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A New Result On $|A,p_n,\delta|_k$-Summabilty

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Abstract: *In this paper we have established a new theorem on* $|A,p_n,\delta|_k$ *-summability which gives some new and interesting results and previous known results as a corollary.*

Keywords: $|\bar{N},p_n|$*-summability,* $|A|_k$ *-summability,* $|A,\delta|_k$ *-summability,* $|A,p_n,\delta|_k$ *-summability and infinite series.*
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I. INTRODUCTION:

Let Σa_n be a given infinite series with the sequence of partial sum (s_n) and let $A=(a_{nv})$ be a normal matrix of non zero diagonal entries. Then A defines the sequence to sequence transformation mapping the sequences $s=(s_n)$ to $A_s=(A_n(s))$,

$$A_n(\mathrm{s})=\sum_{v=1}^{\infty} A_{nv}s_v$$

where (1.1)

The series Σa_n is said to summable $|A|_k, k\ge 1$ if (RHOADES and SAVAS [3])

$$\sum_{n=1}^{\infty} n^{k-1}|\bar{\Delta}A_n(s)|^k<\infty \qquad (1.2)$$

where $\bar{\Delta}A_n(s)=A_n(s)-A_{n-1}(s)$ and it is said to be summable $|A,\delta|_k, k\ge 0$ and $\delta\ge 0$ if

$$\sum_{n=1}^{\infty} n^{\delta k+k-1}|\Delta \mathrm{A}_{n-1}|^k<\infty \qquad (1.3)$$

Let (p_n) be a sequence of positive numbers such that

$$P_n=\sum_{v=0}^{n} p_v\to\infty \text{ as } n\to\infty \qquad (1.4)$$

where $P_{-i}=p_{-i}=0, i\ge 1$ and Σa_n is said to be summable $|A,p_n|_k, k\ge 1$ if (ÖZARSLAN, [2])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|\bar{\Delta}A_n(s)|^k<\infty \qquad (1.5)$$

And is said to be summable $|A,p_n,\delta|_k, .k\ge 1$ if

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|\bar{\Delta}A_n(s)|^k<\infty \qquad (1.6)$$

If $P_n=1,\delta=0$, $|A,p_n,\delta|_k$ -summability is the same as $|A|_k$ -summability also if we take $a_{nv}=\frac{p_v}{p_n}$, then $|A,p_n|_k$ -summability is the same as $|\bar{N},p_n|_k$ -summability (BOR [1]).

A sequence (b_n) of positive numbers is said to be δ-quasi monotone, if $b_n>0$ ultimately and $\Delta b_n\ge -\delta_n$ where (δ_n) is a sequence of positive numbers (SAVAS [4]).

and a sequence (d_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants A and B such that

$$Ac_n\le d_n\le Bc_n \text{ for each } n.$$

II. KNOWN RESULT:

Concerning with absolute matrix summability factor SAVAS [5] has proved the following theorem.

Theorem 2.1

Let A be a lower triangular or Normal matrix with non-negative entries satisfying

$$\overline{a}_{n,0} = 1 \tag{2.1}$$

$$a_{n-1}, v \geq a_{nv} \text{ for } n \geq v+1 \tag{2.2}$$

$$na_{nn} = O(1) \tag{2.3}$$

$$\sum_{n=v+1}^{m+1} n^{\delta k} |\Delta_v \hat{a}_{nv}| = O(v^{\delta k} a_{vv}) \tag{2.4}$$

$$\sum_{n=v+1}^{m+1} n^{\delta k} |\hat{a}_{n,v+1}| = O(v^{\delta k}) \tag{2.5}$$

where A associates with two lower triangular matrices $\overline{A}$ & $\hat{A}$ defined.

$$\overline{a}_{nv} = \sum_{r=v}^{n} a_{nr}, \quad n, v = 0,1,2 \quad \text{and}$$

$$\hat{a}_{nv} = \overline{a}_{nv} - \overline{a}_{n-1,v}, \quad n = 1,2,3$$

If (X_n) is an almost increasing sequence such that,

$$|\Delta X_n| = O\left(\frac{X_n}{n}\right) \text{ and} \tag{2.6}$$

and

$$\lambda_n \to 0 \text{ as } n \to \infty \tag{2.7}$$

Suppose that there exist a sequence of numbers (A_n) such that it is δ-quasi monotone with $\Sigma n X_n \delta_n < \infty$, $\Sigma A_n X_n$ is convergent and

$$\sum_{n=1}^{\infty} \frac{|\lambda_n|}{n} < \infty \tag{2.8}$$

$$\sum_{n=1}^{\infty} n^{\delta k-1} |t_n|^k = O(X_m) \tag{2.9}$$

where $t_n = \frac{1}{n+1} \sum_{k=1}^{n} k a_k$,

then the series $\Sigma a_n \lambda_n$ is summable $|A, \delta|_k, k \geq 1$ and $\delta \geq 0$.

III. MAIN RESULT:

The goal of this paper is to generalize the theorem (2.1) for $|A, p_n, \delta|_k$ -summability.

Theorem 3.1

If $A = (a_{nv})$ is any normal matrix associated with two lower sub-matrices $\overline{A} = (\overline{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows

$$\overline{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0,1,2 \tag{3.1}$$

and

$$\hat{a}_{nv} = \overline{a}_{nv} - \overline{a}_{n-v,v} \tag{3.2}$$

where $\hat{a}_{0,0} = \overline{a}_{0,0} = a_{0,0}$.

If the conditions

$$\overline{a}_{n,0} = 1 \tag{3.3}$$

$a_{n-1,v} \geq a_{n,v}$ for $n \geq v+1$

and let (p_n) be the sequence of positive numbers such that,

$$P_n = O(np_n) \text{ as } n \to \infty \tag{3.4}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right) \tag{3.5}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_v \hat{a}_{nv}| = O\left(\frac{P_v}{p_v} a_{vv}\right) \tag{3.6}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\hat{a}_{n,v+1}| = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k}\right) \tag{3.7}$$

If $\{X_n\}$ is an almost increasing sequence such that $\left(\frac{P_n}{p_n}|\Delta X_n|\right) = O(X_n)$ and $\lambda_n \to 0$ as $n \to \infty$.

Suppose that there exist a sequence of numbers (A_n) such that it is δ-quasi monotone with $\Sigma n X_n \delta_n < \infty$, $\Sigma A_n X_n$ is convergent and $|\Delta\lambda_n| \leq |A_n|$ for all n, if

$$\sum_{n=1}^{\infty} \frac{p_n |\lambda_n|}{P_n} < \infty \tag{3.8}$$

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k-1} |t_n|^k = O(X_m) \tag{3.9}$$

where $t_n = \frac{1}{n+1}\sum_{k=1}^{n} k a_k$

are satisfied then the series $\Sigma a_n \lambda_n$ issummable$|A, p_n, \delta|_k, k \geq 1, \delta \geq 0.$

IV. LEMMA:

We need the following lemmas for the proof of theorem (3.1).

Lemma 4.1.

Under the condition of theorem, we have ((SAVAS [4])

$$|\lambda_n| X_n = O(1) \tag{4.1}$$

Lemma 4.2 (SAVAS [5])

Let $\{X_n\}$ is an almost increasing sequence such that

$$|\Delta X_n| = O\left(\frac{X_n}{n}\right)$$

If (A_n) is δ-quasi monotone with $\Sigma_n X_n \delta_n < \infty, \Sigma A_n X_n$ is convergent, then

$\sum_{n=1}^{\infty} n X_n |\Delta A_n| < \infty$ and

$n A_n X_n = O(1)$

V. PROOF OF THEOREM:

Let $\{y_n\}$ be the nth term of the A-transform of $\sum_{i=0}^{n} \lambda_i a_i$ then,

$$Y_n = \sum_{i=0}^{n} a_{ni} s_i$$
$$= \sum_{i=0}^{n} a_{ni} \sum_{i=0}^{i} \lambda_v a_v$$
$$= \sum_{v=0}^{n} \lambda_a a_v \sum_{i=v}^{n} a_{n,i}$$
$$= \sum_{v=0}^{n} \bar{a}_{nv} \lambda_v a_v \qquad (5.1)$$

and

$$\bar{y}_n = y_n - y_{n-1} = \sum_{v=0}^{n} (\bar{a}_{nv} - \bar{a}_{n-1,v}) \lambda_v a_v$$

$$= \sum_{v=0}^{n} \hat{a}_{nv} \lambda_v a_v \qquad (5.2)$$

we may write

$$y_n = \sum_{v=1}^{n} \left(\frac{\hat{a}_{nv} \lambda_v}{v} \right) v a_v$$
$$= \sum_{v=1}^{n} \left(\frac{\hat{a}_{nv} \lambda_v}{v} \right) \left[\sum_{r=1}^{v} r a_r - \sum_{r=1}^{v-1} r a_r \right]$$
$$= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} \lambda_v}{v} \right) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn} \lambda_n}{n} \sum_{v=2}^{n} v a_v$$
$$= \sum_{v=1}^{n-1} | \Delta_v \hat{a}_{nv} | \lambda_v \frac{v+1}{v} t_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} (\Delta \lambda_v) \frac{v+1}{v} t_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \frac{1}{v} t_v + (n+1) \frac{a_{nn} \lambda_n t_n}{n}$$
$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} \text{ (say)} \qquad (5.3)$$

To complete the proof, it is sufficient, by Minkowski's inequality, to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} | T_{n,r} |^k < \infty, \text{ for } r = 1, 2, 3, 4 \qquad (5.4)$$

Using Hölder's inequality and (5.3), we get

$$I_1 = \sum_{n=1}^{m} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} | T_{n,1} |$$
$$= \sum_{n=1}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} \left| \sum_{v=1}^{n-1} \Delta_v \hat{a}_{n,v} \lambda_v \frac{v+1}{v} t_v \right|^k$$
$$= O(1) \sum_{n=1}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} \left(\sum_{v=1}^{n-1} | \Delta_v \hat{a}_{vv} | \lambda_v || \mathrm{t}_v | \right)^k$$
$$= O(1) \sum_{n=1}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} \left(\sum_{v=1}^{n-1} | \Delta_v \hat{a}_{vv} | \lambda_v |^k | \mathrm{t}_v |^k \right) \left(\sum_{v=1}^{n-1} \Delta_v a_{vv} \right)^{k-1}$$
$$= O(1) \sum_{n=1}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k} \left(\frac{P_n}{p_n} a_{nn} \right)^{k-1} \sum_{v=1}^{n-1} | \Delta_v \hat{a}_{vv} | \lambda_v |^k | \mathrm{t}_v |^k$$

$$= O(1)\sum_{n=1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\left(\frac{P_n}{p_n}a_{nn}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\lambda_v|^{k-1}|\lambda_v||\Delta\hat{a}_{vv}||t_v|^k\right)$$

$$= O(1)\sum_{v=1}^{m}|\lambda_v||t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\left(\frac{P_n}{p_n}a_{nn}\right)^{k-1}|\Delta_v\hat{a}_{n,v}|$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}|\lambda_v|a_{vv}|t_v|^k$$

$$= O(1)\sum_{v=1}^{m}|\lambda_v|\left[\sum_{r=1}^{v}a_{rr}|t_r|^k\left(\frac{P_r}{p_r}\right)^{\delta k}+\sum_{r=1}^{v-1}a_{rr}|t_r|^k\left(\frac{P_r}{p_r}\right)^{\delta k}\right]$$

$$= O(1)\sum_{v=1}^{m-1}\Delta(|\lambda_v|\sum_{r=1}^{v}|t_r|^k\left(\frac{P_r}{p_r}\right)^{\delta k-1}+|\lambda_m|\sum_{r=1}^{m}|t_r|^k\left(\frac{P_r}{p_r}\right)^{\delta k-1}$$

$$= O(1)\sum_{v=1}^{m-1}|A_v|X_v+O(1)|\lambda_m|X_m$$

$$= O(1). \qquad (5.5)$$

Again, using the hypothesis of the theorem (3.1) and Lemma (4.1), using Hölder's inequality

$$I_2=\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,2}|^k$$

$$=\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v+1}(\Delta\lambda_v)\frac{v+1}{v}t_v\right|$$

$$\le\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left[\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v|\left|\frac{v+1}{v}\right||t_v|\right]^k$$

$$=O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left[\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right]^k$$

$$=O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left[\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|^k\right]\left[\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v|\right]^{k-1}$$

from (Rhoades and Savas[3]).

$$\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v|\le Ma_{nn}$$

Hence

$$I_2=O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\left(\frac{P_n}{p_n}a_{nn}\right)^{k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|^k$$

$$=O(1)\sum_{v=1}^{m}||\Delta\lambda_v||t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\left(\frac{P_n}{p_n}a_{nn}\right)^{k-1}|\hat{a}_{n,v+1}|$$

$$=O(1)\sum_{v=1}^{m}|\Delta\lambda_v||t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}|$$

$$=O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}|\Delta\lambda_v||t_v|^k$$

$$= O(1)\sum_{v=1}^{m}\Delta\left(\frac{P_v}{p_v}\right)^{\delta k}\left(\frac{P_v}{p_v}\right)|\Delta\lambda_v||t_v|^k\left(\frac{p_v}{P_v}\right)$$

$$= O(1)\sum_{v=1}^{m}\Delta\left(\frac{P_v}{p_v}|\Delta\lambda_v|\right)\sum_{r=1}^{r}\left(\frac{P_r}{p_r}\right)^{\delta k-1}|t_r|^k + O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}\left(\frac{P_r}{p_r}\right)^{\delta k-1}|t_r|^k .$$

$$= O(1)\sum_{v=1}^{m}\Delta\left(\frac{P_v}{p_v}|\Delta\lambda_v|\right)X_v + O(1)m|\Delta\lambda_m|X_m$$

$$= O(1)\sum_{v=1}^{m}\Delta\left(\frac{P_v}{p_v}|\Delta\lambda_v|\right)X_v + O(1)\sum_{v=1}^{m-1}|A_{v-1}|X_{v-1} + O(1)m|A_m|X_m$$

$$= O(1)$$

Next using the hypothesis of the theorem (3.1) and Hölder's inequality

$$I_3 = \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,3}|^k$$

$$= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\lambda_{v+1}\frac{t_v}{v}\right|^k$$

$$\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left[\frac{|\lambda_{v+1}|}{v}|\hat{a}_{n,v+1}||t_v|\right]^k$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left[\sum_{v=1}^{n-1}\frac{|\lambda_{v+1}|}{v}|t_v|^k|\hat{a}_{n,v+1}|^k\right]\left[\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\frac{|\lambda_{v+1}|}{v}\right]^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\left(\frac{P_n}{p_n}a_{nn}\right)^{k-1}\left[\sum_{v=1}^{n-1}\frac{|\lambda_{v+1}|}{v}|t_v|^k|\hat{a}_{n,v+1}|\right]\left[\sum_{v=1}^{n-1}\frac{|\lambda_{v+1}|}{v}\right]^{k-1}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}\left(\frac{P_n}{p_n}a_{nn}\right)^{k-1}\sum_{v=1}^{n-1}\frac{|\lambda_{v+1}|}{v}|t_v|^k|\hat{a}_{n,v+1}|$$

$$= O(1)\sum_{v=1}^{m}\frac{|\lambda_{v+1}|}{v}|t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\left(\frac{P_n}{p_n}a_{nn}\right)^{k-1}|\hat{a}_{n,v+1}|$$

$$= O(1)\sum_{v=1}^{m}\frac{|\lambda_{v+1}|}{v}|t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}|$$

$$= O(1)\sum_{v=1}^{m}\frac{|\lambda_{v+1}|}{v}\left(\frac{P_v}{p_v}\right)^{\delta k}|t_v|^k$$

$$= O(1)\sum_{v=1}^{m}(|\lambda_{v+1}|\left(\frac{P_v}{p_v}\right)^{\delta k-1}|t_v|^k$$

$$= O(1)\sum_{v=1}^{m-1}(|\Delta\lambda_{v+1}|)X_v + O(1)|\lambda_{m+1}|X_m$$

$$= O(1)$$

Finally

$$I_4 = \sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} | T_{n,4} |^k$$

$$= \sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left|\frac{(n+1)a_{nn}\lambda_n t_n}{n}\right|^k$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} | a_{nn} |^k | \lambda_n |^k | t_n |^k$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left(\frac{P_n}{p_n} a_{nn}\right)^{k-1} a_{nn} | \lambda_n |^{k-1} | \lambda_n || t_n |^k$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k} a_{nn} | \lambda_n || t_n |^k$$

$= O(1)$, as in the proof of I_1.

This completes the proof of theorem.

VI. COROLLARY:

This theorem have the following results as a corollary.

Corollary 6.1

Taking $\left(\frac{P_n}{p_n}\right) = n$ the theorem (3.1) reduces to theorem (2.1).

Corollary 6.2

Taking $\frac{P_n}{p_n} = n$, and $\delta = 0$ the theorem (3.1) is $| A|_k$ -summable.

Corollary 6.3

Taking $a_{nv} = \frac{p_v}{P_n}$, and $\delta = 0$ then theorem (3.1) is $| \bar{N}, p_n |_k$ -summable.

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