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Some Generalized Difference Sequence Spaces Defined by Orlicz Functions

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Abstract: *The idea of difference sequence spaces was introduced by Kizmaz [1] and then this subject has been studied and generalized by various mathematicians. In this paper we define some difference sequence spaces by Orlicz space of bounded sequences and establish some inclusion relations. Some properties of these spaces are studied*

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I. Introduction

A complex sequence, whose k[th] term is x_k is denoted by $\{x_k\}$ or simply x. Let Φ be the set of all finite sequences. A sequence $x = \{x_k\}$ is said to be bounded if $\sup_k |x_k| < \infty$. The vector space of all bounded sequences will be denoted by l_∞ .

Throughout the article $(l_\infty)_M$ denote the Orlicz space of bounded sequences respectively.

Throughout m denotes an arbitrary positive integer. Kizmaz [1] introduced the notation of difference sequence spaces as follows: $X(\Delta) = \{x = (x_k) : (\Delta x_k) \in X\}$; for $X = l_\infty,\ c, c_0$, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$.

Later on the notion was generalized by Et and Colak [2] as follows:

$X(\Delta^m) = \{x = (x_k) : (\Delta^m x_k) \in X\}$ for $X = l_\infty,\ c, c_0$, where $m \in N, \Delta^0 x = (x_k)$ and

$\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$

$= \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$ for all $k \in N$

Later on difference sequence spaces have been studied by Et [3], Et and Nuray [4], Colak Et al [5], Isik [6], Altin and Et [7] and many others.

Orlicz [8] used the idea of Orlicz function to construct the space (L^M). Lindenstrauss and Tzafriri [9] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space l_M contains a subspace isomorphic to l_p $(1 \le p < \infty)$.

An Orlicz function is a function M : [0, ∞) → $[0, \infty)$ which is continuous, non-decreasing and convex with M(0) = 0, M(x) > 0, for x > 0 and $M(x) \to \infty$ $as\ x \to \infty$. If convexity of Orlicz function M is replaced by $M(x + y) \le M(x) + M(y)$, then this function is called modulus function, defined and discussed by Ruckle [10] and Maddox [11].

Lindenstrauss and Tzafriri [9], S.D.Parashar [12] used the idea of Orlicz function to construct Orlicz sequence space

$l_M = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, for\ some\ \rho > 0\right\}$ where $w = \{all\ complex\ sequences\}$.

The space l_M with the norm $\|x\| = inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1\right\}$, becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p, 1 \le p < \infty$, the spaces l_M coincide with the classical sequence space l_p.

Definition 1.1. The space consisting of all those sequences x in w such that $\left\{\mathrm{Sup}_k \left[M\left(\frac{|x_k|}{\rho}\right)\right]\right\} < \infty$ as for some arbitrary fixed $\rho > 0$ is denoted by $(l_\infty)_M$, M being an Orlicz function. In other words $\left\{M\left(\frac{|x_k|}{\rho}\right)\right\}$ is a bounded sequence. $(l_\infty)_M$ is called the Orlicz space of bounded sequences. The space $(l_\infty)_M$ is a metric space with the metric $d(x, y) = \mathrm{Sup}_k\, M\left[\frac{|x_k - y_k|}{\rho}\right]$ for all $x = \{x_k\}$ and $y = \{y_k\}$ in $(l_\infty)_M$.

Definition 1.2. If M is a convex function and M(0) = 0, then M(λx) ≤ λM(x) for all λ with $0 < \lambda < 1$.

Definition 1.3. A sequence space E is said to be solid or normal if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequences of scalars (α_k) with $|\alpha_k| \le 1$.

Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k < sup\ p_k = G$ an let D = Max(1, 2^{G-1}). Then for $a_k, b_k \in C$, the set of complex numbers for all $k \in N$, we have

$$|a_k + b_k|^{p_k} \le D\{|a_k|^{p_k} + |b_k|^{p_k}\}$$

Let M be an Orlicz function, X be locally convex Hausdorff topological linear space whose topology is determined by a set Q of continuous semi norms q. The symbols $l_\infty(X)$ denote the space of all bounded sequences defined over X. We define the following sequence spaces:

$$(l_\infty)_M(\Delta^m, p, q) = \left\{x \in l_\infty(X) : Sup \sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^m x_k}{\rho}\right)\right)\right]^{p_k} < \infty, for\ some\ \rho > 0\right\}$$

II. Main Results

Theorem 2.1 If M is an Orlicz function, then $(l_\infty)_M(\Delta^m, p, q)$ is a linear set over the set of complex numbers C.

Proof. Let $x, y \in (l_\infty)_M(\Delta^m, p, q)$ and $\alpha, \beta \in C$.

In order to prove the result, we need to find some ρ_3 such that

$$\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^m(\alpha x_k + \beta y_k)}{\rho_3}\right)\right)\right]^{p_k} < \infty \qquad (1.1)$$

Since $x, y \in (l_\infty)_M(\Delta^m, p, q)$, there exists some positive ρ_1 and ρ_2 such that

$$\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^m x_k}{\rho_1}\right)\right)\right]^{p_k} < \infty \qquad (1.2)$$

$$\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^m y_k}{\rho_2}\right)\right)\right]^{p_k} < \infty \qquad (1.3)$$

Define $\rho_3 = max(2|\alpha|\rho_1\ ,\ 2|\beta|\rho_2)$

Since M is a non-decreasing and convex function, q seminorm and Δ^m is linear then

$$\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^m(\alpha x_k + \beta y_k)}{\rho_3}\right)\right)\right]^{p_k} \le \sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^m \alpha x_k}{\rho_3}\right) + q\left(\frac{\Delta^m \beta y_k}{\rho_3}\right)\right)\right]^{p_k}$$

$$\le \sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^m x_k}{\rho_1}\right)\right) + M\left(q\left(\frac{\Delta^m y_k}{\rho_2}\right)\right)\right]^{p_k}$$

$$\le D\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^m x_k}{\rho_1}\right)\right)\right]^{p_k} + D\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^m y_k}{\rho_2}\right)\right)\right]^{p_k}$$

$$\le \infty$$

By (1.2) and (1.3)

$$\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^m(\alpha x_k + \beta y_k)}{\rho_3}\right)\right)\right]^{p_k} \le \infty$$

So $(\alpha x + \beta y) \in (l_\infty)_M(\Delta^m, p, q)$.

Therefore $(l_\infty)_M(\Delta^m, p, q)$ is a linear space.

Theorem 2.2 Let M_1 and M_2 be two Orlicz functions.

Then $(l_\infty)_{M_1}(\Delta^m, p, q) \cap (l_\infty)_{M_2}(\Delta^m, p, q) \subseteq (l_\infty)_{M_1+M_2}\big((\Delta^m, p, q)\big)$.

Proof.

Let $x \in (l_\infty)_{M_1}(\Delta^m, p, q) \cap (l_\infty)_{M_2}(\Delta^m, p, q)$.

Then there exists ρ_1 and ρ_2 such that

$$\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^m x_k}{\rho_1}\right)\right)\right]^{p_k} \le \infty \quad \text{and} \quad \sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^m x_k}{\rho_2}\right)\right)\right]^{p_k} \le \infty$$

Let $\rho = min\left(\frac{1}{\rho_1}, \frac{1}{\rho_2}\right)$. Then we have

$$\sum_{n=1}^{\infty}\left[(l_\infty)_{M_1+M_2}\left(q\left(\frac{\Delta^m x_k}{\rho}\right)\right)\right]^{p_k} \le$$

$$D\left[\sum_{n=1}^{\infty}\left[(l_\infty)_{M_1}\left(q\left(\frac{\Delta^m x_k}{\rho_1}\right)\right)\right]^{p_k}\right] + D\left[\sum_{n=1}^{\infty}\left[(l_\infty)_{M_2}\left(q\left(\frac{\Delta^m x_k}{\rho_1}\right)\right)\right]^{p_k}\right]$$

$$\le \infty$$

Therefore $\sum_{n=1}^{\infty}\left[(l_\infty)_{M_1+M_2}\left(q\left(\frac{\Delta^m x_k}{\rho}\right)\right)\right]^{p_k} \le \infty$.

Hence $x \in (l_\infty)_{M_1+M_2}\big((\Delta^m, p, q)\big)$.

Thus $(l_\infty)_{M_1}(\Delta^m, p, q) \cap (l_\infty)_{M_2}(\Delta^m, p, q) \subseteq (l_\infty)_{M_1+M_2}\big((\Delta^m, p, q)\big)$.

Theorem 2.3 Let $m \ge 1$. Then we have the following inclusion

$$(l_\infty)_M(\Delta^{m-1}, p, q) \subseteq (l_\infty)_M(\Delta^m, p, q).$$

Proof.
$\boldsymbol{x} \in (l_\infty)_M(\Delta^{m-1}, p, q)$. Then we have

$$\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^{m-1}x_k}{\rho}\right)\right)\right]^{p_k} < \infty\,, for\ some\ \rho > 0$$

Since M is non-decreasing convex function and q is seminorm, we have

$$\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^{m}x_k}{\rho}\right)\right)\right]^{p_k} \leq \sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1}}{\rho}\right)\right)\right]^{p_k}$$

$$\leq D\left\{\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^{m-1}x_k}{\rho}\right)\right)\right]^{p_k} - \sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^{m-1}x_{k+1}}{\rho}\right)\right)\right]^{p_k}\right\}$$

$$\leq \infty$$

Therefore $\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^{m}x_k}{\rho}\right)\right)\right]^{p_k} \leq \infty$.
Hence $x \in (l_\infty)_M(\Delta^m, p, q)$.
Thus $(l_\infty)_M(\Delta^{m-1}, p, q) \subseteq (l_\infty)_M(\Delta^m, p, q)$.

Theorem 2.4
a) If $p_k \leq 1$ for all $k \in N$ then $(l_\infty)_M(\Delta^m, p, q) \subset (l_\infty)_M(\Delta^m,\ q)$.
b) If $p_k \geq 1$ for all $k \in N$ then $(l_\infty)_M(\Delta^m,\ q) \subset (l_\infty)_M(\Delta^m,\ p,\ q)$.
Proof.
Proof for (a) $x \in (l_\infty)_M(\Delta^m, p, q)$. Then

$$\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^{m}x_k}{\rho}\right)\right)\right]^{p_k} < \infty \tag{4.1}$$

since $p_k \leq 1$,

$$\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^{m}x_k}{\rho}\right)\right)\right] \leq \sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^{m}x_k}{\rho}\right)\right)\right]^{p_k} < \infty \tag{4.2}$$

From (4.1) and (4.2) it follows that
$x \in (l_\infty)_M(\Delta^m,\ q)$. Thus $(l_\infty)_M(\Delta^m, p, q) \subset (l_\infty)_M(\Delta^m,\ q)$.
Proof for (b) Let $p_k \geq 1$ for all k and
Let $\in (l_\infty)_M(\Delta^m,\ q)$. Then

$$\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^{m}x_k}{\rho}\right)\right)\right] < \infty \tag{4.3}$$

Since $1 \leq p_k \leq \sup p_k < \infty$, we have

$$\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^{m}x_k}{\rho}\right)\right)\right]^{p_k} \leq \sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^{m}x_k}{\rho}\right)\right)\right]$$

$$< \infty \quad \text{using (4.3)}$$

Therefore $\in (l_\infty)_M(\Delta^m, p, q)$.
Thus $(l_\infty)_M(\Delta^m,\ q) \subset (l_\infty)_M(\Delta^m,\ p,\ q)$.

Theorem 2.5 $l_\infty \subset (l_\infty)_M(\Delta^m, p, q)$, with the hypothesis that $\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^{m}x_k}{\rho}\right)\right)\right]^{p_k} \leq |x_k|$.

Proof: Let $x \in l_\infty$. Then we have the following implication $|x_k| < \infty$

But $\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^{m}x_k}{\rho}\right)\right)\right]^{p_k} \leq |x_k|$.

By our assumption, implies that

$$\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^{m}x_k}{\rho}\right)\right)\right]^{p_k} < \infty.$$

Then $x \in (l_\infty)_M(\Delta^m, p, q)$ and $l_\infty \subset (l_\infty)_M(\Delta^m, p, q)$.
Theorem 2. 6 $(l_\infty)_M(\Delta^m, p, q)$ is solid.
Proof: Let $|x_k| \leq |y_k|$ and Let $y = (y_k) \in (l_\infty)_M(\Delta^m, p, q)$
Because M is non-decreasing

$$\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^{m}x_k}{\rho}\right)\right)\right]^{p_k} \leq \sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^{m}y_k}{\rho}\right)\right)\right]^{p_k}$$

And because $y \in (l_\infty)_M(\Delta^m, p, q)$

$\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^m x_k}{\rho}\right)\right)\right]^{p_k} \in l_\infty$

That is $\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^m y_k}{\rho}\right)\right)\right]^{p_k} < \infty$ and

$$\sum_{k=1}^{\infty}\left[M\left(q\left(\frac{\Delta^m x_k}{\rho}\right)\right)\right]^{p_k} < \infty$$

Therefore $x = (x_k) \in (l_\infty)_M(\Delta^m, p, q)$ is solid.

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