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A Generalized Zero-divisible Graph of a Commutative Ring with Respect to an Ideal

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Abstract
In this article, we generalize the notion of an ideal graph of the zero divisor of a commutative ring. Let R be a commutative ring and let I be an ideal of R. Here we define a generalized zero divisor graph of R with respect to I and denote this graph by (R). We show that (R) is associated with a mean of at most three. If (R) has a cycle, we show that the circumference of (R) is at most four. We also investigate the existence of cut vertices (R). Additionally, we investigate certain situations where (R) is a complete bipartite graph. In this chapter, all circles are commutative, not necessarily with unity. Unless otherwise stated. For a commutative ring R with identity, the zero divisor graph of R, denoted by (R), is a graph whose vertices are nonzero zero divisors of R with two distinct vertices connected by an edge when the product of the vertices is zero. We generalize this notion by replacing elements whose product is zero with elements whose product lies in some ideal I of R. We also determine (up to isomorphism) all rings R such that Γ(R) is a graph on five vertices.
Keywords :- generalized zero-divisible graph ,commutative ring etc.

Definitions and preliminaries

Here, we define a generalized zero-divisor graph of a commutative ring with respect to an ideal as follows:

Definition. Let R be a commutative ring and let I be an ideal of R. We define a generalized zero-divisor graph of R with respect to I, denoted by (R), as the undirected graph whose vertex set is $\{ \ \in R-I \mid$ there exists $\ \in R-I$ such that $\ \in I$ for some $\ \in \langle \ \rangle - I$ and for some $\ \in \langle \ \rangle - I\}$, and two distinct vertices and are adjacent if and only if $\ \in I$ for some $\ \in \langle \ \rangle - I$ and for some $\ \in \langle \ \rangle - I$. If $I = \{0\}$, then (R) is denoted by (R).

S.P. Redmond introduced the definition of the ideal based zero-divisor graph of a commutative ring R.

Definition Let R be a commutative ring with unity and let I be an ideal of R. Then the ideal based zero-divisor graph of R, denoted by $\Gamma\ (R)$, is the undirected graph whose vertex set is $\{ \ \in R-I \mid \ \in I$ for some $\ \in R-I\}$, and two distinct vertices and are adjacent if and only if $\ \in I$. If $I = \{0\}$, then $\Gamma\ (R)$ is the zero-divisor graph $\Gamma(R)$ which is defined by D. F. Anderson and P. S. Livingston.

Theorem. Let I be a nonzero ideal of a commutative ring R. Then Γ (R) is an empty graph if and only if I is a prime ideal of R.

Proof. Suppose that Γ (R) is an empty graph. If possible assume that I is not a prime ideal of R. Then there exists two elements , $\in R - I$ such that $\in$ I. So the vertex set of Γ (R) is non-empty, a contradiction. Hence I is a prime ideal of R.

Conversely, suppose that I is a prime ideal of R. Then $\in$ I implies $\in$ I or $\in$ I. So the vertex set of Γ (R) is empty. Hence Γ (R) is an empty graph.

Remark The Theorem is equivalent to saying that Γ (R) is an empty graph if and only if R / I is an integral domain.

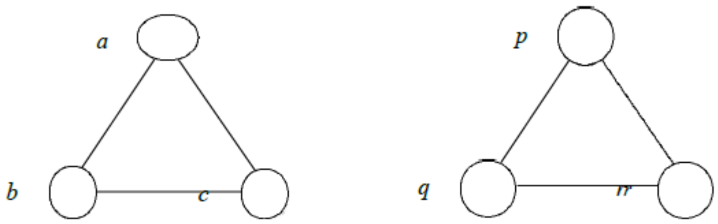
Theorem Let R be a commutative ring with unity and let I be an ideal of R. Then Γ (R) is connected and $(\Gamma\ (R)) \leq 3$.

By definitions it follows that every edge of Γ (R) is an edge of Γ (R). But converse is not true in general, as the following example shows. Thus Γ (R) is a

Example Let $R = \mathbb{Z}$ and $I = \{0\}$. Since $2 \in \langle 2\rangle$, $4 \in \langle 6\rangle$ and $2.4 = 0$, we have $2 — 6$ is an edge of Γ (R). But since $2.6 \neq 0$, we have $2 — 6$ is not an edge of $\Gamma(R)$.

The following example shows that non-isomorphic commutative rings may have isomorphic generalized zero-divisor graph.

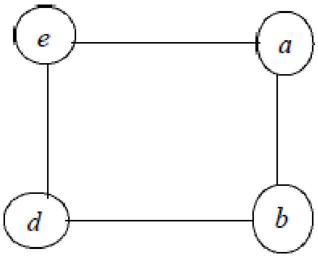
Example Let $R = \mathbb{Z} \times \mathbb{Z}$ and $I_1 = \{(0, 0)\}$ and $R_2 = \mathbb{Z}\ [X]/\langle\ \rangle$ and $I_2 =$ $\{\langle\ \rangle\}$. Then the graphs Γ $(R\ )$ and Γ (R_2) are as follows, where $= (0, 1)$, $=$ $(1, 0)$, $= (1,1)$, $= 1 + \langle\ \rangle$, $= + \langle\ \rangle$ and $= 1 + + \langle\ \rangle$.

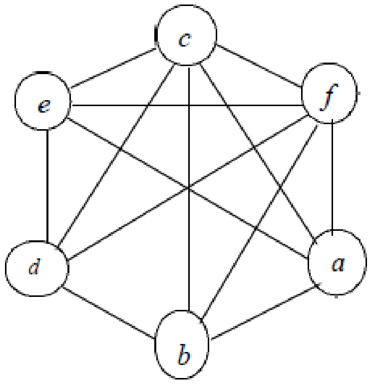


Γ $(R\ )$ Γ (R_2)

The next example shows that the graph Structure (R) and Γ (R) are not isomorphic.

Example Let $R = \mathbb{Z} \times \mathbb{Z}$ and $I = \langle(0, 2)\rangle$. Then the graphs $\Gamma (R)$ and $\Gamma (R)$ are as follows, where $= (0,1)$, $= (1,0)$, $= (1,1)$, $= (0,3)$, $= (1,2)$ and $= (1,3)$.





In our discussion, we assume $I \neq R$ for an ideal I of R. To avoid trivialities when $\Gamma (R)$ is the empty graph we will implicitly assume when necessarily that I is not a prime ideal of R. For any subset U and ideal I of a commutative ring R, we define $[I : U] = \{ \in R| \quad U \subseteq I \}$. Then $[I : U]$ is an ideal of R containing I. If $U = \{ \}$, then $[I : \{ \}]$ is simply denoted by $[I : \quad]$.

Some basic properties of (R)

Some characteristics of (R) are studied in this section. We show that $\Gamma (R)$ is connected with diameter at most 3. If (R) has a cycle, we show that the girth of (R) is at most 4. We also investigate the existence of cut vertices of (R).

Theorem Let I be an ideal of a commutative ring R. If — is an edge of (R) for any $,\ \in V((R))$, then — is an edge of (R) for each $\in R - I$ or — is an edge of (R) for some $\in \langle \ \rangle - I$.

Proof. Suppose that — is an edge of $\Gamma (R)$ for any $,\ \in V(\Gamma (R))$. Suppose that — is not an edge of $\Gamma (R)$ for some $\in R - I$. Then $\in I$ for some $\in \langle \ \rangle - I$ and for some $\in \langle \ \rangle - I$ and $\notin I$. Let $=$. Then $\in \langle \ \rangle - I$. Since I is an ideal of R, $(\quad) \in I$. This implies $(\quad) \in I$. Thus $\in I$. Hence — is an edge of $\Gamma (R)$.

Theorem Let I be an ideal of a commutative ring R. Then Γ (R) is connected

and $((R)) \leq 3$.

Proof. Let and be any two distinct vertices of Γ (R). Consider the following cases:

Case 1. If $\in I$ for some $\in \langle\ \rangle - I$ and for some $\in \langle\ \rangle - I$, then — is an edge of (R).

Case 2. Let $\notin I$ for all $\in \langle\ \rangle - I$ and for all $\in \langle\ \rangle - I$. Then $\notin I$ and $\notin I$ for all $\in \langle\ \rangle - I$ and for all $\in \langle\ \rangle - I$. Since , $\in V(\Gamma$ $(R))$ there exists $\in \langle\ \rangle - I$, $\in \langle\ \rangle - I$ and , $\in R - (I \cup \{\ ,\ \})$ such that $\in I$ and $\in I$. If $=$, then — — is a path of length 2. So assume that $\neq$. If $\in I$, then — — — is a path of length 3. If $\notin I$, then $\langle\ \rangle \cap \langle\ \rangle \not\subseteq I$. Now for each $\in \langle\ \rangle \cap \langle\ \rangle - (I \cup \{\ ,\ \})$, we have $\in \langle\ \rangle\langle\ \rangle \subseteq \langle\ \rangle\langle\ \rangle \subseteq I$ and $\in \langle\ \rangle\langle\ \rangle \subseteq \langle\ \rangle\langle\ \rangle \subseteq I$. Hence — — is a path of length 2.

Thus we conclude that Γ (R) is connected and $(\Gamma$ $(R)) \leq 3$.

Theorem Let I be an ideal of a commutative ring R. If — — is a path in Γ (R), then $I \cup \{\ _1\}$ is an ideal of R for some $\in \langle\ \rangle - I$ or — — lies on a cycle of Γ (R) with length ≤ 4.

Proof. Suppose that — — is a path in Γ (R). Then there exists , $\in \langle\ \rangle - I$ and $\in \langle\ \rangle - I$, $\in \langle\ \rangle - I$ such that $\in I$ and $\in I$. Consider the following cases:

Case 1. If $'\ ' \in I$ for some $' \in \langle\ \rangle - I$ and for some $' \in \langle\ \rangle - I$, then C : — — — is a cycle of length 3 and hence — — lies on the cycle C of length 3.

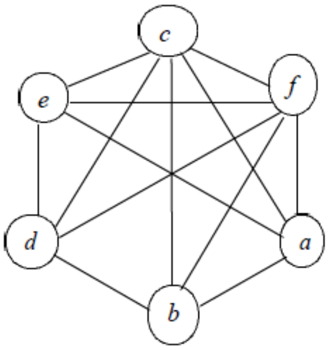
Case 2. Let $'\ ' \notin I$ for all $' \in \langle\ \rangle - I$ and for all $' \in \langle\ \rangle - I$. Suppose that $=$. Then $I \cup \{\ \} \subseteq [I:\ ] \cap [I:\ ]$. If $[I:\ ] \cap [I:\ ] = I \cup \{\ \}$, then $I \cup \{\ \}$ is an ideal of R. Otherwise, there exists a $\in [I:\ ] \cap [I:\ ]$ such that $\notin I \cup \{\ \}$. Then $\in I$ and $\in I$. Thus C : — — — — is a cycle of length 4 and hence — — lies on the cycle C of length 4. Suppose that

$\neq$. Then we have $\langle \ \rangle \cap \langle \ \rangle \not\subseteq I$. Then for each $\in \langle \ \rangle \cap \langle \ \rangle - I$,we have h a v $\in \langle \ \rangle \langle \ \rangle \subseteq I$ and $\in \langle \ \rangle \langle \ \rangle \subseteq I$. Clearly either $\neq$ or $\neq$. Without loss of generality assume that $\neq$. Then — — is a path and C : — — — — is a cycle of length 4, and hence — — lies on the cycle C of length 4.

Thus we conclude that $I \cup \{ \ _1\}$ is an ideal of R for some $\in \langle \ \rangle - I$ or — — lies on a cycle of Γ (R) with length ≤ 4. □

The bound for the length of the cycle is sharp in Theorem, as the following example shows.

Example Consider the commutative ring $R = \mathbb{Z} \times \mathbb{Z}$ and $I = \langle(0, 2)\rangle$. Then the graph Γ (R) is as follows where $= (0,1)$, $= (1,0)$, $= (1,1)$, $= (0,3)$, $=$ $(1,2)$ and $= (1,3)$.



We have $I \cup \{ \ \}$ is not an ideal of R for any $\in \langle \ \rangle - I$ and — — does not lie on any cycle of length 3.

Corollary Let I be an ideal of a commutative ring R with $|V(\Gamma \ (R))| \geq 3$. If $I \cup \{ \ \}$ is not an ideal of R for any $\in R - I$, then every edge of Γ (R) lies on a cycle of Γ (R) with length ≤ 4 and Γ (R) is a union of $3 -$ cycles or $4 -$ cycles.

Proof. Suppose that $I \cup \{ \ \}$ is not an ideal of R for any $\in R - I$. Since $|V(\Gamma \ (R))| \geq 3$, the graph Γ (R) contains at least three vertices. Then every path of Γ (R) of length 2 lies on a cycle of Γ (R) with length ≤ 4 by Theorem. Thus every edge of Γ (R) lies on a cycle of Γ (R) with length ≤ 4 and hence Γ (R) is a union of $3 -$ cycles or $4 -$ cycles.

Theorem Let I be an ideal of a commutative ring R with $|V(\Gamma\ (R))| \geq 3$. If $I \cup \{\ \}$ is not an ideal of R for any $\in R - I$, then every pair of vertices in $\Gamma\ (R)$ lies on a cycle of $\Gamma\ (R)$ with length ≤ 6.

Proof. Suppose that $I \cup \{\ \}$ is not an ideal of R for any $\in R - I$. Since $|V(\Gamma\ (R))| \geq 3$, the graph $\Gamma\ (R)$ contains at least three vertices. Let , be any two distinct vertices of $\Gamma\ (R)$. If — is an edge of $\Gamma\ (R)$, then — lies on a cycle of $\Gamma\ (R)$ with length ≤ 4 by Corollary. If — — is a path in $\Gamma\ (R)$, then — — lies on a cycle of $\Gamma\ (R)$ with length ≤ 4 by Theorem. If — — — is a path in $\Gamma\ (R)$, then we have the cycles — — — — and — — — — , where $\neq$ and $\neq$ by Theorem. This implies : — — — — — — is a cycle of length 6 in $\Gamma\ (R)$. Thus every pair of vertices in $\Gamma\ (R)$ lies on a cycle of $\Gamma\ (R)$ with length ≤ 6. ☐

Theorem Let I be an ideal of a commutative ring R. If $\Gamma\ (R)$ has a cycle, then any cycle of length ≥ 5 is not an induced subgraph of $\Gamma\ (R)$ and $(\Gamma\ (R)) \leq 4$.

Proof. Suppose that $\Gamma\ (R)$ has a cycle C : — — — — — ... — — of length ≥ 5 which is an induced subgraph of $\Gamma\ (R)$. Then — — is a path which lies on a cycle of $\Gamma\ (R)$ with length ≥ 5. Thus $I \cup \{\ /\ \}$ is an ideal of R for some $/ \in \langle\ \rangle - I$ by Theorem. Since — is an edge there exists $/ \in \langle\ \rangle - I$ and $/ \in \langle\ \rangle - I$ such that $/\ / \in I$. Since $I \cup \{\ /\ \}$ is an ideal, we have $/\ / = / = /\ /$. Thus $/ (\ /\ /\ ) \in I$. This implies $(\ /\ /\ )\ / \in I$. This implies $/\ / \in I$. This implies $/ \in I$, which is a contradiction. Thus any cycle of length ≥ 5 is not an induced subgraph of $\Gamma\ (R)$, and hence $(\Gamma\ (R)) \leq 4$. ☐

Remark Let I be an ideal of a commutative ring R. Then $\Gamma\ (R)$ cannot be realized as a cycle of length ≥ 5 by Theorem

Theorem Let I be an ideal of a commutative ring R. Then the following results hold:

(1) If R is a commutative ring with unity, then Γ (R) has no cut vertices;

(2) If R is a commutative ring without unity and I is a nonzero ideal of R, then Γ (R) has no cut vertices.

Proof. Suppose that the vertex of Γ (R) is a cut vertex. Let — — be a path in Γ (R). Since is a cut vertex, lies in every path connecting and .

(1) Suppose that R is a commutative ring with unity. Then for any , $\in$ $V(\Gamma$ $(R))$, there exists a path —1— in Γ (R). Thus $(\neq 1)$ is not a cut vertex of Γ (R). Suppose that $= 1$. Then there exists $\in \langle \ \rangle - I$, $\in \langle \ \rangle - I$ and , $\in$ $R - I$ such that $\in I$ and $\in I$, which shows that , $\in V(\Gamma$ $(R))$. Since Γ (R) is connected, there exists , $\in R - (I \cup \{ \ \})$ such that — — or — — — is a path in Γ (R) by Theorem This implies — — —1— or — — — —1— is a cycle in Γ (R), which contradicts that $= 1$ is a cut vertex.

(2) Suppose that R is a commutative ring without unity and I is a nonzero ideal of R. Since — — is a path from to , there exists $\in \langle \ \rangle - I$, $\in \langle \ \rangle - I$ and , $\in \langle \ \rangle - I$ such that $\in I$ and $\in I$.

Case 1. Suppose that $=$. If $+ I =$ $+ I$, then $\in I$. This implies and are adjacent. If $+ I =$ $+ I$, then $\in I$. This implies and are adjacent. So assume that $+ I \neq$ $+ I$ and $+ I \neq$ $+ I$. Since I is nonzero, there is a $\in$ I such that $\neq 0$. Since $\in I$ and $\in I$, then it implies that (+), (+

) ∈ *I*. If = + , then ≠ . Thus we have — — is a path in Γ (*R*). Otherwise —(+)— is a path in Γ (*R*). Therefore there is a path from to that is not passing throw , which is a contradiction.

Case 2. Suppose that either = or = . Without loss of generality assume that = and ≠ . Then ∈ *I* and ∈ *I*. This implies that ∈ *I* and ∈ *I*. Thus — — is a path in Γ (*R*). Therefore there is a path from to that is not passing through , which is a contradiction.

Case 3. Suppose that ≠ and ≠ such that ≠ . If ∈ *I*, then — — — is a path in Γ (*R*). Therefore there is a path from to that is not passing throw , which is a contradiction. Otherwise we have ∉ *I*. If = , then ∈ *I* and ∈ *I*. If + *I* = + *I*, then ∈ *I*. This implies and are adjacent. If + *I* = + *I*, then ∈ *I*. This implies and are adjacent. So assume that + *I* ≠ + *I* and + *I* ≠ + *I*. Since *I* is nonzero, there is a ∈ *I* such that ≠ 0. Since ∈ *I* and ∈ *I*, then it implies that (+), (+) ∈ *I*. Then + ∉ *I* and ≠ + . Thus we have —(+)— is a path in Γ (*R*). Therefore there is a path from to that is not passing through , which is a contradiction. If ≠ . Then — — is a path in Γ (*R*). Therefore there is a path from to that is not passing through , which is a contradiction.

Thus can not be a cut vertex of Γ (*R*).

Recall that, the core of a graph *G* is the union of all cycles of *G*.

Theorem Let *I* be an ideal of a commutative ring *R*. If Γ (*R*) has a cycle, then the core *K* of Γ (*R*) is a union of 3 – cycles or 4 – cycles. Moreover, any vertex in Γ (*R*) is either a vertex of the core *K* of Γ (*R*) or is an end vertex of Γ (*R*).

Proof. Suppose that Γ (*R*) has a cycle. Then any cycle of length ≥ 5 is not an induced subgraph of Γ (*R*) and (Γ (*R*)) ≤ 4 by Theorem Thus the core *K* of Γ (*R*) is a union of 3 – cycles or 4 – cycles.

For the second statement we assume that |*V*(Γ (*R*))| ≥ 3. Let be any vertex of Γ (*R*). Then we have the followings and one of them is true.

Case 1. is in the core *K* of Γ (*R*).

Case 2. is an end vertex of Γ (R).

Case 3. — — is a path in Γ (R), where is an end vertex, $\notin K$ and $\in K$.

Case 4. — — — or — — — is a path in Γ (R), where is an end vertex, , $\notin K$ and $\in K$.

In the first and second cases, there is nothing to prove.

Suppose that Case 3 holds. Assume that — — is a path in Γ (R), where is an end vertex, $\notin K$ and $\in K$. Then $I \cup \{\ _1\}$ is an ideal of R for some $\in \langle\ \rangle - I$ by Theorem. Since $\in$ K we have — — — — or — — — — — is a path in Γ (R). Then $\in I$ for some $\in \langle\ \rangle - I$ and for some $\in \langle\ \rangle - I$. Since $I \cup \{\ _1\}$ is an ideal of R, we have $=$. ( ) $\in I$. This implies ( ) $\in I$. This implies $\in I$. Thus is a vertex of the cycle — — — — , which is a contradiction.

Suppose that Case 4 holds. Without loss of generality assume that — — — is a path in Γ (R), where is an end vertex, , $\notin K$ and $\in K$.

Since $\in K$, there is some $\in K$ such that $\neq$ and — lies on a cycle of Γ (R) with length ≤ 4. Then we have — — — — is a path in Γ (R). Since $(\Gamma\ (R)) \leq 3$, we have — or — is an edge. If — is an edge, then $\in K$. Then $\in K$ by Case 3. Thus we get a contradiction. Again if — is an edge, then — — — — is a cycle. Thus , $\in K$, a contradiction.

Hence any vertex in Γ (R) is either a vertex of the core K of Γ (R) or is an end vertex of Γ (R).

Corollary Let I be an ideal of a commutative ring R. If R has unity with $|V(\Gamma\ (R))| \geq 3$ or if R has no unity and I is a nonzero ideal of R with $|V(\Gamma\ (R))| \geq 3$, then Γ $(R) = K$, where K is the core of Γ (R).

Proof. Suppose that R has unity with $|V(\Gamma\ (R))| \geq 3$. Then Γ (R) has no cut vertices by Theorem, and hence Γ (R) has no end vertex. Then every vertex of Γ (R) is a vertex of the core K of Γ (R) by Theorem Thus Γ $(R) = K$. Next suppose that R

has no unity and I is a nonzero ideal of R with $|V(\Gamma\ (R))| \geq 3$. Then $\Gamma\ (R)$ has no cut vertices by Theorem, and hence $\Gamma\ (R)$ has no end vertex. Then every vertex of $\Gamma\ (R)$ is a vertex of the core K of $\Gamma\ (R)$ by Theorem Thus $\Gamma\ (R) = K$.

Conclusion :-

In this research work, we defined a generalized graph of the zero divisor (R) of the commutative ring R with respect to the ideal I and discussed some basic properties of (R). This chapter is just an opening for creating a zero divisor graph generalization bridge. Studying the connected, intersecting vertices of this generalized graph will develop many circle-theoretic concepts. We also investigated some properties of prime and semiprime ideals when (R) is a complete bipartite graph.

Refrence :-

[1]. S. Akbari and A. Mohammadian, On the zero-divisor graph of a commutative ring, J. Algebra 274 (2004), 847 – 855.
[2]. D.D. Anderson and M. Naseer, Beck's coloring of a commutative ring, J. Algebra 159 (1993), 500 – 514.
[3]. D.F. Anderson and P.S. Livingston, The zero-divisor graph of a commutative ring, J. Algebra 217 (1999), 434 – 447.
[4]. D.F. Anderson, A. Frazier, A. Laave and P.S. Livingston, The zero-divisor graph of a commutative ring II, Lecture Notes in Pure and Appl. Math., 220 (2001), Marcel Dekker, New York—Basel, 61 –72.
[5]. I. Beck, Coloring of commutative rings, J. Algebra 116 (1988), 208 – 226.
[6]. P. Dheena and B. Elavarasan, An ideal-based zero-divisor graph of 2-primal near-rings, Bull. Commun. Korean Math. Soc. 46 (6) (2009), 1051 – 1060.
[7]. P. Dheena and B. Elavarasan, A generalized ideal-based zero-divisor graph of a near-ring, Commun. Korean Math. Soc. 24 (2) (2009), 161 – 169.
[8]. R. Diestel, Graph Theory, Springer-Verlag, New York (1997).
[9]. N. Herstein, Topics in Algebra, 2nd edition, John Wiley & Sons (Asia) Pte.Ltd. (1999).
[10]. H. Maimani, M.R. Pouranki and S. Yassemi, Zero-divisor graph with respect to an ideal, Communications in Algebra 34 (3) (2006), 923 – 929.
[11]. S.P. Redmond, An ideal-based zero-divisor graph of a commutative ring, Comm. Algebra 31 (9) (2003), 4425 – 4443.