

Family of cubic-order Newton methods based on symmetric means for nonlinear equations

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Abstract: A family of modified Newton methods for calculating a simple root of a nonlinear equation is presented. These methods employ symmetric means and achieve cubic convergence, requiring fewer iterations than the classical Newton method in the examples studied.

Background: Newton's method is a widely studied technique for solving nonlinear equations; it exhibits quadratic convergence. Although it generally converges rapidly, there are cases where convergence is linear meaning it is slower. Various modifications aim to increase its speed through higher-order iterative procedures.

Materials and Methods: Newton's method is a widely studied technique for solving nonlinear equations; it exhibits quadratic convergence. Although it generally converges rapidly, there are cases where convergence is linear meaning it is slower. Various modifications aim to increase its speed through higher-order iterative procedures.

Results: All methods yielded the same approximate roots. The cubic variants required fewer iterations than Newton's method. The harmonic mean variant stood out in the first problem, while the Weerakoon-Fernando and geometric mean variants excelled in the second; the remaining variants matched the best performance in the third case.

Conclusion: The Symmetric means allow for the construction of methods based on Newton's method that exhibit cubic convergence. There is no single best universal variant, as efficiency depends on the function, the initial value, the computational cost, and domain constraints

Key Word: Nonlinear equations, Newton's method, symmetric means, cubic convergence, iterative methods.

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I. Introduction

This article aims to develop two-step recursive methods for finding a simple root of a nonlinear function. The construction begins with a Newton predictor and evaluates the corrector derivative at a symmetric mean of the current point and the predictor. Arithmetic, harmonic, geometric, and quadratic means are examined. Unlike formulations that conflate the meaning of a derivative evaluated at a mean of points with other approaches, this study distinguishes between the two procedures and establishes their respective hypotheses. It is demonstrated that any consistent, symmetric, and twice-differentiable mean yields local cubic convergence, provided the simple root and the relevant expressions remain within the domain. Asymptotic error equations are derived for all four means. Three test problems show that the variables achieve the required precision in fewer iterations than Newton's method in the cases considered, although they require an additional derivative evaluation per iteration. The results outline the computational advantages and limitations of each scheme. Solving the equation $f(x) = 0$ is an essential procedure in mathematical modeling, as it enables the determination of models describing scientific

and technological phenomena, as well as applications in engineering and computer science. If f is differentiable and x_n is sufficiently close to a simple root α , Newton's method generates

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (1)$$

Under standard local assumptions, the convergence of Equation (1) is quadratic. A classic approach to increasing the order consists of replacing the rectangular approximation implicit in Newton's method with a trapezoidal approximation. Weerakoon and Fernando proposed a two-stage, third-order method based on the arithmetic meaning of two derivative values. Inequalities among the harmonic, geometric, and quadratic means allow for the comparison of measures of central tendency for positive arguments. Furthermore, the relationship between these means and the Mean Value Theorem helps clarify when an identity involving a derivative holds true and when it merely constitutes an approximation. These observations motivate the study of schemes in which the derivative is evaluated at the meaning of two points.

Multi-point methods enable an increase in the order of convergence by utilizing additional information obtained within each iterative cycle. In this context, efficiency must be assessed alongside the number and type of function evaluations, local stability, and domain constraints. The theory of iterative processes and subsequent developments regarding high-order methods provide the framework for this comparison.

The objective of this work is to unify and refine a family of modifications to Newton's method constructed using symmetric means. The main contributions are: (i) rigorously distinguishing between the mean of derivative values and the derivative evaluated at the mean of arguments; (ii) formulating arithmetic, harmonic, geometric, and quadratic variants; (iii) demonstrating local convergence of at least third order under explicit hypotheses; (iv) incorporating and demonstrating the Weerakoon-Fernando method as a cubic benchmark; and (v) comparing accuracy, iterations, evaluation costs, and domain limitations using three reproducible problems.

1.1 Mathematical foundations and reference methods

Consider a differentiable function $f: I \rightarrow \mathbb{R}$ y $\alpha \in I$ defined on an interval $I \subset \mathbb{R}$, and let $\alpha \in I$. Assume that $\alpha \in I$ is a simple root satisfying $f(\alpha) = 0$ that meets the conditions specified by Equation 2.

$$f(\alpha) = 0, \quad f'(\alpha) \neq 0 \quad (2)$$

The Weerakoon-Fernando method is written as

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \quad (3)$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)} \quad (4)$$

This procedure, hereinafter referred to as the Weerakoon-Fernando method, is carried out in two stages: first, the Newtonian predictor y_n is obtained using equation (3); then, the correction from equation (4) is calculated using the arithmetic mean of the derivatives evaluated at x_n y y_n . The process stops when the iteration and residual conditions established for the other methods are simultaneously met. Each iteration requires one evaluation of f and two evaluations of f' .

Equation (4) uses the arithmetic mean of $f'(x_n)$ and $f'(y_n)$. In general, it should not be identified with $f'\left(\frac{x_n+y_n}{2}\right)$; the two quantities coincide only for certain classes of functions. The present paper analyzes these different constructions.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(M(x_n, y_n))} \quad (5)$$

where M is a mean of points.

For $u, v > 0$, the means employed are:

$$\begin{aligned} (u, v) &= \frac{2uv}{u+v}, & G(u, v) &= \sqrt{uv}, \\ A(u, v) &= \frac{u+v}{2}, & Q(u, v) &= \sqrt{\frac{u^2+v^2}{2}} \end{aligned} \quad (6)$$

And they satisfy

$$H(u, v) \leq G(u, v) \leq A(u, v) \leq Q(u, v) \quad (7)$$

The inequalities in equation .7. do not imply that the four means are equal or that identical iterations occur.

II. Material And Methods

2.1 Proposed iterative

For a mean M , an initial value x_0 and a tolerance, the following procedure is applied:

1. Evaluate $f(x_n)$ and $f'(x_n)$.
2. y_n Calculate the Newton predictor y_n using equation .3.
3. Calculate $m_n = M(x_n, y_n)$ and verify that $m_n \in I$ and $f'(m_n) \neq 0$.
4. Check if $|x_{n+1} - x_n| \leq \varepsilon$ and $|f(x_{n+1})| \leq \varepsilon$.

The four specific realizations are as follows, where the subscripts A, H, G , and Q identify, respectively, the arithmetic, harmonic, geometric, and quadratic means:

$$x_{n+1}^{(A)} = x_n - \frac{f(x_n)}{f'(M_A(x_n, y_n))}, \quad M_A(x_n, y_n) = \frac{x_n + y_n}{2} \quad (8)$$

$$x_{n+1}^{(H)} = x_n - \frac{f(x_n)}{f'(M_H(x_n, y_n))}, \quad M_H(x_n, y_n) = \frac{2x_n y_n}{x_n + y_n} \quad (9)$$

$$x_{n+1}^{(G)} = x_n - \frac{f(x_n)}{f'(M_G(x_n, y_n))}, \quad M_G(x_n, y_n) = \sqrt{x_n y_n} \quad (10)$$

$$x_{n+1}^{(Q)} = x_n - \frac{f(x_n)}{f'(M_Q(x_n, y_n))}, \quad M_Q(x_n, y_n) = \sqrt{\frac{x_n^2 + y_n^2}{2}} \quad (11)$$

The harmonic, geometric, and quadratic variants require arguments compatible with their real domains. In particular, the local analysis presented for them assumes $\alpha > 0$; furthermore, the condition $x_n + y_n \neq 0$ must be satisfied in Equation 9, and $x_n y_n \geq 0$ in Equation 10.

2.2 Local convergence analysis

The error $e_n = x_n - \alpha$ and the normalized constants are defined

$$c_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)} \quad k = 2, 3, \dots \quad (12)$$

Theorem 1. Let $f \in C^4(I)$ and let $\alpha \in I$ be a simple root. Let M be a mean that is consistent, symmetric, and of class C^2 in a neighborhood of (α, α) ; that is.

$$M(t, t) = t, \quad M(u, v) = M(v, u) \quad (13)$$

If the iterations in equation .3 and equation .5 are well defined, and if the initial approximation x_0 is sufficiently close to α , the method converges locally with an order of convergence of at least three.

$$M(\alpha + e_n, \alpha + e_{y,n}) = \alpha + \frac{1}{2}e_n + A_M e_n^2 + O(e_n^3) \quad (14)$$

Its error equation is

$$e_{n+1} = \left(2c_2A_M - \frac{c_3}{4}\right)e_n^3 + O(e_n^4) \quad (15)$$

Proof. Using Taylor expansions for f and f' in a neighborhood around α , we obtain the following expressions:

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + O(e_n^4)] \quad (16)$$

Using Taylor for f' around α , the equation is obtained. 17.:

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + O(e_n^3)] \quad (17)$$

Upon dividing equation .16. by equation .17.:

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + (2c_2^2 - 2c_3)e_n^3 + O(e_n^4) \quad (18)$$

Therefore, the predictor error is

$$e_{y,n} = y_n - \alpha = c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + O(e_n^4) \quad (19)$$

The consistency and symmetry of M imply $M_1(\alpha, \alpha) = M_2(\alpha, \alpha) = \frac{1}{2}$. Using Equation 14 and expanding the derivative at $m_n = M(x_n, y_n)$, one obtains

$$\frac{f'(m_n)}{f'(\alpha)} = 1 + c_2e_n + \left(2c_2A_M + \frac{3c_3}{4}\right)e_n^2 + O(e_n^3) \quad (20)$$

Substituting Equation .16. into Equation .20. and Equation .5. yields Equation .15. If the cubic constant does not vanish, the order is exactly three; if it does vanish, the order is higher.

For the four means, we have

Table no 1: The orders of convergence for the four means are shown.

Average	A_M	Cubic constant C_M in $e_{n+1} = C_M e_n^3 + O(e_n^4)$
Arithmetic	$\frac{c_2}{2}$	$c_2^2 - \frac{c_3}{4}$
Harmonica	$\frac{c_2}{2} - \frac{1}{4\alpha}$	$c_2^2 - \frac{c_2}{2\alpha} - \frac{c_3}{4}$
Geometric	$\frac{c_2}{2} - \frac{1}{8\alpha}$	$c_2^2 - \frac{c_2}{4\alpha} - \frac{c_3}{4}$
Quadratic	$\frac{c_2}{2} + \frac{1}{8\alpha}$	$c_2^2 + \frac{c_2}{4\alpha} - \frac{c_3}{4}$

2.3 Design of numerical experiments

Three scalar problems were considered:

$$F_1(x) = x^3 + 4x^2 - 10 \quad x_0 = 1.5 \quad (21)$$

$$F_2(x) = \cos x - x \quad x_0 = 1.0 \quad (22)$$

$$F_3(x) = e^x - 3x \quad x_0 = 1.0 \quad (23)$$

Double-precision arithmetic and the joint criterion were used.

$$|x_{n+1} - x_n| \leq 10^{-12}, \quad |f(x_{n+1})| \leq 10^{-12} \quad (24)$$

A Newton iteration requires one evaluation of f and one of f' . Each mean-based variant requires one evaluation of f and one of f' . The theoretical efficiency index $p^{1/d}$ with order p and d evaluations is $22^{1/2} \approx 1.4142$ for Newton and $3^{1/3} \approx 1.4422$ for each variant.

2.4 Convergence of the Weerakoon-Fernando method

Theorem 2. Let $f \in C^4(I)$, and let α be a simple root of f . If the initial approximation x_0 is sufficiently close to α and the denominators in equations .22. and equations .23. do not vanish, the Weerakoon–Fernando method has an order of convergence of at least three.

Proof. Let $e_n = x_n - \alpha$, and use the constants c_k defined in equation .12., . The Taylor expansions in equations .16. and equation .19., derived from the proof presented in the original article show that the predictor satisfies.

$$e_{y,n} = c_2 e_n^2 + (2c_3 - 2c_2^2) e_n^3 + O(e_n^4) \quad (25)$$

Upon evaluating the derivative at the predictor and expanding around the root:

From the expression for $f'(y_n)$, normalized by $f'(\alpha)$, equation .26. is obtained.

$$\frac{f'(y_n)}{f'(\alpha)} = 1 + 2c_2^2 e_n^2 + (4c_2 c_3 - 4c_2^3) e_n^3 + O(e_n^4) \quad (26)$$

On the other hand, from equation .17. and equation .26., the arithmetic mean of the two derivatives is

$$\frac{f'(x_n) + f'(y_n)}{2f'(\alpha)} = 1 + c_2 e_n + \left(c_2^2 + \frac{3}{2} c_3\right) e_n^2 + O(e_n^3) \quad (27)$$

Upon substituting Equation .16. and Equation .27. into the correction for Equation .4. and expanding the quotient.

$$\frac{2f(x_n)}{f'(x_n) + f'(y_n)} = e_n - \left(c_2^2 + \frac{c_3}{2}\right) e_n^3 + O(e_n^4) \quad (28)$$

Consequently

$$e_{n+1} = \left(c_2^2 + \frac{c_3}{2}\right) e_n^3 + O(e_n^4) \quad (29)$$

Equation .29. demonstrates cubic convergence when $c_2^2 + \frac{c_3}{2} \neq 0$; if this constant vanishes, the order is higher. This proof employs a Taylor series expansion and applies the definition $e_n = x_n - \alpha$ specifically to the Weerakoon-Fernando method.

III Result

Table no 2. Shows the results for the proposed problems, as well as the known and proposed methods.

Problem	Method	Iterations	Final approach	Absolute residual
F_1	Newton	5	1.36523001341410	0.00
F_1	Weerakoon-Fernando	4	1.36523001341410	0.00
F_1	Arithmetic	4	1.36523001341410	0.00
F_1	Harmonica	3	1.36523001341410	0.00
F_1	Geometric	4	1.36523001341410	0.00
F_1	Quadratic	4	1.36523001341410	0.00
F_2	Newton	5	0.739085133215161	0.00
F_2	Weerakoon-Fernando	3	0.739085133215161	0.00
F_2	Aritmética	4	0.739085133215161	0.00
F_2	Armónica	4	0.739085133215161	0.00
F_2	Geométrica	3	0.739085133215161	0.00
F_2	Cuadrática	4	0.739085133215161	0.00
F_3	Newton	7	0.619061286735945	2.22×10^{-16}
F_3	Weerakoon-Fernando	5	0.619061286735945	0.00

F_3	Aritmética	5	0.619061286735945	2.22×10^{-16}
F_3	Armónica	6	0.619061286735945	2.22×10^{-16}
F_3	Geométrica	5	0.619061286735945	2.22×10^{-16}
F_3	Cuadrática	5	0.619061286735945	2.22×10^{-16}

In all three problems, the Weerakoon-Fernando method and all variants involving point means converged to the same root as Newton's method. Weerakoon-Fernando required 4, 3, and 5 iterations for F_1 , F_2 and F_3 , respectively. The reduction in iterations was most noticeable for the harmonic meaning in the case of F_1 , whereas Weerakoon-Fernando and the geometric mean tied for the fewest iterations for F_2 and F_3 , Weerakoon-Fernando matched the arithmetic, geometric, and quadratic mean variants. Residuals reported as zero result from cancellation in double-precision arithmetic rather than algebraic exactness.

IV Discussion

The theoretical result shows that cubic order does not stem from treating the means as “equivalent.” It arises from three local properties: consistency, symmetry, and smoothness. These properties force the linear term of the expansion of $M(x_n, y_n)$ to be $\frac{e_n}{2}$ and eliminate the quadratic term of the iterative error. The specific curvature of each mean modifies A_M and, consequently, the cubic error constant.

The comparison based on the number of iterations favors several variants in the examples, but it does not, by itself, demonstrate computational superiority. The effective computational cost depends on the evaluation of the function and its derivatives, the degree of calculation reuse, and the characteristics of the computing architecture employed. According to Ostrowski's efficiency index, Newton's method yields $\left(\frac{1}{2}\right)^2 \approx 1.414$, whereas a cubic method with three evaluations per cycle yields $\left(\frac{1}{3}\right)^3 \approx 1.4423$. This theoretical difference is moderate and should be complemented by execution time, observed computational order, and robustness analysis⁶⁻⁸.

The harmonic, geometric, and quadratic means impose constraints relative to the arithmetic mean. While these conditions can be guaranteed locally near a positive root, trajectories far from the root may exit the domain or approach a zero denominator. Therefore, a robust implementation must validate each iteration and incorporate a fallback strategy, such as a Newton step or a bounded method.

V Conclusion

The Weerakoon-Fernando method was explicitly incorporated, and its complete proof of cubic convergence was presented. Additionally, a family of modified Newton methods based on point means was corrected and systematized. The formulation avoids confusing the Weerakoon-Fernando method—which relies on a mean of derivatives—with the evaluation of the derivative at a mean of arguments. Local cubic convergence was demonstrated for means that are consistent, symmetric, and sufficiently smooth, and error constants were derived for the arithmetic, harmonic, geometric, and quadratic means. Experiments confirm the expected behavior across three problems involving positive roots. There is no universally superior mean; performance depends on the function, the initial value, the asymptotic constant, and the cost of evaluating the derivative. Data availability and reproducibility the functions, initial values, tolerance, iterative rules, and stopping criteria are specified in equations (21) and (24), allowing the experiments to be directly reproduced on any computing system using double-precision arithmetic.

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