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Periodic Solutions of abstract neutral functional differential equations

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Abstract

We charaterize the existence of periodic solutions for a class of abstract neutral functional differential equations described in the form :

$$\frac{d}{dt}x(t) = A[x(t) - Bx(t-r)] + L(x_t) + f(t), t \in R \qquad (1)$$

Keywords : functional differential equations

1. Introduction :

Let X be a Banach space endowed with a norm |.| and r be non negative real number.

The main objective of this paper is to study the existence of periodic solutions for the class of linear abstract neutral differential equations (1) :

C = C([- r,0] ; X) be the Banach space of continuous functions mapping the interval [-r,0] into X. the function x_t given by $x_t(\theta)$ = x(t+θ) for θ in appropriate domain,denotes the segment or the "history" of the function x(.) at t.

L is a bounded linear map defined on an appropriate space, and f :R $\rightarrow$ X is a locally p-integrable and 2π-periodic function for $1 \leq$ p $\prec +\infty$

we assume that A : D(A) $\subseteq$ X $\rightarrow$ X and B$\subseteq$ X $\rightarrow$ X are closed linear operator

We denote

$$H^{1,p}(T;X) = \{u \in L^p(T : X) :\ni v \in L^p(T; X), \hat{v}(k) = ik\hat{u}(k) forallk \in Z\}$$

2. Preliminaries :

We denote by T the group defined as the quotient R/2πZ. There is an obvious identification between functions on T and 2π-periodic functions on R. We consider the interval [0,2π) as a model for T.

For a function f$\in L^1(T;$X), we denote by $\hat{f}$(k), k$\in$Z the k-th Fourier coefficient of f :

$$\hat{f}(\mathrm{k}) = \frac{1}{2\pi}\int_0^1 e^{-ikt}\mathrm{f}(\mathrm{t})\mathrm{dt} \text{ for } \mathrm{k}\in\mathrm{Z} \text{ and } \mathrm{t}\in\mathrm{R}.$$

Denote f_τ(t) :=f(t+τ), $\tau\in$Z ; then it the follows from the definition that $\hat{f}_\tau(\mathrm{k}) = e^{ik\tau}\hat{f}(\mathrm{k})$, $\tau\in T$.

Let f$\in L^p(T$,X). Then by Fefer's theorem, one has

$$\mathrm{f} = lim_{n\to\infty}\sigma_n(\mathrm{f})$$

in $L^p(T$,X) where

$$\sigma_n(\mathrm{f}) := \frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m} e_k\hat{f}(\mathrm{k})$$

with e_k(t) :=e^{ikt}

A Banach space X is said to be UMD, if the Hilbert transform is bounded on $L^p(R$,X) for all p $\in(1,\infty)$.

Definition 1 *: Let X and Y be a Banach spaces. A family of operators $T\subset B(X,Y)$ is called R-bounded, if there is a constant $C\succ 0$ and p$\in[1,\infty)$ such that for each $N\in N, T_j\in T$, $x_j\in X$ and for all independent, symmetric, $\{-1,1\}$-valued random variables r_j on a probability space (Ω,M,μ) the inequality $\left\|\sum_{j=1}^N r_jT_jx_j\right\|_{L^p(\Omega,Y)}\leq C\left\|\sum_{j=1}^N r_jx_j\right\|_{L^p(\Omega,Y)}$ is valid. The smallest such C is called R-bounded of T, we denot it by $R_p(T)$.*

Definition 2 : *For 1≤p∞ we say that a sequence* $\{M_k\}_{k\in Z} \subset B(X,Y)$ *is an* L^p*-multiplier if, for each* $f\in L^p(T,X)$*, there exists* $u\in L^p(T,Y)$ *such that* $\hat{u}(k) = M_k\hat{f}(k)$ *for all* $k \in Z$.

Theorem 1 :[3, theorem1.3]

Let X, Y be UMD space and let $\{M_k\}_{k\in Z} \subset B(X,Y)$*. If the sets* $\{M_k\}_{k\in Z}$ *and* $\{k(M_{k+1} - M_k)\}_{k\in Z}$ *are R-bounded, then* $\{M_k\}_{k\in Z}$ *is an* L^p*-multiplier for* $1 \prec p \prec \infty$.

3. A Criterion for Periodic Solutions :

We consider : Δ_k = ikI - ikB_k - A(I-B_k) - L_k , for all k ∈ Z.

Denote by B_k :=e^{-ikr}B ; L_k(x) := L($e^{ik\theta}$x) and e_k(t) :=e^{ikt} for all k ∈ Z

and $\sigma_Z(\Delta) = \{k \in Z : \Delta_k \text{ has no inverse}\}$

And we define : $D_k = (ikI - A(I - B_k) - L_k)^{-1}$

3.1. Existence of Strong Solution :

Definition 3 *Let A be a closed linear operator on X. A function x(.) solution of the problem (1) if* $x\in H^{1,p}(T;X)\cap L^p(T;X)$ *and (1) holds for almost all* $t\in[0,2\pi]$

Theorem 2 *: Let X be a Banach space and* $1 \prec p \prec +\infty$ *.Suppose that for every* $f \in L^p(T,X)$ *there exists a unique strong solution of Eq (1).Then*

1. *for every* $k \in Z$ *the operator* $(ikI\text{-}A(I\text{-}B_k)\text{-}L_k)$ *has bounded inverse*
2. *The set is R-bounded and* $\{ikD_k\}_{k \in Z}$ *is R-bounded.*

Lemma 1 *:[2, Lemma 4.2]*

Let $u \in C(T,X)$*. Then*

$$L(\hat{X_s}(k) = L_k \hat{x}(k).$$

proof of theorem 2 :
1) Let $k \in Z$, $y \in X$

for $f(t) = e^{ikr}$ y , $\exists x \in H^{1,p}(T,X)$ such that :

$\frac{dx}{dt}(t) = A(x(t)\text{-}Bx(t\text{-}r)) + L(x_t) + f(t)$

Taking fourier transform , L is linear and bounded, we obtain

$ik\hat{x}(k) = A(I\text{-}B_k)\hat{x}(k) + L_k\hat{x}(k) + \hat{f}(k)$

$(ikI\text{-}A(I\text{-}B_k)\text{-}L_k)\hat{x}(k) = \hat{f}(k) = y \Rightarrow)$ $(ikI\text{-}A(I\text{-}B_k)\text{-}L_k)$ is surjective.

Let $x \in Ker((ik - A(I\text{-}B_k)\text{-}L_k))$, that is $A(I\text{-}B_k)x + L_k x = ikx$, then $u(t)=e_k x$ defines a periodic solution of (1) corresponding to the the function f(t)=0.Consequently, u(t)=0 and x=0.

2) let $f \in L^p(T,X)$. By hypothesis, there exists a unique $x \in H^{1,p}(T,X)$ such that (1) equation is valid.Taking

Fourier transforms, we deduce that $(ikI\text{-}A(I\text{-}B_k)\text{-}L_k)\hat{x}(k) = \hat{f}(k)$ for all

$k \in Z$. Hence

$ik\hat{x}(k) = ik\ (ikI - A(I - B_k) - L_k)^{-1}\ \hat{f}(k)$ for all $k \in Z$

On the other hand, since $x \in H^{1,p}(T,X)$, there exists $v \in L^p(T,X)$ such that

$\hat{v}(k) = ik\hat{x}(k)$.This proves claim.

3.2. Existence of weak solution :

Definition 4 *: Let A be a closed linear operator on X. A function x(.) is called a weak solution of the problem (1) if :*$\int_0^t (x(s)\text{-}Bx(s\text{-}r))ds \in D(A)$ *and* $x(t) - x(0) = A\int_0^t (x(s)\text{-}Bx(s\text{-}r))ds + \int_0^t (Lx_s + f(s))ds, \quad 0 \leq t \leq 2\pi.$

Theorem 3 *: Let* $f \in L^p(T,X)$*, Assume that* $\overline{D(A)}=X$*; if x(.) is said to be a weak solution of Eq (1) then* $(ikI - A(I\text{-}B_k)\text{-} L_k)\hat{x}(k)= \hat{f}(k)$ *for all* $k \in Z$

proof : x(.) is a weak solution of Eq (1) then

$x(t) - x(0) = A\int_0^t Dx(s)ds + \int_0^t (Gx_s + f(s))ds$

$t = 2\pi$

$x(2\pi) - x(0) = A\int_0^{2\pi} (x(s)\text{-}Bx(s\text{-}r))ds + \int_0^{2\pi} (Lx_s + f(s))ds$; or $x(2\pi) = x(0)$ then

$A\int_0^{2\pi} (x(s)\text{-}Bx(s\text{-}r))ds + \int_0^{2\pi} (Lx_s + f(s))ds=0$

$(AI - B_0 + L_0)\hat{x}(0) + \hat{f}(0) = 0$

$(0\text{-}AI - B_0\text{-} L_0)\hat{x}(0)= \hat{f}(0)$ which shows that the assertion holds for $k = 0$.

Define $v(t) = \int_0^t (x(s)\text{-}Bx(s\text{-}r))ds$

And $g(t) = x(t) - x(0) - \int_0^t (Lx_s + f(s))ds$

by lemma 3.1 [2]

We have $\hat{v}(k)=\frac{i}{k}(\hat{x}(0)\text{- } B\hat{x}(0)) - \frac{i}{k}(\hat{x}(k)\text{-}B\hat{x}(k))$ (remark 2.3 [2])

$\hat{g}(k)=\hat{x}(k)\text{-}[\frac{i}{k}L_0\hat{x}(0) - \frac{i}{k}L_k\hat{x}(k)]\text{-}[\frac{i}{k}\hat{f}(0) - \frac{i}{k}\hat{f}(k)]$

$\hat{g}(k)=\hat{x}(k)\text{-}\frac{i}{k}L_0\hat{x}(0) + \frac{i}{k}L_k\hat{x}(k)\text{-}\frac{i}{k}\hat{f}(0) + \frac{i}{k}\hat{f}(k)$

$A\hat{v}(k)=\frac{i}{k}A(I\text{-}B_0)\hat{x}(0) - \frac{i}{k}A(I\text{-}B_k\hat{x}(k)$

Then

ik$\hat{x}$(k) + $L_0\hat{x}(0)$ - $L_k\hat{x}$(k)+$\hat{f}(0)$- $\hat{f}$(k)= -A(I-B_0)$\hat{x}(0)$+A(I-B_k)$\hat{x}$(k)

$\Leftrightarrow$ [ik$\hat{x}$(k) -A(I-B_k)$\hat{x}$(k)-$L_k\hat{x}$(k)-$\hat{f}$(k)]-[A(I-B_0)$\hat{x}(0)$+$L_0\hat{x}(0)$+$\hat{f}(0)$]=0

$\Leftrightarrow$ ik$\hat{x}$(k)-A(I-B_k)$\hat{x}$(k)-$L_k\hat{x}$(k)-$\hat{f}$(k) = 0

$\Leftrightarrow$ ik$\hat{x}$(k)-A(I-B_k)$\hat{x}$(k)-$L_k\hat{x}$(k) = $\hat{f}$(k).

Theorem 4 *Let $f \in L^p(T,X)$, Assume that $\overline{D(A)}=X$; if $x(.)$ is said to be a weak solution of Eq (2) and (ikD_k - AD_k - G_k)has a bounded inverse. Then $(ikI - A(I - B_k) - L_k)^{-1}$ is an L^p-multiplier.*

proof; from theorem (1) we have $\hat{x}(k) = (ikI - A(I - B_k) - L_k)^{-1}\hat{f}(k)$, for all f $\in L^p$(T,X)

Main result :

Our main result in this paper, establish that the converse of theorem (2) and the give the definition of Mild solution

Theorem 5 *:*

Let X be a UMD space and let $A : D(A) \subset X \to X$ be a closed linear operator.The following assertions are equivalent for $1 \prec p \prec \infty$.

1. *for every $f \in L^p(T,X)$ there exists a unique strong solution of Eq (1)*
2. *for every $k \in Z$ the operator (ikI-A(I-B_k)-L_k) has bounded inverse and the set is R-bounded and $\{ikD_k\}_{k\in Z}$ is R-bounded.*

proof :

1⇐2) Let f $\in L^p$(T,X) . Define D_k= $(ikI - A(I - B_k) - L_k)^{-1}$, the family

$\{ikD_k\}_{k\in Z}$ is an L^p-multiplier it is equivalent to the family $\{D_k\}_{k\in Z}$

is an L^p-multiplier that maps L^p(T,X) into $H^{1,p}$(T,X),[i.e. there exists

x$\in H^{1,p}$(T,x) such that

(1.1) $\hat{x}(k)=D_k\hat{f}(k)= (ikI - A(I - B_k) - L_k)^{-1}\hat{f}(k)$

In particular, x$\in L^p$(T,X) and there exists v$\in L^p$(T,X) such that

(1.2) $\hat{x'}(k) := \hat{v}(k) = ik\ \hat{x}(k)$

By Fejer's theorem one has in $L^p([-r_{2\pi},0],X)$

$$x_t(\theta) = x(t+\theta) = \lim_{n\to+\infty}\frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m}e^{ikt}e^{ik\theta}\hat{x}(k)$$

Hence in L^p(T,X) we obtain

$$x_t = \lim_{n\to+\infty}\frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m}e^{ikt}e_k\hat{x}(k)$$

Then, since L is linear and bounded

$$Lx_t = \lim_{n\to+\infty}\frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m}e^{ikt}L(e_k\hat{x}(k))$$

$$=\lim_{n\to+\infty}\frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m}e^{ikt}L_k\hat{x}(k)$$

By (1.1) and (1.2) we have

$$\hat{x'}(k) = ik\hat{x}(k) = A(I-B_k)\hat{x}(k) + L_k\hat{x}(k) + \hat{f}(k). \quad \text{for all } k \in Z.$$

Then using that A and B are closed we conclude tat (x(t)-Bx(t-r))∈D(A), and from the uniqueness theorem of Fourier coefficients, that equation (2) is valid for t ∈ T.[3. lemma 3.1]

Definition 5 *: of Mild solution about convert of weak solution*

Introduction :

Assume that A generates a C_0-semigroup T(.) on X; and x(.) is a weak solution, then we have

$$x(t) - x(0) = A\int_0^t (x(s)-Bx(s-r))ds + \int_0^t (Gx_s + f(s))ds$$

$$\int_0^t T(t-s)(x(s)-x(0))ds =$$
$$\int_0^t T(t-s)A \int_0^s (x(\xi)-Bx(\xi-r))d\xi ds + \int_0^t T(t-s)\int_0^s (L(x_\xi) + f(\xi))d\xi ds$$

$$= \int_0^t (T(t-s)-I)(x(s)-Bx(s-r))ds + \int_0^t T(t-s)\int_0^s (L(x_\xi) + f(\xi))d\xi ds$$

Then

$$\int_0^t T(t-s)(Bx(s-r)-x(0))ds = -\int_0^s (x(s)-Bx(s-r))ds + \int_0^t T(t-s)\int_0^s (L(x_\xi) + f(\xi))d\xi ds$$

$$\int_0^t (x(s)-Bx(s-r))ds + \int_0^s T(t-s)(Bx(s-r)-x(0))ds = \int_0^t T(t-s)\int_0^s (L(x_\xi) + f(\xi))d\xi ds$$

$$A\int_0^t (x(s)-Bx(s-r))ds + A\int_0^s T(t-s)(Bx(s-r)-x(0))ds = A\int_0^t T(t-s)\int_0^s (L(x_\xi)+f(\xi))d\xi ds$$

$$A\int_0^t (x(s)-Bx(s-r))ds + A\int_0^s T(t-s)(Bx(s-r)-x(0))ds = \int_0^t (T(t-s)-I)\int_0^s (L(x_\xi)+f(\xi))d\xi ds$$

$$A\int_0^t (x(s)-Bx(s-r))ds + \int_0^t (L(x_s)+f(s))ds =$$
$$\int_0^t T(t-s)(L(x_s)+f(s))ds + A\int_0^t T(t-s)(x(0)-Bx(s-r))ds$$

or x(.) is a weak solution then

$$x(t) - x(0) = A\int_0^t T(t-s)(x(0)-Bx(s-r))ds + \int_0^t T(t-s)(L(x_s)+f(s))ds$$

Our object, establish the convese of this result

Definition 6 *: Assume that A generates a C_0-semigroup T(.) on X. A function x(.) is called a mild solution of the problem (1) if :*

$$\int_0^t T(t-s)(x(0)-Bx(s-r))ds \in D(A) \text{ and}$$

$$x(t) - x(0) = A\int_0^t T(t-s)(x(0)-Bx(s-r))ds + \int_0^t T(t-s)(L(x_s)+f(s))ds \quad 0 \le t \le 2\pi.$$

Corollary 1 *Assume that A generates a C_0-semigroup $T(.)$ on X; let $f \in$*

$L^p(T,X)$

$x(.)$ is a weak solution $\Leftrightarrow$ $x(.)$ is a mild solution

proof :

$\Rightarrow$) by introduction

$\Leftarrow$) suppose that x(.) is a mild solution of Eq (2) then

$x(t) - x(0) = A\int_0^t T(t-s)(x(0)-Bx(s-r))ds + \int_0^t T(t-s)(L(x_s)+f(s))ds$

$\int_0^t (x(s)-x(0))ds = \int_0^t A\int_0^s T(t-\xi)(x(0)-Bx(\xi-r))d\xi ds + \int_0^t\int_0^s T(t-\xi)(L(x_\xi)+f(\xi))d\xi ds$

$\int_0^t (x(s)-x(0))ds = \int_0^t (T(t-s)-I)(x(0)-Bx(s-r))ds + \int_0^t\int_0^s T(t-\xi)(L(x_\xi)+f(\xi))d\xi ds$

$A\int_0^t (x(s)-x(0))ds = A\int_0^t (T(t-s)-I)(x(0)-Bx(s-r))ds + A\int_0^t\int_0^s T(t-\xi)(L(x_\xi)+f(\xi))d\xi ds$

$A\int_0^t (x(s)-x(0))ds = A\int_0^t (T(t-s)-I)(x(0)-Bx(s-r))ds + \int_0^t (T(t-s)-I)(L(x_s)+f(s))ds$

$A\int_0^t (x(s)-x(0))ds + \int_0^t (L(x_s)+f(s))ds + A\int_0^t (x(0)-Bx(s-r))ds = A\int_0^t T(t-s)(x(0)-Bx(s-r))ds + \int_0^t (T(t-s)(L(x_s)+f(s))ds$

$$\underbrace{A\int_0^t T(t-s)(x(0)-Bx(s-r))ds + \int_0^t (T(t-s)(L(x_s)+f(s))ds}_{=x(t)-x(0)} = A\int_0^t (x(s)-Bx(s-r))ds + \int_0^t (L(x_s)+f(s))ds$$

$x(t)-x(0) = A\int_0^t (x(s)-Bx(s-r))ds + \int_0^t (L(x_s)+f(s))ds$ then x(.) is a weak solution.

Proposition 1 : *Assume that A generates a C_0-semigroup $T(.)$ on X. if $(ikI - A(I-B_k) - L_k)^{-1}$ is an L^p-multiplier Then there exists a unique weak(mild) solution of Eq (1).*

proof : let f $\in L^p$(T,X), then f(t) $= lim_{n\to+\infty}\frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m}e^{ikt}\hat{f}$(k)
or $(ikI - A(I-B_k) - L_k)^{-1}$ is an L^p-multiplier then there exists x $\in L^p$(T,X)
such that $\hat{x}$(k) $= (ikI - A(I-B_k) - L_k)^{-1}\hat{f}$(k)
put x_n(t) $= lim_{n\to+\infty}\frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m}e^{ikt}(ikI - A(I-B_k) - L_k)^{-1}\hat{f}$(k)
then x_n(t) $\to$ x(t) and x_n is strong L^p-solution of Eq (1) and x_n verified

$$x_n(t) - x_n(0) = A\int_0^t ((x_n(t-s)) - Bx_n(t-s))ds + \int_0^t (G((x_n)_s) + f_n(s))ds$$

we put $y_n = x_n(0)$ then

$$x_n(t) = y_n + A\int_0^t ((x_n(t-s)) - Bx_n(t-s))ds + \int_0^t (L((x_n)_s) + f_n(s))ds$$

$t = 2\pi$

$$\underbrace{x_n(2\pi)}_{=x_n(0)} = y_n + A\int_0^{2\pi} ((x_n(t-s)) - Bx_n(2\pi-s))ds + \int_0^{2\pi} (L((x_n)_s) + f_n(s))ds$$

(n$\to\infty$)

$$* \; y = y + A\int_0^{2\pi} ((x(s) - Bx(2\pi-r))ds + \int_0^{2\pi} (L(x_s) + f(s))ds$$

$$x(t) = y + A\int_0^t (x(s) - Bx(t-s))ds + \int_0^t (L(x_s) + f(s))ds := g(t)$$

$$x(2\pi) = g(2\pi) = y + A\int_0^{2\pi} ((x(s) - Bx(2\pi-r))ds + \int_0^{2\pi} (L(x_s) + f(s))ds \overbrace{=}^{*}$$
y = g(0)

$\Rightarrow$ x(2π) = x(0), we conclude that x(.) is a 2π- periodic weak (mild) solution of Eq (1).

4 Exemple :

$\frac{d}{dt}$x(t) = A(x(t) - Bx(t-r)) + Lx_t + f(t)

let A be a closed linear operator and X be a UMD space, and

$sup_k\|(ikI - A(I-B_k))^{-1}\| =: M \prec \infty$ and $\|L\| \prec \frac{1}{r_{2\pi}^{1/p}}$then Eq (1) has a unique weak solution.
we have ikI - A(I - B_k) - L_k = [ikI - A(I-B_k)][I - $L_k(ikI - A(I-B_k))^{-1}$]
it follows that ikI - A(I - B_k) - L_k is invertible whenever
$\|L_k(ikI - A(I-B_k))^{-1}\| \prec 1$ [7.Theorem 1.1.7]
observe that $\|L_k\| \leq r_{2\pi}^{1/p}\|L\|$
Hence $\|L_k(ikI - A(I-B_k))^{-1}\| \leq r_{2\pi}^{1/p}\|L\|$M :=$\alpha$
Therefore, under the condition $\|L\| \prec \frac{1}{r_{2\pi}^{1/p}M}$

$$(ikI - A(I-B_k) - L_k)^{-1} = [ikI - A(B_k)]^{-1}[I - L_k(ikI - A(I-B_k)^{-1}]^{-1}$$

$$= [ikI - A(B_k)]^{-1}\sum_{n=0}^{\infty}[L_k(ikI - A(I-B_k)^{-1}]^n$$
it follows that :
$$\|ik(ikI - A(I-B_k) - L_k)^{-1}\| \leq \|ik(ikI - A(I-B_k))^{-1}\|\sum_{n=0}^{\infty}\alpha^n$$

$\leq \frac{M+1}{1-\alpha}$ then ikD_k is R-bounded.

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