}\left\{\int_0^1 \left\{(1-r_2)^{k\left(\frac{1}{u}-\frac{1}{v}\right)-1}\right\} {r_2}^{(m_2+1)k} dr_2\right\}$$

Therefore

$$E_{m_1,m_2}(\varphi,\beta(u,v,k)) \le C'' \beta(m_1,u,v,k)\beta(m_2,u,v,k)\left\{\sum_{i_1=m_1+1}^{\infty}\sum_{i_2=m_2+1}^{\infty}\left|a_{i_1,i_2}\right|^v\right\}^{\frac{1}{v}}, \quad (3.12)$$

Where C'' is constant and $\beta(m,u,v,k)$ is Euler's integral of the first kind. By using (3.2) we have

$$\sum_{i_1=m_1+1}^{\infty}\sum_{i_2=m_2+1}^{\infty}\left|a_{i_1,i_2}\right|^v \le \sum_{i_1=m_1+1}^{\infty}\sum_{i_2=m_2+1}^{\infty}\left\{exp^{[q-1]}\left\{log^{[p-2]}(i_1+i_2)\right\}^{\frac{1}{L'(p,q)+\varepsilon}}\right\}^{-v(i_1+i_2)}$$

$$= O(1)\left[exp^{[q-1]}\left\{log^{[p-2]}(m_1+m_2+2)\right\}^{\frac{1}{L'(p,q)+\varepsilon}}\right]^{-v(m_1+m_2+2)}$$

using this inequality in (3.12), we get

$$E_{m_1,m_2}(\varphi,\beta(u,v,k)) \le C''\beta(m_1,u,v,k)\beta(m_2,u,v,k) \times \left[exp^{[q-1]}\left\{log^{[p-2]}(m_1+m_2+2)\right\}^{\frac{1}{L'(p,q)+\varepsilon}}\right]^{-v(m_1+m_2+2)} \quad (3.13)$$

For $(p,q) = (2,1)$ the result has been proved by Ganti and Srivastava [2] .
Now $(p,q) = (2,2)$we have from (3.13) that

$$E_{m_1,m_2}(\varphi,\beta(u,v,k)) \le C''\beta(m_1,u,v,k)\beta(m_2,u,v,k) \times \left\{exp(m_1+m_2+2)^{\frac{1}{L'(2,2)+\varepsilon}}\right\}^{-(m_1+m_2+2)}$$

Or

$$log\ E_{m_1,m_2}(\varphi,\beta(u,v,k)) - \log\beta(m_1,u,v,k) - \log\beta(m_2,u,v,k)$$
$$\le log\left\{exp(m_1+m_2+2)^{\frac{1}{L'(2,2)+\varepsilon}}\right\}^{-(m_1+m_2+2)}$$

Or

$$\frac{1}{log\ E_{m_1,m_2}(\varphi,\beta(u,v,k)) - \log\beta(m_1,u,v,k) - \log\beta(m_2,u,v,k)} \ge \frac{1}{-(m_1+m_2+2)\log\left\{exp(m_1+m_2+2)^{\frac{1}{L'(2,2)+\varepsilon}}\right\}}$$

Since

$$\beta\left[(n+1)k+1, k\left(\frac{1}{u}-\frac{1}{v}\right)\right] = \frac{\Gamma(n+1)(k+1)\Gamma\left(k\left(\frac{1}{u}-\frac{1}{v}\right)\right)}{\Gamma\left((n+1)(k+1)+k\left(\frac{1}{u}-\frac{1}{v}\right)\right)}$$

and

$$\left\{\beta\left[(n+1)k+1, k\left(\frac{1}{u}-\frac{1}{v}\right)\right]\right\}^{\frac{1}{(n+1)}} \cong 1 . \quad (3.14)$$

Therefore from above inequality, we get

$$\frac{1}{log\ E_{m_1,m_2}(\varphi,\beta(u,v,k))} \ge \frac{1}{-(m_1+m_2)\log\left\{exp(m_1+m_2)^{\frac{1}{L'(2,2)+\varepsilon}}\right\}}$$

Or

$$log\ E_{m_1,m_2}\big(\varphi,\beta(u,v,k)\big) \le \log\left\{exp(m_1+m_2)^{\frac{1}{L^{'}(2,2)+\varepsilon}}\right\}^{-(m_1+m_2)}$$

Or

$$log\big[\,E_{m_1,m_2}\big(\varphi,\beta(u,v,k)\big)\big]^{-\frac{1}{(m_1+m_2)}} \ge (m_1+m_2)^{\frac{1}{L^{'}(2,2)+\varepsilon}}$$

Or

$$L^{'}(2,2)+\varepsilon \ge \frac{\log(m_1+m_2)}{log^{[2]}\big[\,E_{m_1,m_2}\big(\varphi,\beta(u,v,k)\big)\big]^{-\frac{1}{(m_1+m_2)}}}$$

Proceeding to limits, we get

$$L^{'}(2,2) \ge \lim_{m_1,m_2\to\infty} sup \frac{\log(m_1+m_2)}{log^{[2]}\big[\,E_{m_1,m_2}\big(\varphi,\beta(u,v,k)\big)\big]^{-\frac{1}{(m_1+m_2)}}}$$

Or

$$P\left(L^{'}(2,2)\right)-1 \ge \lim_{m_1,m_2\to\infty} sup \frac{\log(m_1+m_2)}{log^{[2]}\big[\,E_{m_1,m_2}\big(\varphi,\beta(u,v,k)\big)\big]^{-\frac{1}{(m_1+m_2)}}}$$

Or

$$P\left(L^{'}(2,2)\right)-1 \ge L^{'}(2,2)$$

Or

$$P\left(L^{'}(2,2)\right) \ge L^{'}(2,2)+1 \quad . \qquad (3.15)$$

Now consider the case $(p,q) \ne (2,1)$ and $(2,2)$ from (3.13) we get

$$log\ E_{m_1,m_2}\big(\varphi,\beta(u,v,k)\big) - \log\beta\big(m_{1,}u,v,k\big) - \log\beta\big(m_{2,}u,v,k\big) - logC^{''}$$

$$\le\ log\left[exp^{[q-1]}\{log^{[p-2]}(m_1+m_2+2)\}^{\frac{1}{L^{'}(p,q)+\varepsilon}}\right]^{-(m_1+m_2+2)}$$

Using (3.14), we obtain

$$\frac{1}{log\ E_{m_1,m_2}\big(\varphi,\beta(u,v,k)\big)} \ge \frac{1}{-(m_1+m_2)log\left[exp^{[q-1]}\{log^{[p-2]}(m_1+m_2)\}^{\frac{1}{L^{'}(p,q)+\varepsilon}}\right]}$$

14.

Or

$$\big[log\ E_{m_1,m_2}\big(\varphi,\beta(u,v,k)\big)\big]^{-\frac{1}{(m_1+m_2)}} \ge exp^{[q-1]}\{log^{[p-2]}(m_1+m_2)\}^{\frac{1}{L^{'}(p,q)+\varepsilon}}$$

Or

$$L^{'}(p,q)+\varepsilon \ge \frac{\log^{[p-1]}(m_1+m_2)}{log^{[q]}\big[\,E_{m_1,m_2}\big(\varphi,\beta(u,v,k)\big)\big]^{-\frac{1}{(m_1+m_2)}}}$$

Proceeding to limits, immediately we get

$$\rho(p,q) \ge P\left(L^{'}(p,q)\right) \qquad (3.16)$$

To prove reverse inequality taking (3.12) into account this gives

$$\big|a_{m_1+1,m_2+1}\big|\beta\big(m_{1,}u,v,k\big)\beta\big(m_{2,}u,v,k\big) \le\ E_{m_1,m_2}\big(\varphi,\beta(u,v,k)\big)$$

Or

$$-log\ E_{m_1,m_2}\big(\varphi,\beta(u,v,k)\big) \le -log\big|a_{m_1+1,m_2+1}\big| - log\beta\big(m_{1,}u,v,k\big) - log\beta\big(m_{2,}u,v,k\big)$$

Again using (3.14), we get

$$\frac{\log(m_1+m_2)}{log^{[2]}\big[\,E_{m_1,m_2}\big(\varphi,\beta(u,v,k)\big)\big]^{-\frac{1}{(m_1+m_2)}}} \ge \frac{\log(m_1+m_2)}{log^{[2]}\big|a_{m_1+1,m_2+1}\big|^{-\frac{1}{(m_1+m_2)}}}$$

In view of lemma (2.1), we obtain

$$L^{'}(2,2) \ge \rho(2,2)-1$$

Or

$$L^{'}(2,2)+1 \geq \rho(2,2) \qquad (3.17)$$

Since $\rho(p,q) \geq 1$ for $p=q$ the inequality for $p=q \geq 3$ gives $\rho(p,p) \leq max\left(1, L^{'}(p,p)\right)$ and for $p>q$ it gives

$$\rho(p,q) \leq L^{'}(p,q) \qquad (3.18)$$

Combining (3.15), (3.16), (3.17) and (3.18) the proof of second step is immediate.
Now consider the third step. Let $0<u<v<2$ and $k,v \geq 1$.
Since

$$E_{m_1,m_2}\left(\varphi,\beta(u_1,v_1,k_1)\right) \leq 2^{\left(\frac{1}{u_1,}-\frac{1}{v_1,}\right)}\left[k\left(\left(\frac{1}{u}-\frac{1}{v}\right)\right)\right]^{\left(\frac{1}{u_1,}-\frac{1}{v_1,}\right)} E_{m_1,m_2}(\varphi,\beta(u,v,k))$$

where $u_1=u, v_1=2$ and $k_1=k$ and the condition (3.1) is already proved for the space $\beta(u,2,k)$ hence

$$\lim_{m_1+m_2\to\infty} sup \frac{log^{[p-1]}\{(m_1+m_2)a_{m_1 1,m_2}\}}{log^{[q]}\left[E_{m_1,m_2}(\varphi,\beta(u,v,k))\right]^{\frac{-1}{(m_1+m_2)}}}$$

$$\geq \lim_{m_1+m_2\to\infty} sup \frac{log^{[p-1]}\{(m_1+m_2)a_{m_1 1,m_2}\}}{log^{[q]}\left[E_{m_1,m_2}(\varphi,\beta(u,2,k))\right]^{\frac{-1}{(m_1+m_2)}}} \qquad (3.19)$$

Now let $0<u\leq 2<v$ .Since

$$M_1(\varphi,r_1,r_2) \leq M_2(\varphi,r_1,r_2)\,, 0<r_1<r_2<1.$$

Therefore

$$E_{m_1,m_2}\left(\varphi,\beta(u_1,v_1,k_1)\right) \geq \left\{\int_0^1\int_0^1 \{(1-r_1)(1-r_2)\}^{k\left(\frac{1}{u}-\frac{1}{v}\right)-1} S\, dr_1 dr_2\right\}$$

$$\geq \left[|a_{m_1,m_2}|\beta(m_1,u,v,k)\beta(m_2,u,v,k)\right]$$

16.

where
$S=inf[M_2^k(\varphi-u,r_1,r_2): p \in P]$.Hence we get

$$\lim_{m_1+m_2\to\infty} sup \frac{log^{[p-1]}\{(m_1+m_2)a_{m_1 1,m_2}\}}{log^{[q]}\left[E_{m_1,m_2}(\varphi,\beta(u,v,k))\right]^{\frac{-1}{(m_1+m_2)}}}$$

$$\geq \lim_{m_1+m_2\to\infty} sup \frac{log^{[p-1]}\{(m_1+m_2)a_{m_1 1,m_2}\}}{log^{[q]}[|a_{m_1,m_2}|]^{\frac{-1}{(m_1+m_2)}}} \qquad (3.20)$$

In view of (3.19), (3.20) and Lemma 2.1, we get the required result. This is complete proof of the theorem.

Theorem 3.2. If $\varphi(z_1 z_2)=\sum_{m_1,m_2=0}^{\infty}\{a_{m_1,m_2}(z_1^{m_1},z_2^{m_2})\}$ be an entire function, then for a pair of integers $(p,q), p\geq 2, q\geq 1$ the function $\varphi(z_1 z_2) \in H$ is of (p,q) −order $\rho(p,q)$ if and only if $\rho(p,q)=P\left(L^{**}(p,q)\right)$
where

$$L^{**}(p,q)=\lim_{m_1+m_2\to\infty} sup \frac{log^{[p-1]}\{(m_1+m_2)a_{m_1,m_2}\}}{log^{[q-1]}\left\{\frac{1}{(m_1+m_2)}\, log\left[E_{m_1,m_2}(\varphi,H_\vartheta)\right]^{-1}\right\}}\,. \qquad (3.21)$$

Proof. Let $\varphi(z_1 z_2)=\sum_{m_1,m_2=0}^{\infty}\{a_{m_1,m_2}(z_1^{m_1},z_2^{m_2})\} \in H_\vartheta$ be an entire transcendental function. Since φ is entire, we have

$$\lim_{m_1,m_2\to\infty}|a_{m_1,m_2}|^{\frac{1}{(m_1+m_2)}}=0, \qquad (3.22)$$

and $\varphi \in H_\vartheta$,therefore

$$M_\vartheta(\varphi,r_1,r_2)<\infty\,,$$

and $\varphi(z_1 z_2) \in \beta(u,v,k), 0<u<v\leq\infty, k\geq 1\,.By$ (1.1) we have

$$E_{m_1,m_2}(\varphi,\beta(v/2\,,v,v)) \leq K\, E_{m_1,m_2}(\varphi,H_\vartheta), 1\leq v<\infty, \qquad (3.23)$$

where K is a constant independent of m_1, m_2 and φ In the case of space $H_{\infty,}$

$$E_{m_1,m_2}(\varphi,\beta(u,\infty,\infty)) \leq E_{m_1,m_2}(\varphi,H_\vartheta), 0<u<\infty. \qquad (3.24)$$

From (3.23) we have

$$\xi(\varphi) = \lim_{m_1+m_2\to\infty} sup \frac{log^{[p-1]}\{(m_1+m_2)a_{m_1,m_2}\}}{log^{[q-1]}\left\{\frac{1}{(m_1+m_2)}\ log\left[E_{m_1,m_2}(\varphi,H_\vartheta)\right]^{-1}\right\}}$$

$$\geq \lim_{m_1+m_2\to\infty} sup \frac{log^{[p-1]}\{(m_1+m_2)a_{m_1,m_2}\}}{log^{[q-1]}\left\{\frac{1}{(m_1+m_2)}\ log\left[E_{m_1,m_2}(\varphi,\beta(v/2,v,v))\right]^{-1}\right\}}$$

$$\geq L^{**}(p,q), 1\leq v\leq\infty, \qquad (3.25)$$

using (3.24) we prove inequality (3.25) for the case $v=\infty$.
For the reverse inequality, we have

$$E_{m_1,m_2}(\varphi,H_\vartheta) \leq O(1)\sum_{i_1=m_1+1}^{\infty}\sum_{i_2=m_2+1}^{\infty}|a_{i_1,i_2}|$$

using (3.2) we have

$$E_{m_1,m_2}(\varphi,H_\vartheta) \leq O(1)\sum_{i_1=m_1+1}^{\infty}\sum_{i_2=m_2+1}^{\infty}\left\{exp^{[q-1]}\{log^{[p-2]}(i_1+i_2)a_{i_1,i_2}\}^{\frac{1}{L'(p,q)+\varepsilon}}\right\}^{-2(i_1+i_2)}$$

Or

$$E_{m_1,m_2}(\varphi,H_\vartheta) \leq O(1)\left[exp^{[q-1]}\{log^{[p-2]}(i_1+i_2)a_{i_1,i_2}\}^{\frac{1}{L'(p,q)+\varepsilon}}\right]^{-2(m_1+1+m_2+1)}$$

18.

which gives

$$L'(p,q)+\varepsilon \geq \frac{log^{[p-1]}\{(m_1+m_2)a_{m_1,m_2}\}}{log^{[q]}\left[E_{m_1,m_2}(\varphi,H_\vartheta)\right]^{\frac{-1}{(m_1+m_2)}}}$$

Proceeding to limits we get

$$L'(p,q) \geq \xi(\varphi) \qquad (3.26)$$

In the consequence of Theorem 3.1 with (3.25) and (3.26) we obtain the result immediately.
Now to prove sufficiency, assume that the condition (3.21) is satisfied. Then it follows that

$$log\left[\frac{1}{E_{m_1,m_2}(\varphi,H_\vartheta)}\right]^{\frac{1}{(m_1+m_2)}} \to \infty \quad as\ m_1+m_2\to\infty.$$

It gives

$$\lim_{m_1+m_2\to\infty}\left[\frac{1}{E_{m_1,m_2}(\varphi,H_\vartheta)}\right]^{\frac{1}{(m_1+m_2)}} = 0\,.$$

This relation and the estimate $|a_{m_1,m_2}(\varphi)| \leq E_{m_1,m_2}(\varphi,H_\vartheta)$ yield the relation (3.22). It follows that $\varphi(z_1, z_2) \in H_\vartheta$ is an entire transcendental function.
Hence the proof is completed.

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