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Square Sum Graph Associated with a Sequence of Positive Integers

Reena Sebastian[1], K.A Germina[2]
[1]*(Department of Mathematics, S.E.S College,Sreekandapuram, India.)*
[2]*(School of Mathematical and Physical Sciences, Central University of Kerala, Kasargode, India.)*

Abstract: *A (p,q)-graph G is said to be square sum, if there exists a bijection f: V(G)→{0,1,2,..,p-1} such that the induced function f*:E(G)→ N defined by $f^*(uv)= (f(u))^2+(f(v))^2$, for every uv∈ E(G) is injective. In this paper, a recursive construction of infinite families of square sum graphs associate with a sequence of positive integers are studied. That is for any sequence of positive integers (a1,a2,..,an) with ai≥ 2, i=1,2,..,n we associate some square sum graphs. In particular we obtain the result of level joined planar grid are square sum as the special case.*
Keywords: *Square sum graphs, Level joined planar grid.*

I. Introduction

If the vertices of the graph are assigned values subject to certain conditions, it is known as graph labeling. A dynamic survey on graph labeling is regularly updated by Gallian[1]. Let G= (V,E) be a (p,q)-graph. Unless mentioned otherwise, by a graph we shall mean in this paper a finite, undirected, connected graph without loops or multiple edges. Terms not defined here are used in the sense of Harary[2]. Acharya and Germina defined a square sum labeling of a (p,q)-graph G [3,4] as follows.

Definition 1.1

A $(p,q)-graph G is said to be square sum, if there exists a bijection f: V(G)→{0,1,2,..,p-1} such that the induced function f*:E(G)→ N defined by $f^*(uv)= (f(u))^2+(f(v))^2$, for every uv∈ E(G) is injective.

Here, for any sequence of positive integers $(n_1,n_2,…,,nt)$ with ni≥ 2, i=1,2,..t, we associate a square sum graph H(<n1,n2,…,nt>) of order n1+n2+…+nt and size n1+n2+2n_i , 2≤ i<t. In particular we obtain the result of level joined planar grids are square sum as a special case. Example of square sum graph is shown in Fig 1.

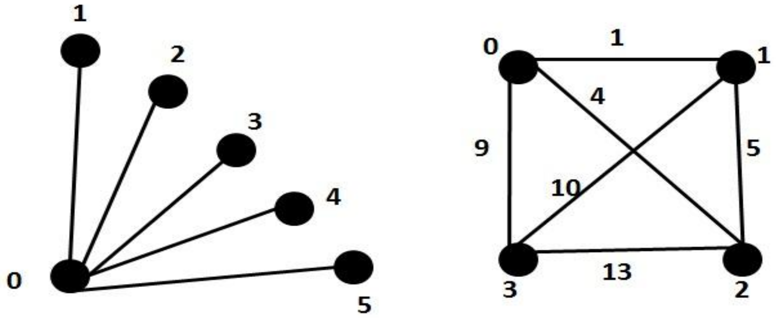

Figure 1

II. Construction of Square Sum Graphs Associated with Sequence of Positive Integers.

Here we present a recursive construction of infinite families of square sum graphs associate with a sequence of positive integers. Let n1,n2,…,nt∈ $N^{>1}$, where $N^{>1}$={2,3,4,....}.Let σ be a sequence of nonzero integers < n1,n2,…,nt> of $N^{>1}$ where t≥2. If n_1≥ 2, we denote G as the class of square sum graph constructed by the following way.

When t=2, G consists of the graph of the form H(<n_1,n_2>) where n_2∈ $N^{>1}$. Let v(1,1),v(1,2),…,v(1,n1),v(2,1),…,v(2,n2) be the vertices of H(<n_1,n_2>). The graph H(<n_1,n_2>) has n_1+n_2 vertices with two layers. The top most layer has vertices v(1,1),v(1,2),..,v(1,n1) and the second layer has vertices v(2,1),…,v(2,n2). The graph H< n1,n2>) has edges in the form :

1) If n2≤ n1, then E(H(<n1,n2>))={(v(1,i),v(2,i)),(v(1,i+1),v(2,i)):1≤i ≤ n_2}∪
{(v(1,i+1),v(2,n2)): n_2<i≤ n_1-1}∪{(v(2,1),v(2,n2))}.

2) If n1 < n2, then
E(H(<n1,n2>))= {(v(1,i),v(2,i)),(v(1,i),v(2,i+1)):1≤i≤ n1}∪

{(v(1,n1),v(2,i+1)):n1<i≤ n2-1}∪{(v(2,1),v(2,n2))}
It is square sum with the following labeling:
f(v(1,i))=i-1 for 1≤i≤ n_1 and f(v(2,i))=n1+i-1 for 1≤i≤ n_2.
(see Fig. 2 for H(<5,3>) and H(<5,8>)

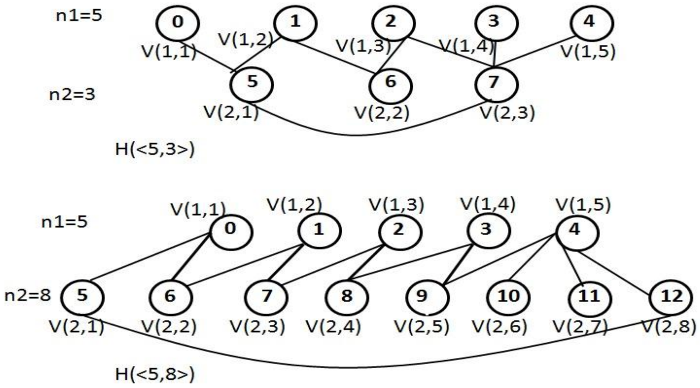

Figure 2.

When t=3, G consists of the graph of the form H(<n1,n2,n3>) where n3∈$N^{>1}$. Let v(1,1),v(1,2),…,v(1,n1),v(2,1),..,v(2,n2),v(3,1),…,v(3,n3) be the vertices of H(<n1,n2,n3>). The graph H(<n1,n2,n3>) has n1+n2+n3 vertices with three layers. The top most layer has vertices v(1,1),v(1,2),…,v(1,n1), the second layer has vertices v(2,1),…,v(2,n2) and third layer has vertices v(3,1),…,v(3,n3). We arrange the vertices of H(<n1,n2,n3>) layer by layer and from left to right as follows.
a) The top most layer is H(<n1,n2>)
b) The second layer has vertices v(3,1),…,v(3,n3). The graph H(<n1,n2,n3>) has edges of the form :
1). If n3≤ n2, then E(H(<n1,n2,n3>))= E(H(<n1,n2>)) ∪{(v(2,i),v(3,i)),(v(2,i+1),v(3,i)): 1≤i≤ n3}∪ {(v(2,i+1),v(3,n3)): n3<i ≤ n2-1}∪ (v(3,1),v(3,n3))}.
2) If n2< n3, then E(H(<n1,n2,n3>))= E(H(<n1,n2>)) ∪{(v(2,i),v(3,i)),(v(2,i),v(3,i+1)): 1≤i≤ n_2}∪{(v(2,n2),v(3,i+1)): n2<i ≤ n3-1}∪ {(v(3,1),v(3,n3))}.
We can extend the labeling of f in G=H(<n1,n2>) to H(<n1,n2,n3>) by defining
f(v(3,i))=n1+n2+i-1 for 1≤i≤ n3. With the above defined vertex labeling no two of the edge labels are same as the edge labels are in an increasing order.
(See Fig.3 and Fig.4 for H(<5,3,4>) and H(<5,8,6>).

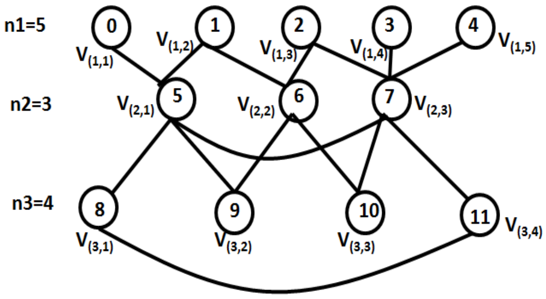

Figure 3

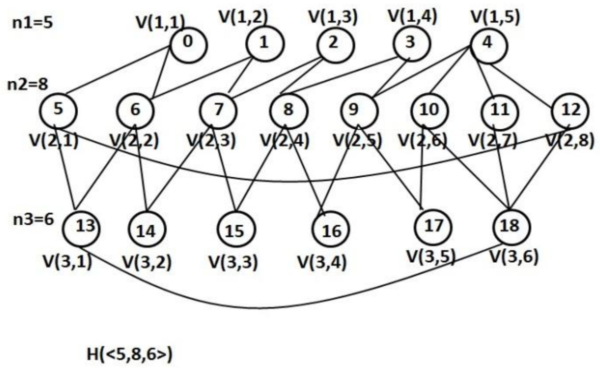

Figure 4

Induction Hypothesis

Assume that $H(<n1,n2,\ldots,nt>)$ is square sum. For any $n\{t+1\}\in N^{>1}$, let W be the graph with $V(W)=\{v(t+1,1),v(t+1,2),..,v(t+1,n\{t+1\})\}$ and $E(W)=\{(v(t+1,1),(v(t+1,n\{t+1\})\}$. we can construct a new graph $H(<n1,n2,..,n\{t+1\}>)$ as follows: The graph $H(<n1,n2,..,n\{t+1\}>)$ has $n1+n2+\ldots+nt+n\{t+1\}$ vertices. $V(H(<n1,n2,\ldots,n\{t+1\}>))=V(H(<n1,n2,..,nt>))\cup \{v(t+1,1),\ldots,v(t+1,n\{t+1\})\}$. We arrange the vertices of $H(<n1,n2,\ldots,n\{t+1\}>)$ layer by layer and from left to right as follows:

a) the top most layer is $H(<n1,n2,\ldots,nt>)$,

b) the second layer has vertices $v(t+1,1),\ldots,v(t+1,n\{t+1\})$,

The graph $H(<n1,n2,..,n\{t+1\}>)$ has edges of the form :

1). If $n\{t+1\}\leq nt$, then $E(H(<n1,n2,\ldots,n\{t+1\}>))=E(H(<n1,n2,\ldots,nt>))\cup E(W)\cup \{(v(t,i),v(t+1,i)),(v(t,i+1),v(t+1,i)):i=1,2,..,n\{t+1\}\}\cup \{v(t,i+1),v(t+1,n\{t+1\})):n\{t+1\}<i\leq nt-1\}$.

2). If $nt< n\{t+1\}$, then $E(H(<n1,n2,..,n\{t+1\}>))= E(H(<n1,n2,..,nt>))\cup E(W)\cup \{v(t,i),v(t+1,i)),(v(t,i),v(t+1,i+1)):i=1,2,\ldots,nt\}\cup \{v(t,nt),v(t+1,i+1):nt<i\leq n\{t+1\}-1\}$. We can extend the labeling of f in $G =H(<n1,n2,..,nt>)$ to $H(<n1,n2,\ldots,n\{t+1\}>)$ by defining $f(v(t+1,i))=n1+n2+\ldots+nt+i-1$ for $i=1,2,\ldots,n\{t+1\}$.

Theorem 2.1

The graph $H(<n1,n2,..,n\{t+1\}>)$ is square sum.

Proof

In fact $H(<n1,n2,..,n\{t+1\}>)$ is $H(<n1,n2,\ldots,nt>)\cup\ell\cup W$, where

1. $\ell=\{(v\{(t,i)\},v\{(t+1,i)\}),(v\{(t,i+1)\}, v\{(t+1,i)\}): i=1,2,..,n\{t+1\}\}\cup \{(v\{(t,i+1)\},v\{(t+1,n\{t+1\})\}):n\{t+1\}<i\leq nt-1\}$, if $n\{t+1\}\leq n_t$.
2. $\ell =\{(v\{(t,i)\},v\{(t+1,i)\}),(v\{(t,i)\},v\{(t+1,i+1)\}): i=1,2,..,nt\}\cup \{(v\{(t,nt)\},v\{(t+1,i+1)\}): nt<i\leq n\{t+1\}-1\}$, if $nt< n\{t+1\}$.

Since labels of $v\{(t,i)\}$, $1\leq i\leq nt$ are in increasing order and strictly less than the labels of $v\{(t+1,i)\}$, $1\leq i\leq n\{t+1\}$, $E(H(<n1,n2,\ldots,nt>))\cup\ell\cup E(W)$ can be arranged in strictly increasing order. Hence no two of the edge labels are same.

Remark 2.2

In the sequence $<n1,n2,\ldots,nt>$, $,ni\geq 2$, $i=1,2,..,t$ of $H(<n1,n2,..,nt>)$, if we change the order of the sequence, then the graphs are isomorphic.The only nonisomorphic classes of graph is one with at least one of these ni=1.

Now we consider the sequence σ on N={1,2,..} with the following property. $\sigma=(<1,n1,n2,..,nt>)$ where $n1,n2,..,nt$ in $N^{>1}$. We construct a new graph $H^*(<n1,n2,\ldots,nt>)$ as follows. The graph $H(<n1,n2,\ldots,nt>)$ has $n1+n2+\ldots +nt$ vertices in G.

$V(H^*(<n1,n2,..,nt>))=V(H(<n1,n2,..,nt>))\cup \{z\}$.

We arrange the vertices of $H(<n1,n2,..,nt>)$ layer by layer and from left to right and label the vertices as before. The lower layer has vertex z. The graph $H^*(<n1,n2,..,nt>)$ has edges in the form:

$E(H^*(<n1,n2,..,nt>))= E(H(<n1,n2,\ldots,nt>))\cup \{(v(t,i),z)): i=1,2,..,nt\}$. We can extend the labeling of f in $G=H(<n1,n2,..,nt>)$ to $H^*(<n1,n2,\ldots,nt>)$ by defining $f(z)=n1+n2+\ldots+nt$.

With the above labeling, no two of the edge labels are same as the edge labels are in strictly increasing order. Fig. 5 depicts $H^*(<5,3>)$

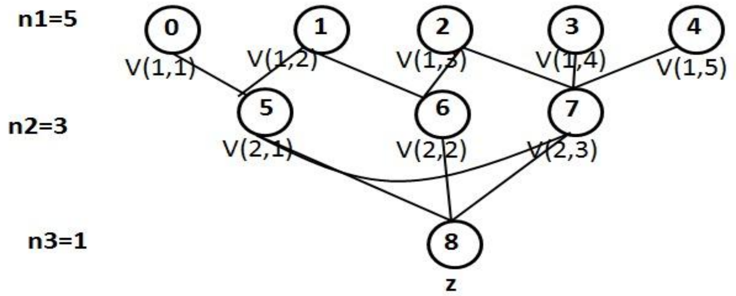

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Figure 5

Hence we have the following theorem.

Theorem 2.3

The graph $H^*(<n1,n2,\ldots,nt>)$ is square sum.

Dually we can construct a new graph $^*H(<n1,n2,..,nt>)$ as follows.

V(*H(<n1,n2,…,nt>))={u}∪ V(H(<n1,n2,..,nt>)). The upper layer has vertex u. We arrange the vertices of H(<n1,n2,..,nt>) layer by layer and from left to right and label the vertices as before. The graph *H(<n1,n2,..,nt>) has edges in the form :
E(*H(<n1,n2,…,nt>))={(u,v(1,i)):i=1,2,..,n1}∪ E(H(<n1,n2,..,nt>)). With the above labeling, no two of the edge labels are same as the edge labels are in increasing order. We illustrate *H(<3,3>), *H(<4,4>) in Fig. 6.

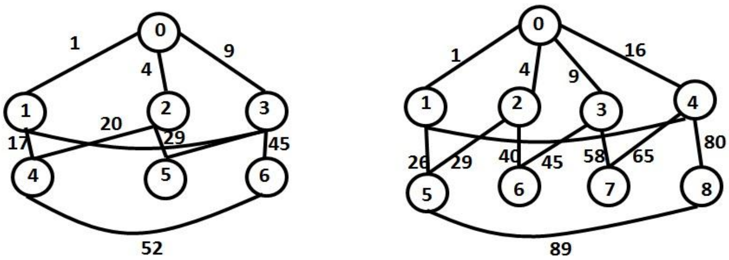

Figure 6

In fact, we have the following theorem.

Theorem 2.4

The graph *H(<n1,n2,..,nt>) is square sum.

We illustrate *H(<2,3,2>), H(<2,3,4,3>) in Fig. 7.

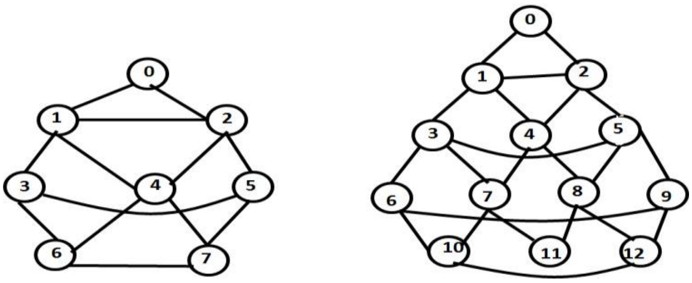

Figure 7

Definition 2.5

Level joined planar grids [5]: Let u be a vertex of Pm ×Pn such that deg(u)=2. Introduce an edge between every pair of distinct vertices v,w with deg(v),deg(w)≠ 4, if d(u,v)=d(u,w), where d(u,v) is the distance between u and v. The graph so obtained is defined as level joined planar grid and is denoted by $LJ_{m,n}$. An example of $LJ_{4,5}$ is illustrated in Fig 8.

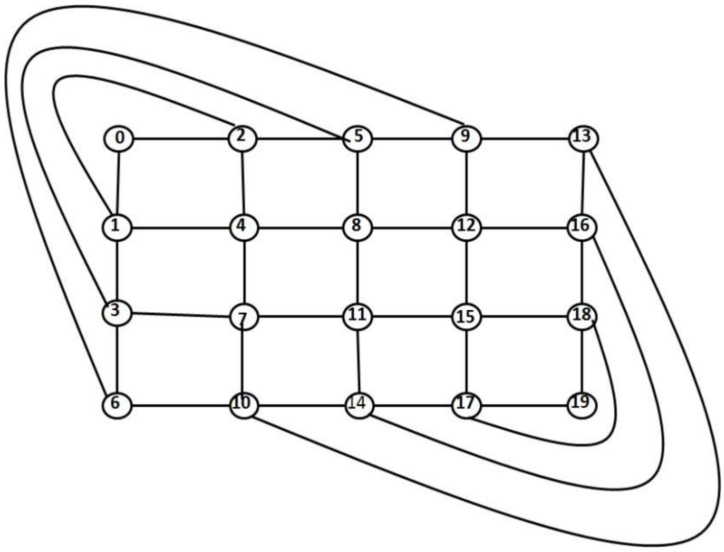

Figure 8

We observe that $LJ_{m,n}$ is the graph *H*<2,3,…,m,.. (n-m)times m, m-1,m-2,..,2>. In Figure 8, m=4 and n=5, and it is *H*<2,3,4,4,3,2>.Hence by theorem 2.3 we have the following.

Corollary 2.6

The graph $LJ_{m,n}$ is square sum.

III. Conclusion

Square sum graphs were studied in [4,6,7]. Our paper produces infinitely many square sum graphs.

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