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On α Generalized Closed Sets In Ideal Topological Spaces

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Abstract: *In this research paper, we are introducing the concept of α-generalized closed sets in Ideal topological space and discussed the characterizations and the properties of α-generalized closed sets in Ideal topological space.*
Keywords: *Ig closed sets, I$\hat{g}$- closed set, αIg- closed sets, Semi- I closed set, Pre- I closed set, α- I closed set,b- I closed set.*

I. Introduction

The notion of α-open sets was introduced and investigated by Njastad[1]. By using α-open sets, Mashhour et al.[2] defined and studied the concept of α-closed sets, α-closure of a set, α-continuity and α-closedness in topology. Ideals in topological spaces have been considered since 1930. This topic has won its importance by the paper of Vaidyanathaswamy[3]. It was the works of Newcomb[4], Rancin[5], Samuels and Hamlet and Jankovic([6, 7, 8, 9, 10]) which motivated the research in applying topological ideals to generalize the most basic properties in General Topology.

II. Preliminaries

An ideal I on a topological space (X, τ) is a nonempty collection of subsets of X, which satisfies the following two conditions:

(i) If A ∈ I and B⊆A implies B ∈ I
(ii) If A ∈ I and B ∈ I , then A∪B ∈ I [11].

An ideal topological space is a topological space (X, τ) with an ideal I on X and it is denoted by (X, τ, I). Given a topological space (X, τ) with an ideal I on X and if $\boldsymbol{\rho(X)}$ is the set of all subsets of X, a set operator (*****) : $\boldsymbol{\rho(X)} \rightarrow \boldsymbol{\rho(X)}$, called a local function[11] of A with respect to τ and I, is defined as follows: for A ⊆ X, $A^*(I,\tau)$={x ∈ X/U∩A∉I for every U∈τ(x)}where τ(x)={U ∈ τ/ x ∈ U}. We simply write A^* instead of $A^*(I,\tau)$. For every Ideal topological space (X, τ, I), there exists a topology $\tau^*(I)$, finer than τ,generated by $\boldsymbol{\beta(I,\tau)}$ = {U-i/U∈τ &i∈I}. But in general $\boldsymbol{(I,\tau)}$ is not always a topology. Additionally $cl^*(A)=A\cup A^*$ defines a kuratowski closure operator for $\tau^*(I)$. If A ⊆ X, cl(A) and int(A) will, respectively, denote the closure and interior of A in (X, τ) and $int^*(A)$ denote the interior of A in (X, τ^*). A subset A of an ideal space (X, τ, I) is ∗-closed(resp. ∗-dense in itself) if $A^*\subseteq A$ (resp. $A \subseteq A^*$).

Definition 2.1[13]
A subset A of a topological space (X, τ) is called a generalized closed set (briefly g-closed) if cl(A) ⊆ A whenever A ⊆ U and U is open in (X, τ).

Definition 2.2[15]
A subset A of a topological space (X, τ) is called a α-generalized closed set (briefly αg-closed set) if αcl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).

Definition 2.3[15]
A subset A of a topological space (X, τ) is called $\hat{\mathbf{g}}$-closed if cl(A) ⊆ U whenever A⊆ U and U is semi-open.

Definition 2.4[15]
A subset A of a topological space (X, τ) is said to be

(i) Pre closed set if cl (int (A)) ⊆ A.
(ii) Semi closed set if int (cl (A)) ⊆ A.
(iii) α- closed set if cl(int (cl (A))) ⊆ A.
(iv) b- closed set if cl(int (A)) ∪ int (cl (A)) ⊆A.

Definition 2.5[12]
Let (X , τ) be a topological space and I be an ideal on X. A subset A of X is said to be Ideal generalized closed set (briefly Ig- closed set) if $A^* \subseteq U$ whenever $A \subseteq U$ and U is open.

Definition 2.6[14]
A subset A of an ideal topological space (X, τ, I) is said to be
(i) pre-I-closed set if $cl^*(int(A)) \subseteq A$.
(ii) semi-I-closed set if $int(cl^*(A)) \subseteq A$.
(iii) α-I-closed set if $cl^*(int(cl^*(A))) \subseteq A$.
(iv) b-I-closed set if $cl^*(int(A)) \cup int(cl^*(A)) \subseteq A$.

Definition 2.7 [11]
Let (X, τ) be a topological space and I be an ideal on X. A subset A of X is said to be I$\hat{g}$-closed set if $A^* \subseteq U$ whenever $A \subseteq U$ and U is semi-open.

Lemma 2.8:[12]
Let (X, τ, I) be an ideal topological space and A, B subsets of X. Then the following properties hold:
(i) $A \subseteq B \Rightarrow A^* \subseteq B^*$,
(ii) $A^* = cl(A^*) \subseteq cl(A)$,
(iii) $(A^*)^* \subseteq A^*$,
(iv) $(A \cup B)^* = A^* \cup B^*$,
(v) $(A \cap B)^* \subseteq A^* \cap B^*$.

III. α-Ideal Generalized Closed sets

Definition 3.1
Let (X, τ) be a topological space and I be an ideal on X. A subset A of X is said to be α -Ideal generalized closed set (briefly αIg- closed set) if $A^* \subseteq U$ whenever $A \subseteq U$ and U is α-open.

Example 3.2
Let X={a,b,c} with topology τ ={Φ,X,{a},{b},{a,b}} and I={Φ,{b}}. The set A={a,c}, where A^* ={a,c} is an αIg- closed set.

Definition 3.3
Let (X, τ) be a topological space and I be an ideal on X. A subset A of X is said to be α -Ideal generalized open set (briefly αIg- open set) if X– A is αIg- closed set.

Theorem 3.4
If (X, τ, I) is any ideal space and $A \subseteq X$, then the following are equivalent.
(a) A is αIg-closed.
(b) $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in X.
(c) For all $x \in cl^*(A)$, $\alpha cl(\{x\}) \cap A \neq \Phi$.
(d) $cl^*(A) - A$ contains no nonempty α-closed set.
(e) $A^* - A$ contains no nonempty α-closed set.

Proof
(a)⇒(b): If A is αIg-closed, then $A^* \subseteq U$ whenever $A \subseteq U$ and U is α-open in X and so $cl^*(A) = A \cup A^* \subseteq U$ whenever $A \subseteq U$ and U is α-open in X. This proves (b).
(b)⇒(c): Suppose $x \in cl^*(A)$. If $\alpha cl(\{x\}) \cap A = \Phi$, then $A \subseteq X - \alpha cl(\{x\})$. By (b), $cl^*(A) \subseteq X - \alpha cl(\{x\})$, which is a contradiction to $x \in cl^*(A)$. This proves (c).
(c)⇒(d): Suppose $F \subseteq cl^*(A) - A$, F is α-closed and $x \in F$. Since $F \subseteq X - A$ and F is α-closed, then $A \subseteq X - F$ and hence $\alpha cl(\{x\}) \cap A = \Phi$. Since $x \in cl^*(A)$ by (c), $\alpha cl(\{x\}) \cap A \neq \Phi$. Therefore, $cl^*(A) - A$ contains no nonempty α-closed set.
(d)⇒(e) : Since $cl^*(A) - A = (A \cup A^*) - A = (A \cup A^*) \cap A^c = (A \cap A^c) \cup (A^* \cap A^c) = A^* \cap A^c = A^* - A$. Therefore, $A^* - A$ contains no nonempty α-closed set.

(e)⇒(a): Let $A \subseteq U$ where U is α-open set. Therefore $X–U \subseteq X–A$ and so $A^* \cap (X–U) \subseteq A^* \cap (X–A) = A^*– A$. Therefore $A^* \cap (X–U) \subseteq A^*– A$. Since A^* is always closed set, $A^* \cap (X–U)$ is a α-closed set contained in $A^*– A$. Therefore, $A^* \cap (X–U) = \Phi$ and hence $A^* \subseteq U$. Therefore, A is αIg-closed.

Theorem 3.5
Every ∗-closed set is αIg-closed set but not conversely.
Proof:
Let A be a ∗-closed, then $A^* \subseteq A$. Let $A \subseteq U$, and U is α-open. This implies $A^* \subseteq U$. Hence A is αIg-closed.

Example 3.6
Let X={a,b,c} with topology τ ={Φ,X,{a},{b,c}} and I={Φ,{c}}. It is clear that A= {b} is αIg-closed set since A^*={b,c} ⊆ U where U is α-open. But A is not a ∗-closed set.

Remark 3.7
αIg- closed set and αI-closed set are independent to each other, as seen from the following examples.

Example 3.8
Let X={a,b,c} with topology τ ={Φ,{a},{b,c},X} and I={Φ,{c}}. Clearly, the set A={b} which is an αIg-closed set is not an αI- closed set since $cl^*(int(cl^*(A)))$={b,c}⊄ A.

Example 3.9
Let X={a,b,c} with topology τ ={Φ,{a},{a,c},X} and I={Φ,{b}}. It is clear that A={c} is an αI- closed set. But A is not an αIg-closed set since A^*={b,c}⊄ U.

Remark 3.10
αIg- closed set and semi I-closed set are independent to each other, as seen from the following examples.

Example 3.11
Let X={a,b,c} with topology τ ={Φ,{a},{b,c},X} and I={Φ,{c}}. Clearly, the set A={b} is an αIg-closed set but not semi I- closed set since $int(cl^*(A))$={b,c}⊄ A.

Example 3.12
Let X={a,b,c} with topology τ ={Φ,{a},{b},{a,b},X} and I={Φ,{a}}. It is clear that A={b} which is semi I-closed set. But A is not an αIg-closed set since A^*={b,c}⊄ U .

Remark 3.13
Every pre I-closed set need not be an αIg- closed set.

Example 3.14
Let X={a,b,c} with topology τ ={Φ,{a},{a,c},X} and I={Φ,{b}}. Clearly, the set A={c} is pre I- closed set but not an αIg- closed set since A^*={b,c} ⊄ U.

Remark 3.15
αIg- closed set and b I-closed set are independent to each other, as seen from the following examples.

Example 3.16
Let X={a,b,c} with topology τ ={Φ,{a},{b,c},X}and I={ Φ,{c}}. Clearly, the set A={a,b} is an αIg-closed set, but not a b I-closed set, since $cl^*(int(A)) \cup int(cl^*(A))$ =X⊄A.

Example 3.17
Let X={a,b,c} with topology τ ={Φ,{a},{a,c},X} and I={ Φ, {b}}. It is clear that A={c} is bI-closed set. But A is not an αIg-closed set since A^*={b,c}⊄ U.

Theorem 3.18
Every αIg-closed set is an Ig- closed set but not conversely.

Proof
Let $A \subseteq U$ and U is open. Clearly every open set is α-open. Since A is αIg-closed set, $A^* \subseteq U$, which implies that A is an Ig-closed set.

Example 3.19
Let X={a,b,c} with topology τ ={Φ,{a},{a,c},X} and I={Φ,{b}}. Clearly, the set A={a,b} is Ig- closed set but not an αIg- closed set since $A^*=X \not\subset U$.

Theorem 3.20
Every I$\hat{g}$-closed set is an αIg-closed set.

Proof
Let $A \subseteq U$ and U is α-open. Clearly, every α-open set is semi-open. Since A is I$\hat{g}$-closed set, $A^* \subseteq U$, which implies that A is an αIg- closed set.

Theorem 3.21
Let (X,τ,I) be an ideal space. For every A ∈ I, A is αIg-closed set.

Proof
Let $A \subseteq U$ where U is α-open set. Since $A^*=\Phi$ for every A ∈ I, then $A^* \subseteq A$. This implies $A^* \subseteq U$. Hence for every A ∈ I, A is an αIg-closed set.

Theorem 3.22
If A and B are αIg-closed sets in (X,τ,I), then $A \cup B$ is also an αIg-closed set.

Proof
Let $A \cup B \subseteq U$ where U is α-open in X. Then $A \subseteq U$ and $B \subseteq U$. Since A and B are αIg-closed set, then $A^* \subseteq U$ and $B^* \subseteq U$ and so $A^* \cup B^* \subseteq U$. By Lemma 2.8(iv), $(A \cup B)^* = A^* \cup B^* \subseteq U$. Hence $A \cup B$ is an αIg-closed set.

Remark 3.23
The intersection of αIg-closed sets need not be an αIg-closed set as shown from the following example.

Example 3.24
Let X={a,b,c,d} with topology τ ={Φ,{a},{d},{a,d},X} and I={Φ,{d}}. If A={b,c}, B={b,d}, then A and B are αIg-closed sets but their intersection $A \cap B$ ={b} is not an αIg- closed set.

Theorem 3.25
If (X, τ, I) is an ideal space, then A^* is always an αIg-closed set for every subset A of X.

Proof
Let $A^* \subseteq U$, where U is α-open. Since $(A^*)^* \subseteq A^*$ [12], we have $(A^*)^* \subseteq U$ whenever $A^* \subseteq U$ and U is α-open. Hence A^* is an αIg-closed set.

Theorem 3.26
If (X, τ,I) is an ideal space, then every αIg-closed, which is α-open is *-closed set.

Proof
Let A be an αIg-closed and α-open set. Then $A \subseteq A$ implies $A^* \subseteq A$ since A is α-open. Therefore, A is *-closed set.

Theorem 3.27
Let (X, τ, I) be an ideal topological space and A be an αIg-closed set. Then the following are equivalent.
(a) A is a *-closed set.
(b) cl*(A) –A is a α-closed set.
(c) A^*–A is a α-closed set.

Proof

(a)⇒(b): If A is *-closed, then $A^* \subseteq A$ and so $cl^*(A) - A = (A \cup A^*) - A = \Phi$. Hence $cl^*(A) - A$ is α-closed set.
(b)⇒(c): Since $cl^*(A) - A = A^* - A$ and so $A^* - A$ is α-closed set.
(c)⇒(a): If $A^* - A$ is a α-closed set, then by Theorem 3.4, $A^* - A = \Phi$ and so A is *-closed.

Theorem 3.28

Let (X, τ, I) be an ideal space and $A \subseteq X$. Then A is αIg-closed if and only if A=F–N, where F is *-closed and N contains no nonempty α-closed set.

Proof

If A is αIg-closed, then by Theorem 3.4(e), $N = A^* - A$ contains no nonempty α-closed set. If $F = cl^*(A)$, then F is *-closed such that $F - N = (A \cup A^*) - (A^* - A) = (A \cup A^*) \cap (A^* \cap A^c)^c = (A \cup A^*) \cap ((A^*)^c \cup A) = (A \cup A^*) \cap (A \cup (A^*)^c) = A \cup (A^* \cap (A^*)^c) = A$.
Conversely, suppose A=F –N where F is *-closed and N contains no nonempty α-closed set. Let U be a α-open set such that $A \subseteq U$. Then $F - N \subseteq U \Rightarrow F \cap (X - U) \subseteq N$. Now $A \subseteq F$ and $F^* \subseteq F$ then $A^* \subseteq F^*$ and so $A^* \cap (X - U) \subseteq F^* \cap (X - U) \subseteq F \cap (X - U) \subseteq N$. By hypothesis, since $A^* \cap (X - U)$ is α-closed, $A^* \cap (X - U) = \Phi$ and so $A^* \subseteq U$. Hence A is αIg-closed.

Lemma 3.29[11]

Let (X, τ, I) be an ideal space and $A \subseteq X$. If $A \subseteq B \subseteq A^*$, then $A^* = B^*$ and B is *-dense in itself.

Theorem 3.30

Let (X, τ, I) be an ideal space. If A and B are subsets of X such that $A \subseteq B \subseteq cl^*(A)$ and A is αIg-closed , then B is αIg-closed.

Proof

Since A is αIg-closed then by Theorem 3.4(d), $cl^*(A) - A$ contains no nonempty α-closed set. Since $cl^*(B) - B \subseteq cl^*(A) - A$ and so $cl^*(B) - B$ contains no nonempty α-closed set. Hence B is αIg-closed set.

Theorem 3.31

Let (X, τ, I) be an ideal space and $A \subseteq X$. Then, A is αIg-open if and only if $F \subseteq int^*(A)$ whenever F is α-closed and $F \subseteq A$.

Proof

Suppose that A is αIg-open. Let $F \subseteq A$ and F be α-closed. Then $X - A \subseteq X - F$ and X –F is α-open. Since X –A is αIg-closed, then $(X - A)^* \subseteq X - F$ and $X - int^*(A) = cl^*(X - A) = (X - A) \cup (X - A)^* \subseteq X - F$ and hence $F \subseteq int^*(A)$.
Conversly, Let $X - A \subseteq U$ where U is α-open. Then $X - U \subseteq A$ and X– U is α-closed. By hypothesis, we have $X - U \subseteq int^*(A)$ and hence $(X - A)^* \subseteq cl^*(X - A) = X - int^*(A) \subseteq U$. Therefore X–A is αIg closed and A is αIg open.

Theorem 3.32

Let (X, τ, I) be an ideal space and $A \subseteq X$. If A is αIg open and $int^*(A) \subseteq B \subseteq A$, then B is αIg-open.

Proof

Since A is αIg open, then X – A is αIg closed. By Theorem 3.4(d), $cl^*(X - A) - (X - A)$ contains no nonempty α-closed set. Since $int^*(A) \subseteq int^*(B)$, $X - cl^*(X - A) \subseteq X - cl^*(X - B)$ which implies that $cl^*(X - B) \subseteq cl^*(X - A)$ and so $cl^*(X - B) - (X - B) \subseteq cl^*(X - A) - (X - A)$. Hence B is αIg open.

Theorem 3.33

Let (X, τ, I) be an ideal space. Then, every subset of X is αIg-closed if and only if every α-open set is *-closed.

Proof

Suppose every subset of X is αIg-closed. If $U \subseteq X$ is α-open, then U is αIg-closed and so $U^* \subseteq U$. Hence U is *-closed. Conversely, suppose that every α-open set is *-closed. If U is α-open set such that $A \subseteq U \subseteq X$, then $A^* \subseteq U \subseteq X$ and so A is αIg-closed.

Theorem 3.34

If A and B are αIg-open sets in (X, τ, I), then A ∩ B is an αIg-open set.

Proof

If A and B are αIg-open sets, then X – A and X – B are αIg-closed sets. By Theorem 3.22, (X – A) ∪ (X – B) is an αIg-closed set, which implies that X – (A ∩ B) is an αIg-closed set. Hence A ∩ B is an αIg-open set.

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