IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN:2319-765X. Volume 10, Issue 2 Ver. II (Mar-Apr. 2014), PP 17-25
www.iosrjournals.org

On certain generalized difference sequence spaces with some of their topological properties defined by a sequence of moduli

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Abstract: *The idea of difference sequences* $X(\Delta)=\{x=(x_k):\Delta x\in X\}$ with $X=l_\infty, c$ and c_0 *was introduced by kizmaz. In this paper, using the sequence of moduli we define some generalized sequence spaces and give various topological properties of these spaces with some of their inclusion relations. Furthermore, we study some of their properties, such as solidity, symmetricity, convergence free and so on.*
Keywords: *Difference sequence space, Paranorm, Seminorm, Sequence of moduli*

I. Introduction

Let ω be the set of all sequences of real or complex numbers (scalar). It is not difficult to see that any arbitrary sequence space X can be shown to be a vector space by defining its vector addition and multiplication as follows:
Let

$$x=(x_1,x_2,x_3,..),\ y=(y_1,y_2,y_3,..)$$

be two arbitrary elements of X i.e $x,y\in X$ *and* $\alpha\in K$ then

(i.) $x+y=(x_1+y_1,x_2+y_2,...)\in X$

(ii.) $\alpha x=\alpha(x_1,x_2,x_3,...)$
$=(\alpha x_1,\alpha x_2,\alpha x_3,...)\in X$

In view of this, a sequence space is any linear subspace of ω. The spaces l_∞, c *and* c_0 are of particular interest in this research work.

In 1981, Kizmaz [1] introduces the notion of difference sequence spaces as follows:

$$X(\Delta)=\{x=(x_k)\in\omega:(\Delta x_k)\in X\}$$

$$X=l_\infty,\ c \text{ and } c_0\ X(\Delta)=\{x=(x_k)\in\omega:(\Delta x_k)\in X\}$$

where

$$\Delta x=\Delta x_k=(x_k-x_{k+1})$$

They are Banach spaces with the norm

$$\| x\|_\Delta=|x_1|+\|\Delta_x\|_\infty .$$

Later on, this notion was generalized in the following way:

$$X(\Delta^m)=\{x=(x_k)\in\omega:(\Delta^m x_k)\in X\}$$

for $X=l_\infty$, c and c_0, where $m\in\mathbb{N}$, where

$$\Delta^m x=\Delta^m x_k=(\Delta^{m-1}x_k-\Delta^{m-1}x_{k+1})$$
$$\Delta x_k=(x_k-x_{k+1})$$
$$\text{and } \Delta^0 x=(x_k).$$

such that

$$\Delta^m x_k=\sum_{i=0}^{m}(-1)^i\binom{m}{i}x_{k+i}$$

They are also Banach spaces with norm defined by

$$\| x\|_\Delta=\sum_{i=1}^{m}|x_i|+\|\Delta^m x\|_\infty$$

(Et and Colak, [2])

Worthy it is to mention that recently, the sequence spaces $X(\Delta^m)$ were generalized by Et and Esi [3] to the following sequence spaces

$$X(\Delta_v^m) = \left\{ x = (x_k) \in \omega : (\Delta_v^m x_k) \in X \right\}$$

for $X = l_\infty$, c and c_0, where

$$\Delta_v^0 x = (v_k x_k),\ \Delta_v x_k = (v_k x_k - v_{k+1} x_{k+1})$$

and

$$\Delta_v^m x = \Delta_v^m x_k = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})$$

such that

$$\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+1} x_{k+i}$$

and $v = (v_k) \neq 0$ is any fixed sequence of non-zero numbers for all $k \in \mathbb{N}$. (Et and Esi, [3])

Subsequently, difference sequence spaces have been studied by Asma and Et [4], Bektas et al [5], Colak [6], Isik [7] and Mursaleen [8].

Definition (See Nakano [9]): A function $f : [0,\infty) \to [0,\infty)$ is called a modulus if

- i.) $f(t) = 0$ if and only if $t = 0$
- ii.) $f(t+u) \leq f(t) + f(u)$ for all $t, u \geq 0$
- iii.) f is increasing, and
- iv.) f is continuous from the right of 0.

Let X be a sequence space. Then the sequence space $X(f)$ is defined by

$$X(f) = \left\{ x = (x_k) \in \omega : \left(f\left(|x_k|\right)\right) \in X \right\}.$$

Kolk [10] gave an extension of $X(f)$ by considering a sequence of moduli $F = (f_k)$ i.e

$$X(F) = \left\{ x = (x_k)_{k=1}^\infty \in \omega : \left(f_k\left(|x_k|\right)\right) \in X \right\},$$

where $X = l_\infty$, c or c_o.

Later, Gaur and Mursaleen [11] defined the following sequence spaces

$$l_\infty(F, \Delta) = \left\{ x = (x_k) \in \omega : (x_k) \in l_\infty(F) \right\}$$

$$c_0(F, \Delta) = \left\{ x = (x_k) \in \omega : (x_k) \in c_0(F) \right\}$$

Recently Khan [12] defined the following sequence spaces

$$X(F, p) = \left\{ x = (x_k)_{k=1}^\infty \in \omega : \left(f_k\left(|x_k|\right)\right) \in X(p) \right\}$$

In particular,

$$l_\infty(F, p) = \left\{ x = (x_k)_{k=1}^\infty \in \omega : \sup_k f_k |x_k|^{p_k} < \infty \right\}$$

$$c_0(F, p) = \left\{ x = (x_k)_{k=1}^\infty \in \omega : f_k\left(|x_k|^{p_k}\right) \to 0; (k \to \infty) \right\},$$

$$l_\infty(F, p, \Delta^m) = \left\{ x = (x_k)_{k=1}^\infty \in \omega : \left(\Delta^m x_k\right) \in l_\infty(F, p) \right\},$$

$$c_0(F, p, \Delta^m) = \left\{ x = (x_k)_{k=1}^\infty \in \omega : \left(\Delta^m x_k\right) \in c_0(F, p) \right\}.$$

For any sequence of moduli $F = (f_k)$

Khan [12] used the following lemmas to prove few results on the inclusion of the above spaces.

Lemma 1: The condition $\sup_k f_k(t) < \infty, t > 0$ holds if and only if there exist a point $t_0 > 0$ such that $\sup_k f_k(t_0) < \infty.$

Lemma 2: The condition $\inf_k f_k(t) > 0, t > 0$ holds if and only if there exist a point $t_0 > 0$ such that $\inf_k f_k(t_0) > 0$.

A modulus may be bounded or unbounded. Maddox [13]) and Ruckle [14] used a modulus function to construct some sequence spaces.

Later on, modulus function was investigate by Bhardwaj [15], Bilgin 16], Connor [17], Esi [18] and many others.

Let $P = (p_k)$ be a sequence of strictly positive real numbers and $s \geq 0$. Let *X* be a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm q. The symbol $\omega(X)$ denotes the space of all sequences defined over *X*. Let $V = (v_k)$ be any fixed sequences of non-zero complex numbers. We define the following sequence spaces as follows.

$$c(\Delta_v^m, f, p, q, s) = \left\{ x = (x_k) \in \omega : \lim_{k\to\infty} k^{-s} \left[f\left(q\left(\Delta_v^m x_k - l\right)\right)\right]^{p_k} = 0, l \in X \right\}$$

$$c_0(\Delta_v^m, f, p, q, s) = \left\{ x = (x_k)_{k=0}^{\infty} \in \omega : \lim_{k\to\infty} k^{-s} \left[f\left(q\left(\Delta_v^m x_k\right)\right)\right]^{p_k} = 0, \right\}$$

$$l_\infty(\Delta_v^m, f, p, q, s) = \left\{ x = (x_k)_{k=0}^{\infty} \in \omega : \sup_k k^{-s} \left[f\left(q\left(\Delta_v^m x_k\right)\right)\right]^{p_k} < \infty \right\},$$

where f is a modulus function.

The following inequality will be used throughout this research.

Let $P = (p_k)$ be a sequence of positive real number with

$0 < p_k \leq \sup_k p_k = H,\ D = \max\left(1, 2^{H-1}\right)$. then, for $a_k, b_k \in \mathbb{C}$ we have.

$$|a_k + b_k|^{p_k} \leq D\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\},.$$

Some well known spaces as we shall see later, are obtained by specializing f, s, q, v and m.

i.) If $f(x) = x$, $m = 0$, $V = (v_k) = (1,1,1,...)$ and $q(x) = |x|$, then $c(\Delta_v^m f, p, q, s) = c(p, s)$, $c_0(\Delta_v^m f, p, q, s) = c_0(p, s)$, and $l_\infty(\Delta_v^m f, p, q, s) = l_\infty(p, s)$ ….(See Basarir, [19])

ii.) If $f(x) = x$, $m = 0$, $V = (v_k) = (1,1,1,...)$, $s = 0$ and $q(x) = |x|$ then $c(\Delta_v^m f, p, q, s) = c(p)$, $c_0(\Delta_v^m f, p, q, s) = c_0(p)$ and $l_\infty(\Delta_v^m f, p, q, s) = l_\infty(p)$ (See Maddox, [20])

If $m = 0$ and $q(x) = |x|$, $V = (v_k) = (1, 1, 1, \dots)$, then

$c(\Delta_v^m f, p, q, s) = c(p, f, s)$, $c_0(\Delta_v^m f, p, q, s) = c_0(p, f, s)$ and $l_\infty(\Delta_v^m f, p, q, s) = l_\infty(p, f, s)$ (Esi, [18]).

Definition (Gulcan and Cigdem, [21]): Let $f = (f_k)$ be sequence of moduli *X* be a seminormed space over the field $\mathbb{C}$ of complex number with the seminorm *q*, $p = (p_k)$ be a sequence of strictly positive real numbers and $u \in U$. By $\omega(X)$ we shall denote the space of all sequences defined over *X*. let $v = (v_k)$ be any fixed sequence of nonzero complex number. Now we define the following sequence spaces

$$l_\infty\left(\Delta_v^m, F, p, q, u\right) = \left\{ x \in \omega : \sup_k u_k \left[F_k\left(q\left(\Delta_v^m x_k\right)\right)\right]^{p_k} < \infty \right\}$$

$$c\left(\Delta_v^m, F, p, q, u\right) = \left\{ x \in \omega : \lim_{k\to\infty} u_k \left[f_k\left(q\left(\Delta_v^m x_k - l\right)\right)\right]^{p_k} = 0 \right\}$$

and

$$c_0\left(\Delta_v^m, F, p, q, u\right) = \left\{ x \in \omega : \lim_{k\to\infty} u_k \left[f_k\left(q\left(\Delta_v^m x_k\right)\right)\right]^{p_k} = 0 \right\}$$

For $p_k = 1$ for all $k \in \mathbb{N}$ we write these spaces as $l_\infty\left(\Delta_v^m, F, q, u\right)$, $c\left(\Delta_v^m, F, q, u\right)$ and $c_0\left(\Delta_v^m, F, q, u\right)$

For $u_k = 1$ for all $k \in \mathbb{N}$, we write these spaces as $l_\infty\left(\Delta_v^m, F, p, q\right)$, $c\left(\Delta_v^m, F, p, q\right)$ and $c_0\left(\Delta_v^m, F, p, q\right)$.

A sequence $x \in l_\infty$ is said to be almost convergent (Lorentz. [22]) if all Banach limits of x coincide. Lorentz [22] proved that

$$\hat{c} := \left\{ x = (x_k) \in \omega : \lim_n \frac{1}{n} \sum_{k=1}^{n} x_{k+s}, \textit{uniformly in } S \right\}$$

Maddox ([23] and [24]) has defined x to be strongly almost convergent to l if

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - l| = 0, \textit{ uniformly in S}, \text{ for some } l > 0$$

Let $p = (p_k)$ be a sequence of strictly positive real number. Nanda [25] defined.

$$[\hat{c}, p] := \left\{ x = (x_k) \in \omega : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - l|^{p_k} = 0, \textit{ uniformly in S, for some } l > 0 \right\},$$

$$[\hat{c}, p]_0 := \left\{ x = (x_k) \in \omega : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+s}|^{p_k} = 0, \textit{uniformly in } S \right\},$$

$$[\hat{c}, p]_\infty := \left\{ x = (x_k) \in \omega : \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - l|^{p_k} = 0, \textit{uniformly in } S \right\}$$

It was extended by Khan and Ayaz [26] that if $F = (f_k)$ is a sequence of moduli, $u = (u_i)$ be any sequence such that $u_k \neq 0$ for all k and $p = (p_k)$ be any sequence space of strictly positive real number then w define the following sequence spaces:

$$[\hat{c}, F, p](\Delta_u^m) := \left\{ x = (x_k) \in \omega : \lim_n \frac{1}{n} \sum_{k=1}^{n} \left[f_k \left(|u_k \Delta^m x_{k+s} - l| \right) \right]^{p_k} = 0, \textit{uniformly in } S \right\}, \text{ for some } l > 0,$$

$$[\hat{c}, F, p]_0 (\Delta_u^m) := \left\{ x = (x_k) \in \omega : \lim_n \frac{1}{n} \sum_{k=1}^{n} \left[f_k \left(|u_k \Delta^m x_{k+s}| \right) \right]^{p_k} = 0, \textit{ uniformly in } S \right\},$$

$$[\hat{c}, F, p]_\infty (\Delta_u^m) := \left\{ x = (x_k) \in \omega : \sup_{s,n} \frac{1}{n} \sum_{k=1}^{n} \left[f_k \left(|u_k \Delta^m x_{k+s}| \right) \right]^{p_k} < \infty, \textit{uniformly in } S \right\}$$

For if $f_k(x) = x$ for every k, then $[\hat{c}, F, p](\Delta_u^m) = [\hat{c}, p]$,

$$[\hat{c}, F, p]_0 (\Delta_u^m) = [\hat{c}, p]_0 \text{ and } [\hat{c}, F, p]_\infty (\Delta_u^m) = [\hat{c}, p]_\infty .$$

And for if $p_k = 1$ for all K it yields

$$[\hat{c}, F, p](\Delta_u^m) = [\hat{c}, F](\Delta_u^m),$$

$$[\hat{c}, F, p]_0 (\Delta_u^m) = [\hat{c}, F]_0 (\Delta_u^m)$$

and,

$$[\hat{c}, F, p]_\infty (\Delta_u^m) = [\hat{c}, F]_\infty (\Delta_u^m).$$

II. Main Results

Suppose that $F = (f_k)$ is a sequence of moduli, $p = (p_k)$ be any sequence of strictly positive real numbers with q, a seminorm and $u = (u_k) \neq 0$ be any sequence of all K. Let $v = (v_k)$ be any fixed sequence of nonzero complex numbers. Then we define the following sequence spaces:

$$\left[\hat{c}_{\Delta_v^m}, F, p, q, u \right]_c := \left\{ x = (x_k) \in \omega(X) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} u_k \left[f_k \left(\left| q \left(\Delta_v^m x_{k+s} - l \right) \right| \right) \right]^{p_k} = 0, \textit{ uniformly in } S \right\}, \text{ for some } l > 0,$$

$$\left[\hat{c}_{\Delta_v^m}, F, p, q, u \right]_{c_0} := \left\{ x = (x_k) \in \omega(X) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} u_k \left[f_k \left(\left| q \left(\Delta_v^m x_{k+s} \right) \right| \right) \right]^{p_k} = 0, \textit{ uniformly in } S \right\}$$

and

$$\left[\hat{c}_{\Delta_v^m}, F, p, q, u \right]_{l_\infty} := \left\{ x = (x_k) \in \omega(X) : \sup_{s,n} \frac{1}{n} \sum_{k=1}^{n} u_k \left[f_k \left(\left| q \left(\Delta_v^m x_{k+s} \right) \right| \right) \right]^{p_k} < \infty, \textit{ uniformly in } S \right\}.$$

If $q(x) = |x|$, $u_k = 1$ then

$$\left[\hat{c}_{\Delta_v^m}, F, p, q, u \right]_c = \left[\hat{c}_{\Delta_v^m}, F, p \right]_c,$$

$$\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{c_0} = \left[\hat{c}_{\Delta_v^m}, F, p\right]_{c_0}$$

and,

$$\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{l_\infty} = \left[\hat{c}_{\Delta_v^m}, F, p\right]_{l_\infty} \qquad \text{(Khan and Ayaz, [26])}$$

But if $m = 0$, $v = (v_k) = (1,1,1,...)$, $f_k(x) = x$ for all k, $q(x) = |x|$, $u_k = 1$ then,

$$\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_c = \left[\hat{c}, p\right]_c,$$

$$\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{c_0} = \left[\hat{c}, p\right]_{c_0}$$

and $\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{l_\infty} = \left[\hat{c}, p\right]_{l_\infty}$

And if $m = 0$, $v = (v_k) = (1,1,1,...)$, $f_k(x) = x$, $q(x) = |x|$, $p_k = 1$ and $u_k = 1$ then

$$\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_c = \left[\hat{c}\right]_c,$$

$$\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{c_0} = \left[\hat{c}\right]_{c_0}$$

and $\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{l_\infty} = \left[\hat{c}\right]_{l_\infty}$ (See Maddox [23] and [24])

If $F = (f_k)$ be a sequence of moduli then $\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{c_0} \subset \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_c \subset \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{l_\infty}$ are strict.

Proof. The first inclusion is clear. We shall only establish the second inclusion.

Suppose that $x \in \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_c$, by the definition of modulus function and lemma1, we obtain;

$$\frac{1}{n}\sum_{k=1}^{n} u_k \left[f_k \left(\left| q\left(\Delta_v^m x_{k+s}\right) \right| \right) \right]^{p_k} \le D \frac{1}{n}\sum_{k=1}^{n} u_k \left[f_k \left(\left| q\left(\Delta_v^m x_{k+s} - l + l\right) \right| \right) \right]^{p_k}$$

$$\Rightarrow \frac{1}{n}\sum_{k=1}^{n} u_k \left[f_k \left(\left| q\left(\Delta_v^m x_{k+s}\right) \right| \right) \right]^{p_k} \le D \frac{1}{n}\sum_{k=1}^{n} u_k \left[f_k \left(\left| q\left(\Delta_v^m x_{k+s} - l\right) \right| \right) \right]^{p_k} + D \frac{1}{n}\sum_{k=1}^{n} u_k \left[f_k \left(\left| q\left(l\right) \right| \right) \right]^{p_k}$$

We therefore now may choose an integer D such that $q(l) \le k$. Hence, we have

$$\frac{1}{n}\sum_{k=1}^{n} u_k \left[f_k \left(\left| q\left(\Delta_v^m x_{k+s}\right) \right| \right) \right]^{p_k} \le D \frac{1}{n}\sum_{k=1}^{n} u_k \left[f_k \left(\left| q\left(\Delta_v^m x_{k+s} - l\right) \right| \right) \right]^{p_k} + \max.\left[1, \left((k_l) f_k(1)\right)^H\right]$$

Hence $x \in \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{l_\infty}$

To show the inclusion are strict consider the following example:

Suppose $f_k(x) = x$, $p_k = 1$, $v_k = 1$, $u_k = 1$ for all $k \in \mathbb{N}$ and $q(x) = x$. Then, the sequence $x = (k^m)$ belongs to $\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_c$ but does not belong to $\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{c_0}$ and sequence $x = \left((-1)^k\right)$ belongs to $\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{l_\infty}$ but does not belong to $\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_c$. Therefore the inclusions are strict.

Theorem 2.2: Let the sequence (p_k) be bounded. Then $\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_c$, $\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{c_0}$ and $\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{l_\infty}$ are linear spaces over the complex field $\mathbb{C}$.

Proof:

We shall give the proof for $\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{c_0}$ only, the others can be treated similarly.

Let $x, y \in \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{c_0}$. For $\lambda, \alpha \in \mathbb{C}$, there exit positive integers M_λ and N_α, such that $|\lambda| \le M_\lambda$ and $|\alpha| \le N_\alpha$. Since f is subadditive, q is a seminorm, and Δ_v^m is

$$\frac{1}{n}\sum_{k=1}^{n}u_k\left[f_k\left(\left|q\left(\Delta_v^m(\lambda x_{k+s}+\alpha x_{k+s})\right)\right|\right)\right]^{p_k} \le D\left(\max\left(1,|M_\lambda|^H\right)\right)\frac{1}{n}\sum_{k=1}^{n}u_k\left[f_k\left(\left|q\left(\Delta_v^m x_{k+s}\right)\right|\right)\right]^{p_k}$$

$$+D\left(\max\left(1,|N_\alpha|^H\right)\right)\frac{1}{n}\sum_{k=1}^{n}u_k\left[f_k\left(\left|q\left(\Delta_v^m x_{k+s}\right)\right|\right)\right]^{p_k} \to 0 \,.\, k\to\infty$$

Hence, $\left[\hat{c}_{\Delta_v^m},F,p,q,u\right]_c$ is a linear space.

The space $\left[\hat{c}_{\Delta_v^m},F,p,q,u\right]_{c_0}$ is a paranormed space, paranormed by

$$g(x)=\sum_{i=1}^{m}f_k\left(\left|q\left(v_i x_i\right)\right|\right)+\sup_k\frac{1}{n}\sum_{k=1}^{n}u_k\left[f_k\left(\left|q\left(\Delta_v^m x_{k+s}\right)\right|\right)\right]^{\frac{p_k}{M}}$$

where $M=\max\left(1,\sup_k p_k\right)$; $\left[\hat{c}_{\Delta_v^m},F,p,q,u\right]_c$ and $\left[\hat{c}_{\Delta_v^m},F,p,q,u\right]_{c_0}$ are paranormed by *g* if $\inf_k p_k>0$.

Proof:

Obviously, $g(x)=g(-x)$ for all $x\in\left[\hat{c}_{\Delta_v^m},F,p,q,u\right]_{c_0}$

It is trivial that $\Delta_v^m x_k=\overline{e}$ for $x=\overline{e}$, where $\overline{e}=(e,e,e,...)$ and *e* is the zero element of *X*.

Since $q(\overline{e})=0$ and $f(0)=0$,

We get $g(e)=0$, since $t_k={p_k}/{M}\le 1$,

If a_k and b_k are complex numbers then we have;

$$|a_k+b_k|^{t_k}\le|a_k|^{t_k}+|b_k|^{t_k}$$

Since $M\ge 1$, the above inequality implies that

$$\sum_{i=1}^{m}f_k\left(\left|q\left(v_i x_i+v_i y_i\right)\right|\right)+\sup_k\frac{1}{n}\sum_{k=1}^{n}u_k\left[f_k\left(\left|q\left(\Delta_v^m(x_{k+s}+y_{k+s})\right)\right|\right)\right]^{\frac{p_k}{M}}$$

$$\le\sum_{i=1}^{m}f_k\left(\left|q\left(v_i x_i\right)\right|\right)+\sup_k\frac{1}{n}\sum_{k=1}^{n}u_k\left[f_k\left(\left|q\left(\Delta_v^m x_{k+s}\right)\right|\right)\right]^{\frac{p_k}{M}}+\sum_{i=1}^{m}f_k\left(\left|q\left(v_i y_i\right)\right|\right)+\sup_k\frac{1}{n}\sum_{k=1}^{n}u_k\left[f_k\left(\left|q\left(\Delta_v^m y_{k+s}\right)\right|\right)\right]^{\frac{p_k}{M}}$$

Now, it follows that g is subadditive. Next, let λ be a nonzero scalar. The continuity of scalar multiplication follows from the inequality.

But note that $\lambda g(x)=g(\lambda x)$

$$\text{So, } g(\lambda x)=k_\lambda\sum_{i=1}^{m}f_k\left(\left|q\left(v_i x_i\right)\right|\right)+\sup_k\frac{1}{n}\sum_{k=1}^{n}\left(k_\lambda^{\frac{p_k}{M}}\right)u_k\left[f_k\left(\left|q\left(\Delta_v^m x_{k+s}\right)\right|\right)\right]^{\frac{p_k}{M}}$$

$$\le\max\left(1,k_\lambda^{\frac{H}{M}}\right)g(x)$$

Where k_λ is an integer such that $|\lambda|<k_\lambda$ and $H=\sup_k p_k$

This complete the proof of the theorem other cases will follow by applying similar technique.

Theorem 2.4: Let $F=(f_k)$ and $G=(g_k)$ be two sequences of moduli. For any two sequences $p=(p_k)$ and $t=(t_k)$ of strictly positive real seminorms q,q_1, and q_2 and u,u_1 and $u_2\ge 0$. Then

(i) $\left[\hat{c}_{\Delta_v^m},F,p,q,u\right]_z\subseteq\left[\hat{c}_{\Delta_v^m},G\circ F,p,q,u\right]_z$

(ii) $\left[\hat{c}_{\Delta_v^m},F,p,q,u\right]_z\cap\left[\hat{c}_{\Delta_v^m},G,p,q,u\right]_z\subseteq\left[\hat{c}_{\Delta_v^m},F+G,p,q,u\right]_z$

(iii) $\left[\hat{c}_{\Delta_v^m},F,p,q_1,u\right]_z\cap\left[\hat{c}_{\Delta_v^m},F,p,q_2,u\right]_z\subseteq\left[\hat{c}_{\Delta_v^m},F,p,q_1+q_2,u\right]_z$

(iv) If q_1 is stronger than q_2, then

(v) $\left[\hat{c}_{\Delta_v^m}, F, p, q_1, u\right]_z \subseteq \left[\hat{c}_{\Delta_v^m}, F, p, q_2, u\right]_z$

(vi) If $u_1 \le u_2$, then $\left[\hat{c}_{\Delta_v^m}, F, p, q, u_1\right]_z \subseteq \left[\hat{c}_{\Delta_v^m}, F, p, q, u_2\right]_z$ where $z = l_\infty, c$ and c_0

Proof:

(i) We prove for $z = c$ and the rest cases will follow similarly.

Let $x \in \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_c$, so that $\mu_n = \frac{1}{n}\sum_{k=1}^{n} u_k \left[f_k\left(\left|q\left(\Delta_v^m x_{k+s} - l\right)\right|\right)\right]^{p_k} \to 0, (n \to \infty)$

Let $\in > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \in$

For $0 \le t \le \delta$. Now we write

$$s_1 := \left\{k \in \mathbb{N} : f_k\left(\left|q\left(\Delta_v^m x_{k+s} - l\right)\right|\right) \le \delta\right\},$$

$$s_2 := \left\{k \in \mathbb{N} : f_k\left(\left|q\left(\Delta_v^m x_{k+s} - l\right)\right|\right) > \delta\right\}.$$

If $x \in \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_c$, then for $f_k\left(\left|q\left(\Delta_v^m x_{k+s} - l\right)\right|\right) > \delta$,

$$f_k\left(\left|q\left(\Delta_v^m x_{k+s} - l\right)\right|\right) < f_k\left(\left|q\left(\Delta_v^m x_{k+s} - l\right)\right|\right)\delta^{-1} < 1 + \left[f_k\left(\left|q\left(\Delta_v^m x_{k+s} - l\right)\right|\right)\delta^{-1}\right],$$

Where $k \in s_2$ and $[n]$ denotes the integer part of n.

Given $\in > 0$, by definition of modulus function we have, for

$$f_k\left(\left|q\left(\Delta_v^m x_{k+s} - l\right)\right|\right) > \delta$$

$$g_k\left(f_k\left(\left|q\left(\Delta_v^m x_{k+s} - l\right)\right|\right)\right) \le 1 + \left[f_k\left(\left|q\left(\Delta_v^m x_{k+s} - l\right)\right|\right)\delta^{-1}\right] g_k(1)$$

$$\le 2g_k(1)\left(f_k\left(\left|q\left(\Delta_v^m x_{k+s} - l\right)\right|\right)\delta^{-1}\right)$$

and hence

$$\frac{1}{n}\sum_{k=1}^{n} u_k \left[g_k\left(f_k\left(\left|q\left(\Delta_v^m x_{k+s} - l\right)\right|\right)\right)\right]^{p_k} \le \left[2g_k(1)\delta^{-1}\right]^H \mu_n < \in, \ \left(k \in s_1, k > k_2\right) \quad (1)$$

If $x \in \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_c$, $f_k\left(\left|q\left(\Delta_v^m x_{k+s} - l\right)\right|\right) \le \delta$ (2)

Form (1) and (2) for every $k > \max\{k_1, k_2\}$,

$$\frac{1}{n}\sum_{k=1}^{n} u_k \left[g_k\left(f_k\left(\left|q\left(\Delta_v^m x_{k+s} - l\right)\right|\right)\right)\right]^{p_k} < \in$$

Hence, $x \in \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_c$

Thus,

$$\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_c \subset \left[\hat{c}_{\Delta_v^m}, F \circ G, p, q, u\right]_c.$$

(ii) It follows from the following inequality

$$\frac{1}{n}\sum_{k=1}^{n} u_k \left[(f_k + g_k)\left(\left|q\left(\Delta_v^m x_{k+s} - l\right)\right|\right)\right]^{p_k} \le D\frac{1}{n}\sum_{k=1}^{n} u_k \left[f_k\left(\left|q\left(\Delta_v^m x_{k+s} - l\right)\right|\right)\right]^{p_k} + D\frac{1}{n}\sum_{k=1}^{n} u_k \left[g_k\left(\left|q\left(\Delta_v^m x_{k+s} - l\right)\right|\right)\right]^{p_k}$$

(iii) It follows from the following inequality

$$\frac{1}{n}\sum_{k=1}^{n} u_k \left[f_k\left(\left|(q_1 + q_2)\left(\Delta_v^m x_{k+s} - l\right)\right|\right)\right]^{p_k} \le D\frac{1}{n}\sum_{k=1}^{n} u_k \left[f_k\left(\left|q_1\left(\Delta_v^m x_{k+s} - l\right)\right|\right)\right]^{p_k} + D\frac{1}{n}\sum_{k=1}^{n} u_k \left[f_k\left(\left|q_2\left(\Delta_v^m x_{k+s} - l\right)\right|\right)\right]^{p_k}$$

(iv.) and (v.) follows obviously.

Theorem 2.5: Let $m \ge 1$ then for all $0 < i \le m$, $\left[\hat{c}_{\Delta_v^i}, F, p, q, u\right]_z \subset \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_z$ where $z = l_\infty, c$ or c_0.

The inclusions are strict.

Proof: We shall prove that

$$\left[\hat{c}_{\Delta_v^i}, F, p, q, u\right]_{l_\infty} \subset \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{l_\infty}$$

For any $0 < i \leq m$.
It follows from the following inequality

$$\frac{1}{n}\sum_{k=1}^{n} u_k \left[f_k \left(\left| q\left(\Delta_v^i x_{k+s}\right)\right| \right)\right]^{p_k} \leq \frac{1}{n}\sum_{k=1}^{n} u_k \left[f_k \left(\left| q\left(\Delta_v^{i-1} x_{k+s}\right)\right| \right)\right]^{p_k} + \frac{1}{n}\sum_{k=1}^{n} u_k \left[f_k \left(\left| q\left(\Delta_v^{i-1} x_{k+s+1}\right)\right| \right)\right]^{p_k}$$

That $(x_k) \in \left[\hat{c}_{\Delta_v^{i-1}}, F, p, q, u\right]_{l_\infty}$ implies $(x_k) \in \left[\hat{c}_{\Delta_v^i}, F, p, q, u\right]_{l_\infty}$

On applying the principle of induction, it follows that
$\left[\hat{c}_{\Delta_v^{i-1}}, F, p, q, u\right]_{l_\infty} \subset \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{l_\infty}$ for $i = 0, 1, 2, 3, ..., m-1$.

The proofs for the rest of the cases are similar.
To show that the inclusions are strict, consider the following example.
Let $X = \mathbb{C}$, $f_k(x) = x$, $p_k = 1$, $v_k = (1,1,1,...)$, $u_k = 1$ for all $k \in \mathbb{N}$ and $q(x) = |x|$.

Then for $z = c_0$ the sequence $(x_k) \notin \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_z$

Under the above restrictions, consider the sequence
$x = k^m \in \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_z$ but $x \notin \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_z$ for $z = l_\infty$ and c.

Therefore the inclusions are strict.

Theorem 2.6: for any two sequences $p = (p_k)$ and $t = (t_k)$ of strictly positive real numbers and for any two seminorms q_1 and q_2 on X, we have
$\left[\hat{c}_{\Delta_v^m}, F, p, q_1, u\right]_z \subset \left[\hat{c}_{\Delta_v^m}, F, t, q_2, u\right]_z \neq \emptyset$ for $z = c, c_0$ and l_∞.

Proof: Since the zero element belong to each of the above classes of sequences, the intersection is nonempty.

Theorem 2.7: Let $0 < p_k \leq t_k$ and (t_k / p_k) be bounded. Then $\left[\hat{c}_{\Delta_v^m}, F, t, q, u\right]_z \subset \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_z$. Where $z = l_\infty, c$ and c_0.

Proof: We shall prove only $\left[\hat{c}_{\Delta_v^m}, F, t, q, u\right]_{c_0} \subset \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{c_0}$. The other inclusions can be proved similarly.

Let $x \in \left[\hat{c}_{\Delta_v^m}, F, t, q, u\right]_c$, write

$w_k = \left[f_k \left(\left| q\left(\Delta_v^m x_{k+s}\right)\right| \right)\right]^{t_k}$ and $\mu_k = \dfrac{p_k}{t_k}$, so that $0 < \lambda < \lambda_k \leq 1$ for each k.

We defined the sequence (r_k) and (s_k) as follows:
Let $r_k = w_k$ and $s_k = 0$ if $w < 1$. Then it is clear that for all $k \in \mathbb{N}$, we have $w_k = r_k + s_k$, $w_k = r_k^{\lambda_k} + s_k^{\lambda_k}$.
Now it follows that $r_k^{\lambda_k} \leq r_k \leq w_k$ and $s_k^{\lambda_k} \leq s_k^{\lambda}$.

Now we are $\lim_k w_k^{\lambda_k} \leq \lim_k w_k + \left(\lim_k s_k\right)^{\lambda}$

This implies that $x \in \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{c_0}$ and this completes the proof.

Theorem 2.8: The sequence spaces $\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_z$ for $z = l_\infty, c$ and c_0 are not solid for $m > 0$.

Proof: Let $X = \mathbb{C}$, $f_k(x) = x$, $p_k = 1$, $u_k = 1$, $v_k = 1$ for all $k \in \mathbb{N}$ and $q(x) = |x|$. Then
$(x_k) = k^m \in \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{l_\infty}$ but $(\alpha_k x_k) \notin \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{l_\infty}$ when $\alpha_k = (-1)^k$ for all $k \in \mathbb{N}$. Hence
$\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{l_\infty}$ is not solid. The other cases can be proved on considering similar examples.

Theorem 2.9: The sequence spaces $\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_z$ for $z = l_\infty, c$ and c_0, are not symmetric for $m > 0$.

Proof: Let $X = \mathbb{R}$, $f_k(x) = x$, $p_k = 1$, $u_k = 1$, $v_k = 1$ for all $k \in \mathbb{N}$ and $q(x) = |x|$. Then consider the sequence $(x_k) = (k^m)$, then $x \in \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_z$ for $z = l_\infty, c$ and c_0.

Let (y_k) be a rearrangement of (x_k), which is defined as follows:

$(y_k) = \{x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{14}, x_{10}, ...\}$.

Then $(y_k) \notin \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_z$ for $z = l_\infty, c$ and c_0.

Theorem 2.10: The sequence space $\left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{c_0}$ is not a sequence algebra.

Proof: Let $X = \mathbb{R}$, $f_k(x) = x$, $p_k = 1$, $u_k = 1$, $v_k = 1$ for all $k \in \mathbb{N}$ and $q(x) = |x|$. Then consider sequence $x = (k^{m-2})$ and $y = (k^{m-2})$, then

$x, y \in \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{c_0}$ but $x \cdot y \notin \left[\hat{c}_{\Delta_v^m}, F, p, q, u\right]_{c_0}$.

For other sequences consider $x = (k^m)$ and $y = (k^m)$

Theorem 2.11: The sequence spaces for $z = l_\infty, c$ and c_0 are not convergence free.

Proof: Let $X = \mathbb{R}$, $f_k(x) = x$, for all $x \in [0, \infty)$, $m = 1$, $p_k = 1$, $u_k = 1$, $v_k = 1$, for all $k \in \mathbb{N}$. Then $(x_k) \in z(\Delta)$, for $z = l_\infty, c$ and c_0. Consider the sequence (y_k) defined as $(y_k) = k^2$ for all $k \in \mathbb{N}$. Then the sequence (y_k) neither belongs to $c_0(\Delta)$ nor $c(\Delta)$ nor to $l_\infty(\Delta)$. Hence the sequence spaces are not convergence free.

III. Conclusion

The concept of difference sequence, however introduced recently, has attracted many researchers because of its broad applications. The topological properties of the newly obtained difference sequence spaces are significant in their own right, on one hand and give rise to many questions regarding the inclusion relation among themselves.

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