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Generalized Polya-Aeppli Process and Its Applications in Risk Modelling and Analysis

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Abstract: *In this paper, we introduce a new compound Poisson process with truncated negative binomial compounding distribution, which we will call as the Generalized Polya-Aeppli Process. We derived expressions for its probability mass function (p.m.f) and discuss several properties. We develop a new risk model with the Generalized Polya-Aeppli process as the counting process. Also, we derived the joint distribution of the time to ruin and deficit at the time of ruin. The differential equation of the ruin probability is given. As an example, we consider the case where the claim size follows the exponential distribution. We derive distributions of aggregate claims and gain (loss) from the Generalized Polya-Aeppli risk model and compute the stop loss premium.*

Keywords: *Generalized Polya-Aeppli distribution, Ruin probability, aggregate claims distribution, stop loss moment.*

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I. Introduction

Recently statistical distributions and counting processes are widely used in risk modeling and analysis. Mao and Lia (2005) developed a risk model and ruin probability with compound Poisson - geometric process. Minkova (2004) introduced compound Polya-Aeppli process as a compound Poisson process with the truncated geometric distribution. They showed that Polya-Aeppli process provides greater flexibility in modeling count data when it possesses overdispersion property. Minkova (2011) developed I-Polya Process and discussed its applications. Chukova and Minkova (2013) obtained some characterizations. Stefanka and Minkova (2015) developed a risk model with Pólya–Aeppli distribution of order k . Lazarova and Minkova (2017) introduced I-Delaporte process and disccussed its applications.

The Poisson process is a stochastic counting process that has applications in a large variety of daily life situations. But it is a good fit only when the count data at hand is equi-dispersed, that is, when the mean of the data is equal to the variance. It is found that for many count data, the sample variance is smaller or greater than the sample mean. It is known as under dispersion and over dispersion, respectively. This motivated the researchers to search for alternative models. Accordingly, there are two ways in which the Poisson process be generalized namely, by means of compounding or by mixing of distributions.

Starting from the parameterized distribution $g(x/\theta)$ of a random variable X with unknown parameter θ, we may obtain a new family of distributions if we allow parameter θ to be a random variable with cumulative distribution function $H(\theta)$. Then the unconditional distribution of X is said to be a mixture distribution and is given by

$$g(x) = \int g(x/\theta)dH(\theta).$$

Usually, $g(x/\theta)$ is called the mixed distribution and the distribution of parameter namely, $H(\theta)$ is called the mixing distribution. Mixtures are usually considered as alternative models that offer more flexibility. For discrete distributions "compounding" is commonly used in place of "mixing". The process of compounding creates a wider class of distributions.

The compound distributions can be constructed as follows. Let M be a counting random variable and $X_1, X_2, X_3...$ be i.i.d random variables independent of M. Then the distribution of $S = X_1 + X_2 + ... + X_M$ is called a compound distribution and is given by

$$P_S(z) = \sum_{k=0}^{\infty} P(M = k) g^{*k}(z) \ ,$$

where g^{*k} is the k-fold convolution of distribution of X. In this regard, the distribution of X is called the compounding distribution, while that of M is the compounded distribution.

The compound Poisson process is a generalization of the Poisson process obtained by compounding with a suitable distribution. It has wide applications in various fields such as transport, ecology, radiology, quality control, telecommunications etc. The compound Poisson process assures a better description for clustering of events.

The compound Poisson process $\{M(t), t \geq 0\}$ is given by the sum

$$M(t) = \sum_{i=1}^{N(t)} X_i \quad ,$$

where $N(t)$ is a homogeneous Poisson process and $X_1, X_2, X_3 \ldots$ is a sequence of i.i.d random variables independent of $N(t)$. The distribution of X is called compounding distribution. If the compounding random variable X has truncated geometric distribution, we get the Polya-Aeppli process.

Gerber (1979) gave a detailed account of Mathematical Risk Theory. Johnson et al (2005) discussed in detail various discrete statistical distributions and their applications in risk analysis. Panjer (1981) as well as Willmot and Lin (2001) obtained Lundberg approximation for compound distributions with insurance applications. Geber (1982) obtained the distribution of aggregate claims and its stop-loss premium.Dufresne and Gerber (1989) and Dickson (2007) considered ruin problems and computed the probability of ruin. Rufresne et al. (1991) considered risk analysis with Gamma process. Willmot and Lin (2010) considered risk modeling with mixed Erlang distribution.

The negative binomial model is one of the most popular models to count data. Among specific fields where negative binomial distribution have been found to provide useful representations may be mentioned in accident statistics, Econometrics, quality control and biometrics. In many cases, however, the entire distribution of counts is not observed. In particular, more often zeros are not observed. The negative binomial distribution often arises in practice where the zero group is truncated. Since the truncated geometric distribution is a special case of the truncated negative binomial distribution, we consider the truncated negative binomial as compounding distribution. As a result,the Generalized Polya-Aeppli process will be obtained.

Generalized Polya-Aeppli Distribution

Consider a random variable

$$M = X_1 + X_2 + \ldots + X_N ,$$

where N has a Poisson distribution with parameter λ, independent of the i.i.d random variables $X_1, X_2, X_3 \ldots$. Suppose that $X_1, X_2, X_3 \ldots$ are truncated negative binomial with parameters r and $1-\rho, \rho \in [0,1]$ and $r > 0$.

The PMF and the PGF of the compounding random variable X are given by

$$P(X = x) = \frac{\binom{r+x-1}{x} \rho^x (1-\rho)^r}{1-(1-\rho)^r}, x = 1,2,\ldots, 0 < \rho < 1, r > 0.$$

and

$$P(s) = Es^X = \frac{(1-\rho s)^{-r} - 1}{(1-\rho)^{-r} - 1}.$$

Then, we obtain the PGF of the random variable N as

$$\Psi_M(s) = e^{-\lambda(1-P(s))} = e^{-\lambda\left(1-\frac{(1-\rho s)^{-r}-1}{(1-\rho)^{-r}-1}\right)},$$

Similarly, PMF is obtained as

$$P(M=m) \quad = e^{-\lambda}, \quad m=0$$
$$= e^{-\lambda}\rho^m \sum_{i=1}^{m}\sum_{k=1}^{i} \frac{(-1)^{i+k}\binom{i}{k}\left(\frac{\lambda}{(1-\rho)^{-r}-1}\right)^i \binom{rk+m-1}{m}}{i!}, \ m=1,2,\ldots \tag{1}$$

The above distribution will be called the Generalized Polya-Aeppli distribution withparameters λ, ρ *and* r and is denoted by $GPA(\lambda,\rho,r)$.

The cumulative distribution function of N is given by

$$F(0) \quad = e^{-\lambda},$$
$$F(x) \quad = e^{-\lambda}\sum_{j=0}^{x}\sum_{i=1}^{x-j}\sum_{k=1}^{i} \frac{(-1)^{i+k}\left(\frac{\lambda}{(1-\rho)^{-r}-1}\right)^i \binom{rk+n-j-1}{n-j}\rho^{n-j}}{i!}, \ x \geq 1. \tag{2}$$

Generalized Polya-Aeppli Process

Let $M(t)$ denotes the number of occurrence in the interval $(0,t]$. For the Generalized Polya-Aeppli process, $M(t)$ has a Generalized Polya-Aeppli distribution, $GPA(\lambda t,\rho,r)$ and is given by

$$P(M(t)=m) = \begin{cases} e^{-\lambda t}, & m=0 \\ e^{-\lambda t}\rho^m \sum_{i=1}^{m}\sum_{k=1}^{i} \frac{(-1)^{i+k}\binom{i}{k}\left(\frac{\lambda t}{(1-\rho)^{-r}-1)}\right)^i \binom{rk+m-1}{m}}{i!}, & m=1,2,\ldots \end{cases} \tag{3}$$

To express $\{M(t), t \geq 0\}$ is a Generalized Polya-Aeppli process with parameters λ,ρ and r, we use the notation $M(t) \sim \text{GPAP}(\lambda,\rho,r)$.

Remark 1.*Taking into account the equality* $\sum_{k=1}^{i}(-1)^k\binom{i}{k}\binom{k+m-1}{m} = (-1)^i\binom{m-1}{i-1}$, *when* $r=1$, *the Generalized Polya-Aeppli process* $GPAP(\lambda,\rho,1)$ *reduces to the Polya-Aeppli process. If* $r=1$ *and* $\rho=0$ *, then it is a homogeneous Poisson process with intensity* λ. *Thus the Poisson process and the Polya-Aeppli process are the special cases of the Generalized Polya-Aeppli process.*

Definition 1.*A counting process* $\{M(t), t \geq 0\}$ *is called a Generalized Polya-Aeppli process with parameters* λ,ρ *and* r *if it satisfies* (i) M(0)= 0, i.e, it starts at zero; (ii) M(t) has independent increments; (iii) For each $t>0$, the number of occurrence $M(t)$ in any interval of length t has Generalized Polya-Aeppli distribution with parameters $\lambda t, \rho$ *and* r. We have

$$EM(t) = \frac{r\rho\lambda t}{(1-\rho)(1-(1-\rho)^r)} \tag{4}$$

and

$$Var(M(t)) = \frac{r\rho\lambda t(1+r\rho)}{(1-(1-\rho)^r)(1-\rho)^2}. \tag{5}$$

using (4) and (5) it can be shown that autocovariance between $N(s)$ and $N(t), s<t$ is

$$c(s,t)) = \frac{r\rho\lambda s(1+r\rho)}{(1-(1-\rho)^r)(1-\rho)^2}.$$

Hence the autocorrelation function is

$$\rho(s,t) \quad = \frac{c(s,t)}{\sqrt{\text{var}(M(s)).\text{var}(M(t))}} = \left(\frac{s}{t}\right)^{\frac{1}{2}}.$$

The Fisher index of dispersion is given by

$$FI(M(t)) = \frac{\text{var}(M(t))}{EM(t)} = \frac{1+r\rho}{1-\rho} = 1 + \frac{(1+r)\rho}{1-\rho} > 1.$$

If $r = 0$ the Generalized Polya-Aeppli process is over dispersed, which offer more flexibility in modeling count data compared to the standard Poisson process.

Alternate Definition of Generalized Polya-Aeppli process

In this section we define Generalized Polya-Aeppli process as a pure birth process.

Definition 2.

A counting process $\{M(t), t \geq 0\}$ is called a Generalized Polya-Aeppli process with parameters λ, ρ and r if $M(0) = 0$ and $M(t)$ has stationary independent increments; and the state transition probabilities are defined as follows:

$$P\big(M(t+h) = m/M(t) = n\big) = \begin{cases} 1 - \lambda h + o(h), & m = n \\ \dfrac{\binom{r+i-1}{i}\rho^i (1-\rho)^r}{1-(1-\rho)^r}\lambda h + o(h), & m = n+i, i = 1,2,\ldots \end{cases} \quad (6)$$

for every $n = 0,1,\ldots$ where $o(h) \to 0$ as $h \to 0$.

Let $P_m(t) = P(M(t) = m), \quad m = 0,1,2,\ldots$.

From the above postulates we get the following Kolmogorov forward equations:

$$\begin{aligned} P_0'(t) &= -\lambda P_0(t), \\ P_m'(t) &= -\lambda P_m(t) + \frac{\lambda}{(1-\rho)^{-r} - 1}\sum_{i=1}^{m}\binom{r+i-1}{i}\rho^i P_{m-i}(t), m \geq 1, \end{aligned} \quad (7)$$

with initial conditions.

$$P_0(0) = 1 \text{ and } P_m(0) = 0, \quad m = 1,2,\ldots$$

The equations in (7) yield to the following differential equation for $\psi_{M(t)}$

$$\frac{\partial}{\partial t}\psi_{M(t)}(s) = -\lambda(1 - P(s))\psi_{M(t)}(s).$$

With $\psi_{M(t)}(1) = 1,$ the above differential equation admit of the solution

$$\psi_{M(t)}(s) = \mathrm{e}^{-\lambda t(1-P(s))}$$

But this is the PGF of the GPAP (λ, ρ, r), which leads to (3).

Therefore two definitions of the Generalized Polya-Aeppli Process are equivalent.

Properties of GPAP (λ, ρ, r)

In this section, we discuss some properties of GPAP (λ, ρ, r).

Interarrival Times Distributions

Theorem 1. *For the Generalized Polya-Aeppli process GPAP* (λ, ρ, r), *time interval* Z_1 *to the first occurrence follows exponential distribution with parameter* λ *and the time between first and second occurrences of the process namely* Z_2, *takes value 0 with probability* $1 - \frac{r\rho}{(1-\rho)^{-r} - 1}$ *and follows exponential distribution with parameter* λ *,with probability* $\frac{r\rho}{(1-\rho)^{-r} - 1}$.

Proof. Let Z_k be the time of the k^{th} arrival, for $k=1,2,\ldots$. Let $W_n=\sum_{i=1}^{n} Z_i$ be the waiting time up to the n^{th} occurrence and $M(t)$ denote the number of occurrence up to the instant t. For any $t \geq 0$ and $n \geq 0$, we have the following relation.

$$P(M(t)=n)=P(W_n \leq t)-P(W_{n+1} \leq t), n=0,1,\ldots \tag{8}$$

For $n=0$, equation (8) yields

$$P(M(t)=0)=1-P(Z_1 \leq t)=1-F_{Z_1}(t), \tag{9}$$

where $F_{Z_1}(t)$ is the distribution function of Z_1.

According to (3), $P(M(t)=0)=\mathrm{e}^{-\lambda t}$. (10)

From (9) and (10), we get

$$F_{Z_1}(t)=1-\mathrm{e}^{-\lambda t}.$$

Hence the density function of Z_1 is

$$f_{Z_1}(t)=\lambda \mathrm{e}^{-\lambda t}, t \geq 0.$$

Hence Z_1 is an exponential random variable with parameter λ.

Now from (8), for $n=1$, we have,

$$P(M(t)=1)=P(W_1 \leq t)-P(W_2 \leq t).$$

Then taking Laplace transform on both sides of above equation, we get

$$\frac{r\rho}{(1-\rho)^{-r}-1}\frac{\lambda}{\lambda+s}\frac{s}{s+\lambda}=LS_{W_1}(s)-LS_{W_2}(s).$$

On simplification, we get

$$LS_{Z_1+Z_2}(s)=\frac{\lambda}{\lambda+s}\left[1-\frac{r\rho}{(1-\rho)^{-r}-1}+\frac{r\rho}{(1-\rho)^{-r}-1}\frac{\lambda}{s+\lambda}\right],$$

Itfollows that Z_1 and Z_2 are independent. Furthermore, Z_2 has an exponential distribution with parameter λ and takes value zero with probability $1-\frac{r\rho}{(1-\rho)^{-r}-1}$.

Thus, the P.D.F of of Z_2 is given by

$$f_{Z_2}(t)=\left(1-\frac{r\rho}{(1-\rho)^{-r}-1}\right)\delta_0(t)+\left(\frac{r\rho}{(1-\rho)^{-r}-1}\right)\lambda \mathrm{e}^{-\lambda t}, t \geq 0,$$

where $\delta_0(t)$ is the dirac delta function.

The Waiting Time Distribution

Theorem 2. *The distribution function of the waiting time up to the* n^{th} *occurrence is given by*

$$F_{W_n}(t)=1-\mathrm{e}^{-\lambda t}\left(1+\sum_{m=1}^{n-1} a_{m,t}\rho^m\right),$$

where

$$a_{m,t}=\sum_{i=1}^{m}\sum_{k=1}^{i}\frac{(-1)^{i+k}\binom{i}{k}\left(\frac{\lambda t}{(1-\rho)^{-r}-1}\right)^i}{i!}\binom{rk+m-1}{m}.$$

Proof. Let Z_n denotes the time between $(n-1)^{\text{th}}$ and n^{th} occurrence of the process, $n=2,3,\ldots$. Then for any given integer $n \geq 1$ and time $t>0$, the relation between waiting time up to the n^{th} occurrence W_n and counting random variable M(t), is given by

$$\{W_n \le t\} = \{M(t) \ge n\}$$

Hence

$$P(M(t) \ge n) = P(W_n \le t).$$

The cumulative distribution function is

Martingale Property

Theorem 3. *For* $M(t) \sim GPAP(\lambda, \rho, r)$ *, the process* $N(t) = M(t) - \dfrac{r\rho\lambda t}{(1-\rho)(1-(1-\rho)^r)}$ *is a martingale with respect to* $(\Omega, \mathsf{F}_t, \rho)$ *, where*

$$\mathsf{F}_t = \sigma\{M(s), 0 \le s \le t\}.$$

Proof. Since $E(M(t) - \dfrac{r\rho\lambda t}{(1-\rho)(1-(1-\rho)^r)}) = 0$ and $M(t)$ has independent increments, for any $t \ge s$, we have

$$\begin{aligned}
E(N(t)/\mathsf{F}_s) &= E\left(M(t) - \frac{r\rho\lambda t}{(1-\rho)(1-(1-\rho)^r)} / \mathsf{F}_s \right) \\
&= E\left(M(t) - M(s) - \frac{r\rho\lambda (t-s)}{(1-\rho)(1-(1-\rho)^r)} / \mathsf{F}_s \right) \\
&\quad + M(s) - \frac{r\rho\lambda s}{(1-\rho)(1-(1-\rho)^r)} \\
&= E\left(M(t-s) - \frac{r\rho\lambda (t-s)}{(1-\rho)(1-(1-\rho)^r)} \right) \\
&\quad + M(s) - \frac{r\rho\lambda s}{(1-\rho)(1-(1-\rho)^r)} \\
&= N(s).
\end{aligned}$$

Therefore, $\{N(t), t \ge 0\}$ is a martingale.

Relation between GPAP (λ, ρ, r) and Uniform distribution

Theorem 4. *If the Generalized Polya − Aeppli process,* $\{M(t), t \ge 0\}$ *has occurredonly once in* $[0, t]$ *, then the distribution of the time interval of that occurrence is uniform in* $[0, t]$.

Proof. Let Z_1 denotes the time of first occurrence. Hence it follows that,

$$\begin{aligned}
P(Z_1 \le x / M(t) = 1) &= \frac{P(\text{there is only one occurrence in } (0, x], \text{there was no occurrence in } (x, t])}{P(M(t) = 1)} \\
&= \frac{P(M(x) = 1, M(t-x) = 0)}{P(M(t) = 1)} \\
&= \frac{\dfrac{e^{-\lambda x} r\rho\lambda x \cdot e^{-\lambda(t-x)}}{((1-\rho)^{-r} - 1)}}{\dfrac{e^{-\lambda t} r\rho\lambda t}{(1-\rho)^{-r} - 1}} \\
&= \frac{x}{t}, 0 \le x \le t.
\end{aligned}$$

Application to Risk Theory

Consider the standard risk model $\{X(t), t \geq 0\}$ of an insurance company given by

$$X(t) = ct - \sum_{k=1}^{M(t)} Y_k, \left(\sum_{1}^{0} = 0 \right). \tag{11}$$

Here, we assume that c is a premium income per unit time.The claim size sequence $\{Y_i\}_{i=1}^{\infty}$ are i.i.d random variables having common distribution function F such that $F(0) = 0$, with mean value μ and $\{Y_i\}_{i=1}^{\infty}$ are independent of the counting process $\{M(t), t \geq 0\}$.

We have established that the counting process $\{M(t), t \geq 0\}$ in the risk model given in (11) is a Generalized Polya-Aeppli process and the resulting risk model obtained is called the Generalized Polya-Aeppli risk model.

The relative safety loading θ is given by

$$\theta = \frac{EX(t)}{E\sum_{k=1}^{M(t)} Y_k} = \left(\frac{c(1-\rho)(1-(1-\rho)^r)}{r\lambda\rho\mu} - 1 \right)$$

and in relation to positive safety loading $\theta > 0, c > \dfrac{r\rho\mu\lambda}{(1-\rho)(1-(1-\rho)^r)}$.

Suppose that the time of ruin of the company with the initial capital u is τ so that $\tau = \inf\{t : X(t) + u < 0\}$.
Here by convention we take $\inf \phi = \infty$.
Then the ruin probability of a company having initial capital u is defined as

$$\Psi(u) = P(\tau < \infty). \tag{12}$$

Then the probability ofnon ruin is given by $\Phi(u) = 1 - \Psi(u)$.

The joint probability distribution $W(u, z)$ of the time to ruin τ and deficit at the time of ruin $D = |u + X(\tau)|$ is given by

$$W(u, z) = P(\tau < \infty, D \leq z), \quad z \geq 0, \tag{13}$$

It is obvious that

$$\lim_{z \to \infty} W(u, z) = \Psi(u). \tag{14}$$

We can obtain the following equation by using the postulates in (6).

$$\begin{aligned} W(u,z) \quad &= (1-\lambda h)W(u+ch, z) + \frac{\lambda}{(1-\rho)^{-r} - 1} \sum_{k=1}^{\infty} \binom{r+k-1}{k} \rho^k h \\ &\times \left[\int_0^{u+ch} W(u+ch-x, z) dF^{*k}(x) + (F^{*k}(u+ch+z) - F^{*k}(u+ch)) \right] + o(h), \end{aligned}$$

where $F^{*k)}(x), k = 1,2,\ldots$ is the distribution function of $Y_1 + Y_2 + \ldots + Y_k$.
Equivalently,

$$\begin{aligned} \frac{W(u+ch, z) - W(u,z)}{ch} &= \frac{\lambda}{c} W(u+ch, z) - \frac{\lambda \sum_{k=1}^{\infty} \binom{r+k-1}{k} \rho^k}{c((1-\rho)^{-r} - 1)} \\ &\times \left[\int_0^{u+ch} W(u+ch-x, z) dF^{*k}(x) + (F^{*k}(u+ch+z) - F^{*}(u+ch)) \right] + o(h), \end{aligned}$$

In the limit, as $h \to 0$,

$$\frac{\partial}{\partial u} W(u, y) = \frac{\lambda}{c} \left[W(u, y) - \int_0^u W(u-x, z) dG(x) - (G(u+z) - G(u)) \right] \tag{15}$$

where

$$G(x)=\frac{1}{(1-\rho)^{-r}-1}\sum_{k=1}^{\infty}\binom{r+k-1}{k}\rho^{k}F^{*k}(x),$$

is the nondefective distribution function of the claims with

$$G(0)=0, G(\infty)=1.$$

Related to safety loading, the above equation can be written as

$$\begin{aligned}\frac{\partial}{\partial u}W(u,z) &= \frac{(1-\rho)(1-(1-\rho)^{r})}{rp\mu(1+\theta)}\\ &\times\left[W(u,z)-\int_0^u W(u-x,z)dG(x)-(G(u+z)-G(u))\right]\end{aligned} \tag{16}$$

From (14) and (15) we get the integro differential equation for ruin probability as

$$\frac{d}{du}\Psi(u)=\frac{\lambda}{c}\left[\Psi(u)-\int_0^u\Psi(u-x)dG(x)-(1-G(u))\right], u\geq 0.$$

Theorem 5. *The function* $W(0,z)$ *is given by*

$$W(0,y)=\frac{\lambda}{c}\int_0^y(1-G(u))du. \tag{17}$$

Proof. Integrating (15) from 0 to ∞ and then using $W(\infty,z)=0$, we have

$$\begin{aligned}-W(0,z)=\frac{\lambda}{c}&\left[\int_0^\infty W(u,z)du-\int_0^\infty\int_0^u W(u-x,z)dG(x)du\right.\\ &\left.-\int_0^\infty(G(u+z)-G(u))du\right].\end{aligned}$$

Substitution in the double integral and after simplification, we get

$$W(0,z)=\frac{\lambda}{c}\int_0^\infty(G(u+z)-G(u))du.$$

Hence

$$W(0,z)=\frac{\lambda}{c}\int_0^y(1-G(u))du.$$

Theorem 6. *The ruin probability with no initial capital satisfies*

$$\Psi(0)=\frac{\lambda r\rho\mu}{c(1-\rho)(1-(1-\rho)^{r})} \tag{18}$$

Proof. From (14) and (17) we obtain,

$$\Psi(0)=\lim_{z\to\infty}W(0,z)=\frac{\lambda}{c}\int_0^\infty(1-G(u))du. \tag{19}$$

Suppose that $G(x)$ be the distribution function of a random variable X. Using the result $E(X)=\int_0^\infty(1-G(x))dx$, (19) becomes.

$$\Psi(0)=\frac{\lambda}{c}E(X) \tag{20}$$

Considering the definitions of $G(x)$ and $EY=\mu$, we get

$$\begin{aligned}E(X) &= \frac{\mu}{(1-\rho)^{-r}-1}\sum_{k=1}^{\infty}k\binom{r+k-1}{k}\rho^{k}\\ &=\frac{\mu r\rho}{(1-\rho)(1-(1-\rho)^{r})}.\end{aligned} \tag{21}$$

From the equations (20) and (21), we get the result.

Exponentially Distributed claims

Suppose that the claim sizes have an exponential distribution with mean μ. i.e., $\overline{F}(x) = 1 - e^{-\frac{x}{\mu}}$, $x \geq 0$, $\mu > 0$. Now the density function $g(x) = \frac{d}{dx}G(x)$ obtained is an Erlang mixture and is given by

$$g(x) = \sum_{k=1}^{\infty} \frac{q_k (\frac{x}{\mu})^{k-1} e^{-\frac{x}{\mu}}}{\mu(k-1)!}, \qquad x > 0,$$

where $q_k = P(Y = k) = \frac{\binom{r+k-1}{k}\rho^k}{(1-\rho)^{-r} - 1}, k = 1,2,\dots$ is the mixing distribution, for details see Willmot and Lin (2001).

The survival function corresponding to the above density is given by

$$\overline{G}(x) = \sum_{k=1}^{\infty} \frac{\overline{Q}_{k-1} (\frac{x}{\mu})^{k-1} e^{-x/\mu}}{(k-1)!}, x > 0,$$

where

$$\begin{aligned} \overline{Q}_{k-1} &= P(Y > k-1) \\ &= \sum_{i=k}^{\infty} \frac{\binom{r+i-1}{i}\rho^i}{(1-\rho)^{-r} - 1} \quad k = 1,2,3... \end{aligned}$$

Applying theorem 5, for the case when claim sizes have an exponential distribution with mean μ, $W(0, y)$ is given by

$$W(0, Z) = \frac{\lambda\mu}{c} \sum_{k=1}^{\infty} \frac{\overline{Q}_{k-1}\gamma(k, Z/\mu)}{(k-1)!},$$

where $\gamma(j, y) = \int_0^y u^{j-1} e^{-u} du$ is the incomplete gamma function.

Derivation of Total Claims Distribution, Total Loss(gain) Distribution and Stop Loss Moment

A central problem in risk theory is the modeling of the probability distribution of the aggregate claims. The aggregate claims distribution and its components, the claim count and claim amount distributions are used to compute ruin probabilities and to provide other information of interest to the decision makers. Panjer(1981) found that a compound Poisson process approximately modeled the aggregate claims distribution, based on the collective risk assumption.

Here we concentrate on the case where the aggregate claims distribution has a compound Generalized Polya-Aeppli process. This is equivalent to assuming that the counting process is the Generalized Polya-Aeppli process. We assume that the claim sizes have a continuous distribution with cumulative distribution function F such that $F(0) = 0$, and mean value μ.

Denote by $N(t)$ the number of claims, by Z_i the i^{th} claim amount and by $S(t)$ the aggregate claim amount in a time period of length t given by $S(t) = \sum_{i=1}^{M(t)} Z_i$.

In this case $ES(t)$ corresponds to the pure premium and is given by

$$\begin{aligned} ES(t) &= EM(t)E(Z) \\ &= \frac{r\rho\lambda t}{(1-\rho)(1-(1-\rho)^r)}\mu \end{aligned}$$

Let $H(x,t)$ denotes the cumulative distribution function of the aggregate claims and $F^{*k}(x)$ is the k -fold convolution of claim amount distribution function which can be calculated recursively as

$$F^{*k}(x) = \int_0^x F^{*k-1}(x-y)f(y)dy.$$

with

$$\begin{aligned} F^{*0}(x) &= 1, \qquad x \geq 0 \\ &= 0, \qquad x < 0. \end{aligned}$$

Recalling that the number of claims has Generalized Polya-Aeppli distribution, We have

$$\begin{aligned} H(x,t) &= \sum_{k=0}^{\infty} P(M(t)=k)F^{*k}(x) \\ &= e^{-\lambda t}\left[I_{[0,\infty)}(x) + \sum_{k=1}^{\infty} a_{k,t}\rho^k F^{*k}(x)\right], \qquad x \geq 0, \end{aligned} \tag{22}$$

where $a_{k,t} = \sum_{i=1}^{k}\sum_{j=1}^{i}(-1)^{i+j}\binom{i}{j}\binom{rj+k-1}{k}\left(\frac{\lambda t}{(1-\rho)^{-r}-1}\right)^i$ and $I_A(x)$ is the indicator function of the set A.

Finding the cumulative distribution function of the aggregate gain(loss) is one of the principal problems in the collective risk theory. Here we derive the distribution function of aggregate loss (gain) from Generalized Polya-Aeppli risk model using the distribution function of aggregate claims.

Consider the Generalized Polya-Aeppli risk model

$$X(t) = ct - S(t),$$

mentioned in (11) , where $X(t)$ denotes the aggregate gain (loss) and $S(t)$ is the aggregate claims in a time period of length t. In this model, the number of claims occurring in a period of length t has the Generalized Polya-Aeppli distribution.

Then the cumulative distribution function $G(x,t)$ of $X(t)$ is given by

$$\begin{aligned} G(x,t) &= P(X(t) \leq x) \\ &= P(S(t) \geq ct - x) \\ &= \overline{H}(ct - x, t) \\ &= \sum_{k=1}^{\infty} a_{k,t}\rho^k \overline{F}^{*k}(ct - x), \end{aligned} \tag{23}$$

where $\overline{F}^{*k}(x)$ is the survival function of $F^{*k}(x)$.

Stop-loss moment of any positive order can be obtained using (22) and is given by

$$\int_y^{\infty}(x-y)^m dH(x,t) = e^{-\lambda t}\sum_{k=1}^{\infty} a_{k,t}\rho^k \int_y^{\infty}(x-y)^m f^{*k}(x). \tag{24}$$

where f^{*k} is the k-fold convolutions of pdf of claims.

Note that the case $m=0$ we get $\overline{F}(a)$, the tail function of the aggregate claim amount. When $m=1$, the stop-loss premium results and is given by

$$\int_y^\infty \overline{H}(x,t)dx = \mu e^{-\lambda t}\sum_{k=1}^{\infty} k a_{k,t}\rho^n \overline{F}_I^{*k}(y), \tag{25}$$

where $F_I^{*k}(x) = \frac{1}{EX}\int_0^x \overline{F}_X^{*k}(u)du$, is the integrated tail distribution of F^{*k}.

Now we discuss a particular case in which the claim sizes have an exponential distribution with mean μ . i.e, $F(x) = 1 - e^{-x/\mu}, x \geq 0, \mu > 0$.

In this case, the k fold convolution of claim sizes is given by

$$F^{*k}(x) = \frac{\gamma(k, x/\mu)}{\Gamma k}$$

$$= 1 - e^{-x/\mu}\sum_{j=0}^{k-1}\frac{(\frac{x}{\mu})^j}{j!}$$

$$= 1 - e^{-x/\mu}e_k(x/\mu),$$

where Γk is the gamma function and $\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt$ is the incomplete gamma function.

Hence the cumulative distribution function of aggregate claims with respect to Generalized Polya-Aeppli risk model with exponential claims is

$$H(x,t) = e^{-\lambda t}\left(I_{[0,\infty)}(x) + \sum_{k=1}^{\infty} a_{k,t}\rho^k[1 - e^{-x/\mu}e_k(x/\mu)]\right)$$

Similarly, the cumulative distribution function of aggregate loss(gain) is given by

$$G(x,t) = e^{-\lambda t}\left(I_{[0,\infty)}(x) + e^{-\frac{(ct-x)}{\mu}}\sum_{k=1}^{\infty} a_{k,t}\rho^k e_k(\frac{ct-x}{\mu})\right).$$

Based on (24), for exponential claims, the stop loss moment is

$$\int_y^\infty (x-y)^m dH(x,t) = e^{-(\lambda t + \frac{y}{\mu})}\sum_{k=1}^{\infty}\sum_{i=0}^{m}\frac{(-1)^{m-i}\binom{m}{i}a_{k,t}\rho^k y^{m-i}\mu^i\Gamma(k+i)\, e_{k+i}(y/\mu)}{\Gamma(k)}$$

From (25) we can obtain stop-loss premium as

$$\int_y^\infty \overline{H}(x,t)dx = e^{-(\lambda t + y/\mu)}\sum_{k=1}^{\infty} a_{k,t}\rho^k\left[k\mu e_{k+1}(y/\mu) - y e_k(y/\mu)\right]$$

II. Conclusions

In this paper, we introduced a new compound Poisson process, called the Generalized Polya-Aeppli process, by compounding with truncated negative binomial distribution. We have shown that it is a generalization of Polya-Aeppli process andestablished that this model is capable of handling over-dispersed count data. We introduced a new risk model with Generalized Polya-Aeppli counting process. We have developed ruin theory, the probability of ruin for this model and as a special case, we have derived an expression for the ruin probability with zero initial capital. This model can be applied in insurance, business and acturial sciences as a more versatile one than existing models.

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