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Spinors and Their Applications to Track near Earth Objects

A. Lakshmi[1] and S. N. Hasan[2]
[1]*Department of Mathematics, Osmania University, Hyderabad -500007*
[2]*Department of Astronomy, Osmania University, Hyderabad – 500007.*

Abstract: *In this paper Spinors of Euclidean spaces of Geometric Algebras over a real field are defined and their algebraic and rotational properties are discussed. The advantage of using Spinors in solving problems of Celestial Mechanics is illustrated by studying the tracking problem of near Earth objects. It is shown that this technique of using spinors can replace the conventional methods and also provide a richer formalism.*
Key words: *Euler angles, Geometric Algebra, Euclidean spaces, Rotations, Spinors.*

I. Introduction

Spinors are defined as a product of two vectors in i-plane or i-space (Hestenes, 1986). Spinors can also be viewed as elements of a minimal left ideal (Hestenes, 1966 & Lounesto, 2001). Rotations can be treated as group actions of Spinors on the vector space over which the Geometric Algebra is constructed. There are different parametrizations for Spinors in Geometric Algebra. Thus rotations also have different parametrizations depending upon the form of the Spinor considered.

Sequences of rotations play a key role in tracking near earth orbiting objects such as an aero plane or a spacecraft (Kuipers, 1999). There exists a sequence of Spinors corresponding to every sequence of rotations. The purpose of the present paper is to compare our results with the results obtained by using Quaternions and to show that Geometric Algebra works as an efficient tool to study problems in Celestial Mechanics.

II. Geometric Algebra

Let E_n be an n-dimensional vector space over R, the field of real numbers, together with a symmetric, positive definite, bilinear form $g : E_n \times E_n \to \mathrm{R}$ denoted by $g(\vec{x},\vec{y}) = \vec{x}.\vec{y}$, $\forall \vec{x},\vec{y} \in E_n$. There exists a unique Clifford Algebra $(C(E_n),\rho)$ which is a universal algebra in which E_n is embedded. Henceforth, we shall identify E_n with $\rho(E_n)$. We choose and fix an orthonormal basis $B_n = \{e_1, e_2, ..., e_n\}$ for E_n.

Let $\mathrm{A}_0 =$ span $\{e_\phi\} = R\, 1_A = R$

$\mathrm{A}_1 =$ span $\{e_i\}$ = E_n

And in general, $\mathrm{A}_k = \{\sum_{\substack{s \\ |s|=k}} a_s e_s\}$

Then $C(E_n) = \oplus \mathrm{A}_k$

Clearly dim $\mathrm{A}_k = nC_k$ and $\dim C(E_n) = 2^n$.

Geometric Algebra is constructed by taking the geometric product of the vectors in the *n* - dimensional vector space E_n, giving multivectors as its products. The 'Geometric Product' of vectors denoted by $\vec{a}\vec{b}$, as

$$\vec{a}\vec{b} = \vec{a}\ .\ \vec{b} + \vec{a} \wedge \vec{b} \qquad \text{.............. (1)}$$

$$\vec{b}\vec{a} = \vec{b}\ .\ \vec{a} + \vec{b} \wedge \vec{a} = \vec{a}\ .\ \vec{b} - \vec{a} \wedge \vec{b} \qquad \text{.............. (2)}$$

As every element of $C(E_n)$ is in the form A = $\langle A \rangle_0 \oplus \langle A \rangle_1 \oplus \ ....\ \oplus \langle A \rangle_n$, it is called a multivector. A multivector is said to be even (odd) if $\langle A \rangle_r = 0$ whenever r is odd (even). A detailed construction was given by Hasan (1987).

k-blade: Outer product of 'k' number of 1-vectors is called a k- blade.

Note that $e_S = e_{i_1} e_{i_2} .... e_{i_m}$ is a m - blade.

Define the set $G_n = \{\pm e_S / S \subseteq N\}$. Clearly G_n is a group with respect to the operation 'Geometric Product' of the elements defined by $e_S e_T = \tau(S,T) e_{S\Delta T}$ with

$e_S^{\dagger} = (-1)^{\frac{k(k-1)}{2}} e_S$ as the inverse of e_S and $\{e_\phi\}$ as the identity. G_n is a 'free group' with B_n as a finite basis. $|G_n| = 2^{n+1}$.

2.1 Euclidean nature of Geometric Algebra

2.1.1 Definition: Norm of a multivector The concept of 'norm' of a multivector is very important to define division in Geometric Algebra.

To every A $\in C(E_n)$ the magnitude or modulus of A is defined as $|A| = \langle A^{\dagger}A \rangle_0^{\frac{1}{2}}$.

With this definition of norm, $C(E_n)$ becomes a Euclidean algebra. The inverse of a non zero element of A of $C(E_n)$, is also a multivector, defined by $A^{-1} = \frac{A^{\dagger}}{|A|^2}$.

2.1.2 Definition: k - *space* Every k- vector A_k determines a k - *space*

2.1.3 Definition: n – dimensional Euclidean space $C(E_n)$: For a n-vector A_n, designate a unit n - vector 'i' proportional to A_n. That is $A_n = |A_n|\, i$.

'i' denotes the direction of the space represented by A_n.

2.1.4 *Definition i- space* : The set of all vectors $\vec{x} \in E_n$ which satisfy the equation $\vec{x} \wedge i = 0$, is said to be an *i- space* and is denoted by $C_n(i)$. Such a n - vector 'i' is called the pseudoscalar of the plane as every other n -vector can be expressed as a scalar multiple of it.

Note (a) $\vec{x} = x_1\sigma_1 + x_2\sigma_2 + \ldots.. + x_n\sigma_n$ is a parametric equation of the *i- space*. $x_1, x_2, \ldots, x_n$ are called the rectangular components of vector $\vec{x}$ with respect to the basis $\{\sigma_1, \sigma_2, \ldots\ldots, \sigma_n\}$ (Hestenes, 1986).

(b) $\vec{x} = x_1\sigma_1 + x_2\sigma_2$ is a parametric equation of the *i- plane*. x_1, x_2 are called the rectangular components of vector $\vec{x}$ with respect to the basis $\{\sigma_1, \sigma_2\}$.

2.2 Spinors in 'n' dimensions

2.2.1 Definition: *Spinor* The product of two vectors in the *i- space* is called a Spinor.

2.2.2 Definition: *Spinor i- space* : The Spinor *i- space* 'S' is defined as

$S_n = \{R / R = \vec{x}\vec{y},\ \vec{x}, \vec{y} \in i\text{- space}\}$

$S_n = C_3^{+}(i)$ if $n \le 3$. 'S_2' can be related to complex numbers and 'S_3' can be related to Quaternion Algebra.

Let $\vec{x} = x_1\sigma_1 + x_2\sigma_2 + \ldots. + x_n\sigma_n$ and $\vec{y} = y_1\sigma_1 + y_2\sigma_2 + \ldots.. + y_n\sigma_n$.

Then the elements in the Spinor *i- space* 'S_n' are in the form

$$R = \vec{x}\ \vec{y} = \sum_{j=1}^{n} x_j y_j + \sum_{j,k=1}^{n} (x_j y_k - x_k y_j)\ \sigma_j \sigma_k$$

R can be written as $\alpha + \beta\ i$ if n = 2 or 3

Where $\alpha = \sum_{j=1}^{n} x_j y_j$ and $\beta = \sum_{j,k=1}^{n} (x_j y_k - x_k y_j)$

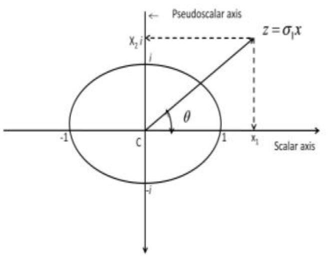

Fig 1: *i- plane* of Spinors

2.3 Algebraic Properties of Spinors

2.3.1 Theorem : $S_n = \{ R / R = \vec{x}\ \vec{y},\ \ \vec{x}, \vec{y} \in i\text{-}space \}$

(i) S_n is an abelian group with respect to the operation '$+$' defined as the addition of the coefficients of the like terms similar to addition of polynomials.
(ii) S_n is a vector space over R .
(iii) dim $S_n = 2^{n-1}$
Spinor spaces S_n in dimensions $n > 2$ do not satisfy commutative property. Hence they form division algebras or associative algebras.
2.4 Euclidean space $C_3(\mathbf{i})$
For a trivector $A_3 \in C_3(\mathbf{i})$, designate a unit trivector '$\mathbf{i}$' proportional to A_3 . That is $A_3 = |A_3|\,\mathbf{i}$.
'$\mathbf{i}$' represents the direction of the space represented by A_3 .
The set of all vectors $\vec{x}$ which satisfy the equation $\vec{x} \wedge \mathbf{i} = 0$, is called the Euclidean 3- dimensional vector space corresponding to '$\mathbf{i}$' and is denoted by '$C_3(\mathbf{i})$'.
$C_3(\mathbf{i})$ can also be called an $\mathbf{i}$- *space*, the trivector $\mathbf{i}$ is called the pseudoscalar of the space as every other pseudoscalar is a scalar multiple of it.
$\vec{x} = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$ is a parametric equation of the $\mathbf{i}$- *space* where x_1, x_2 and x_3 are called the rectangular components of vector $\vec{x}$ with respect to the basis $\{\sigma_1, \sigma_2, \sigma_3\}$.
$\mathbf{i}$- *space* of vectors is a 3 – dimensional vector space with basis $\{\sigma_1, \sigma_2, \sigma_3\}$.
2.4.1 Definition : *bivectors in* $C_3(\mathbf{i})$: $\mathbf{i}_1 = \sigma_1\mathbf{i} = \sigma_2\sigma_3$; $\mathbf{i}_2 = \sigma_2\mathbf{i} = \sigma_3\sigma_1$; $\mathbf{i}_3 = \sigma_3\mathbf{i} = \sigma_1\sigma_2$. The set of bivectors in $C_3(\mathbf{i})$ is a 3-dimensional vector space with basis $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$.
2.4.2 Spinors of Euclidean space $C_3(\mathbf{i})$
2.4.3 Definition: *Spinor* $\mathbf{i}$- *space* The Spinor $\mathbf{i}$- *space* 'S_3' is defined as

$$S_3 = \{ R / R = \vec{x}\ \vec{y},\ \ \vec{x}, \vec{y} \in \mathbf{i}\text{-}space \}$$

S_3 can also be denoted by $C_3^+(E)$ or $C_3^+(\mathbf{i})$. '$R \in S_n$' is a multivector, has a scalar part 'α' and a bivector part '$\gamma\,\mathbf{i}_1 + \delta\,\mathbf{i}_2 + \beta\,\mathbf{i}_3$'.

TABLE 1: Comparison between Spinors of Euclidean plane and Spinors of Euclidean space

Spinors of Euclidean plane $C_2(\mathbf{i})$	Spinors of Euclidean space $C_3(\mathbf{i})$
It is a Field	It is an associative division algebra
Basis is isomorphic to the basis of i-plane of vectors	Basis is not isomorphic to the basis of $\mathbf{i}$- *space* of vectors, as it contains one more element
Reversion is analogous to Complex conjugation.	Reversion is anti-isomorphic to Quaternion conjugation.
Spinor basis is isomorphic to the basis of Complex numbers	Spinor basis is anti-isomorphic to the basis of Quatemion algebra.

2.5 Action of Spinors on Euclidean space, Rotations

Spinors of Euclidean space also can be treated as rotation operators on $\mathbf{i}$- *space* of vectors that is the three dimensional vector space E_3 .
Unlike rotations in two dimensions, rotations in three dimensions are more complex as (i) the operation to be considered is the group action by conjugation, giving Similarity Transformations.
(ii) The axis about which the rotation takes place is also to be specified. The resulting vector changes as the axis of rotation changes. This can be shown in the following examples.
Rotate the vector $\vec{x}$ about the axis σ_1, the axis perpendicular to the plane represented by the bivector $\mathbf{i}_1 = \sigma_1\mathbf{i} = \sigma_2\sigma_3$.

Let $\vec{x} \in \mathbf{i}-\ space$ of vectors and $\vec{x} = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$.

$$\mathbf{i_1}^{\dagger}\vec{x}\ \mathbf{i_1} = \sigma_3\sigma_2\vec{x}\sigma_2\sigma_3 = \sigma_3\sigma_2(x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3)\sigma_2\sigma_3 = x_1\sigma_1 - (x_2\sigma_2 + x_3\sigma_3)$$

Rotate the vector $\vec{x}$ about the axis σ_2, the axis perpendicular to the plane represented by the bivector $\mathbf{i_2} = \sigma_2\mathbf{i} = \sigma_3\sigma_1$.

$$\mathbf{i_2}^{\dagger}\vec{x}\ \mathbf{i_2} = \sigma_1\sigma_3\sigma_1\vec{x}\sigma_3\sigma_1 = \sigma_1\sigma_3(x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3)\sigma_3\sigma_1 = x_2\sigma_2 - (x_1\sigma_1 + x_3\sigma_3).$$

2.6 Diferent parametrizations for Spinors of $C_3(\mathbf{i})$

2.6.1 Spinors of the $\mathbf{i}-\ space$ in half angle form

A unit bivector is treated as a representation of the direction of an area. It can also be treated as a representation of an angle, which is a relation between two directions. Hence for $\vec{x},\vec{y} \in C_3(\mathbf{i})$, let $\hat{x},\hat{y}$ be their directions which are elements of the $\mathbf{i}-\ space$.

From the definition of a Spinor of the $\mathbf{i}-\ space$, the Spinor,

$$R = \hat{x}\ \hat{y} = \hat{x}\ .\ \hat{y} + \hat{x} \wedge \hat{y}.$$

$= \cos\frac{1}{2}|\mathbf{A}| + \hat{\mathbf{A}}\sin\frac{1}{2}|\mathbf{A}|$ is the half angle form of the Spinor R.

2.6.2 Spinors of the $\mathbf{i}-\ space$ in exponential form

$R = \cos\frac{1}{2}|\mathbf{A}| + \hat{\mathbf{A}}\sin\frac{1}{2}|\mathbf{A}|$ can be written as $e^{(1/2)\mathbf{A}}$.

Here $\hat{\mathbf{A}} = \hat{x} \wedge \hat{y}$, the bivector representing the plane of rotation and $|\mathbf{A}|$ gives the magnitude of the angle through which the rotation takes place.

2.6.3 Quaternion form

$$R = \hat{x}\ \hat{y} = \alpha + \beta_1\sigma_2\sigma_3 + \beta_2\sigma_3\sigma_1 + \beta_3\sigma_1\sigma_2$$

$$= \alpha + \beta_1\mathbf{i_1} + \beta_2\mathbf{i_2} + \beta_3\mathbf{i_3} = \alpha + \beta\ \mathbf{i}$$

Where $\alpha = \hat{x}\ .\ \hat{y}$ and $\beta = \beta_1\sigma_2\sigma_3 + \beta_2\sigma_3\sigma_1 + \beta_3\sigma_1\sigma_2 = \hat{x} \wedge \hat{y}$

It is the Quaternion form of a Spinor.

The relations between various parameters are, $\alpha = \cos\frac{|\mathbf{A}|}{2}$, $\vec{\beta} = \hat{\mathbf{A}}\sin\frac{|\mathbf{A}|}{2}$

2.6.4 Euler Angle and axis form

$R = e^{(1/2)\ \mathbf{i}\ \vec{a}}$ is the angle and axis form of the Spinor as $\hat{a}$ is the axis of rotation and $|\vec{a}|$ gives the magnitude of the angle through which the given vector is rotated. This is called Euler parameterization of rotations. The parameters angle and axis are called Euler parameters.

2.6.5 Spinor Matrix form of a rotation

We denote a rotation by $\mathcal{R}$ or Q and rotation through an angle θ by $\mathcal{R}_\theta$ or Q_θ. The use of Spinors to represent a rotation gives the matrix elements directly by the formula $e_{jk} = \sigma_j\ .\ e_k = \sigma_j.(\mathcal{R}\sigma_k)$.

The advantages in using Spinors as a substitute for all the other forms for representing rotations are

(i) Spinors are coordinate free.

(ii) Spinors exists in every dimension, thus make it possible to perform rotations in higher dimensional spaces also.

(iii) Spinors represent the orientation of the rotation but matrices do not.

(iv) It is easy to convert Spinors into the other forms as and when required.

2.7 Sequences of Spinors

Sequence or product of Spinors is also Spinor and hence a rotation. Spinors play an important role in the study of the problems related to Celestial mechanics.

2.7.1 1-2-1 symmetric sequence of rotation

We consider the 1-2-1 symmetric sequence of rotations; the Spinor that represents the required rotation is given as a sequence of three spinors about the base vectors is defined by

$$\mathcal{R} = \mathcal{R}_\phi Q_\theta \mathcal{R}_\psi = R_\phi^{\dagger} Q_\theta^{\dagger} R_\psi^{\dagger} \sigma_k R_\psi Q_\theta R_\phi\ ,$$

Where $R_\psi = e^{(1/2)\,i\sigma_1\psi} = \cos\frac{\psi}{2} + \sigma_2\sigma_3 \sin\frac{\psi}{2}$, $Q_\theta = e^{(1/2)\,i\sigma_2\theta} = \cos\frac{\theta}{2} + \sigma_3\sigma_1 \sin\frac{\theta}{2}$,

$R_\phi = e^{(1/2)\,i\sigma_1\phi} = \cos\frac{\phi}{2} + \sigma_2\sigma_3 \sin\frac{\phi}{2}$

The new set of axes after rotation are given by

$$e_k = \mathcal{R}\sigma_k = R^\dagger \sigma_k R$$
$$= R_\phi^\dagger Q_\theta^\dagger R_\psi^\dagger \sigma_k R_\psi Q_\theta R_\phi$$

This can be converted into the matrix form by calculating the elements of the matrix (e_{jk}) given as

$$e_{jk} = \sigma_j \cdot e_k = \mathcal{R}\sigma_k$$
$$e_1 = \mathcal{R}\sigma_1 = R_\phi^\dagger Q_\theta^\dagger R_\psi^\dagger \sigma_1 R_\psi Q_\theta R_\phi$$
$$= R_\phi^\dagger Q_\theta^\dagger \left(R_\psi^\dagger \sigma_1 R_\psi\right) Q_\theta R_\phi$$
$$= R_\phi^\dagger \left(Q_\theta^\dagger \sigma_1 Q_\theta\right) R_\phi$$
$$= R_\phi^\dagger \left[\sigma_1(\cos\theta + \sigma_3\sigma_1 \sin\theta)\right] R_\phi$$
$$= R_\phi^\dagger \left[\sigma_1 \cos\theta - \sigma_3 \sin\theta\right] R_\phi$$
$$= \left(R_\phi^\dagger \sigma_1 R_\phi\right)\cos\theta - \left(R_\phi^\dagger \sigma_3 R_\phi\right)\sin\theta$$
$$= \sigma_1 \cos\theta - \sigma_3(\cos\phi + \sigma_2\sigma_3 \sin\phi)\sin\theta$$
$$= \sigma_1 \cos\theta + \sigma_2 \sin\theta \sin\phi - \sigma_3 \sin\theta \cos\phi$$

Similarly $e_2 = \mathcal{R}\sigma_2 = R_\phi^\dagger Q_\theta^\dagger R_\psi^\dagger \sigma_2 R_\psi Q_\theta R_\phi$

$$= \sigma_1(\sin\theta \sin\psi) + \sigma_2(\cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi) + \sigma_3(\sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi)$$
$$e_3 = \mathcal{R}\sigma_3 = R_\phi^\dagger Q_\theta^\dagger R_\psi^\dagger \sigma_3 R_\psi Q_\theta R_\phi$$
$$= \sigma_1 \sin\theta\cos\psi + \sigma_2(-\sin\phi\cos\theta\cos\psi - \cos\phi\sin\psi) + \sigma_3(\cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi)$$

$$\begin{vmatrix} \cos\theta & (\sin\theta\sin\psi) & (\sin\theta\cos\psi) \\ \sin\theta\sin\phi & (\cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi) & (-\sin\phi\cos\theta\cos\psi - \cos\phi\sin\psi) \\ -\sin\theta\cos\phi & (\sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi) & (\cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi) \end{vmatrix}$$

III. An application of sequences of spinors

3.1 Tracking problem

Rotation sequences are used to track a remote object such as a spacecraft or an aero plane.

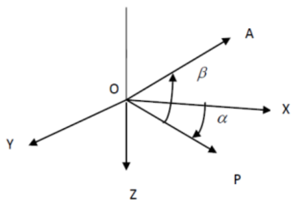

Fig 2 Tracking Problem

In the figure 3, OXYZ is the frame of reference rigidly attached to the Earth. The origin 'O' is a point on Earth from which we are observing the spacecraft.

XY plane is the Tangent plane to the Earth pointing towards North and East directions respectively. Z axis points towards the centre of the Earth (NED frame of reference). 'A' indicates the direction of the spacecraft and 'P' is the projection of A in XY plane. α is the angle between the projection of the position vector of the spacecraft and the X axis.

β is the angle between the projection of the position vector of the spacecraft and the position vector of the spacecraft. α is called the 'Heading angle' and β is called the 'Elevation angle'

To locate the spacecraft we use $\mathcal{R}_\beta \mathcal{R}_\alpha$ sequence of coordinate frame transformations where $\mathcal{R}_\alpha$ rotates the XY plane through an angle α about Z axis in such a way that the X axis coincides with the projection vector OP, and $\mathcal{R}_\beta$ rotates the XZ plane through an angle β about new y axis in such a way that the newest 'x' axis coincides with the direction vector OA. The final direction of the X axis represents the direction of the spacecraft.

$\mathcal{R} = \mathcal{R}_\beta \mathcal{R}_\alpha$

3.2 Spinor Matrix form of the Tracking Transformation

The new set of axes after rotation are given by

$$e_k = \mathcal{R}\sigma_k = \mathcal{R}_\beta \mathcal{R}_\alpha \sigma_k = R_\beta^\dagger R_\alpha^\dagger \sigma_k R_\alpha R_\beta$$

Where $R_\alpha = e^{(1/2)\,\mathrm{i}\sigma_3\alpha} = \cos\dfrac{\alpha}{2} + \sigma_1\sigma_2 \sin\dfrac{\alpha}{2}$, $R_\beta = e^{(1/2)\,\mathrm{i}\sigma_2\beta} = \cos\dfrac{\beta}{2} + \sigma_3\sigma_1 \sin\dfrac{\beta}{2}$

This can be converted into the matrix form by calculating the elements of the matrix (e_{jk}) given as

$$e_{jk} = \sigma_j \cdot e_k = \mathcal{R}\sigma_k$$

Let the final set of coordinate axes be $\{e_k, k = 1,2,3\}$.

$$e_1 = \mathcal{R}\sigma_1 = \mathcal{R}_\beta \mathcal{R}_\alpha \sigma_1$$
$$= \mathcal{R}_\beta\, \sigma_1(\cos\alpha + \sigma_1\sigma_2 \sin\alpha)$$
$$= \mathcal{R}_\beta(\sigma_1 \cos\alpha + \sigma_2 \sin\alpha)$$
$$= \sigma_2 \cos\alpha \quad - \sigma_1(\cos\beta + \sigma_3\sigma_1 \sin\beta)\sin\alpha$$
$$= \sigma_1(\cos\beta + \sigma_3\sigma_1 \sin\beta)\cos\alpha + \sigma_2 \sin\alpha$$
$$= \sigma_1 \cos\alpha\cos\beta + \sigma_2 \sin\alpha - \sigma_3 \cos\alpha \sin\beta$$

Similarly

$$e_2 = \mathcal{R}\sigma_2 = \mathcal{R}_\beta \mathcal{R}_\alpha \sigma_2$$
$$= \quad -\sigma_1 \sin\alpha\cos\beta + \sigma_2\cos\alpha + \sigma_3 \sin\beta \sin\alpha$$

$$e_3 = \mathcal{R}\sigma_3 = \mathcal{R}_\beta \mathcal{R}_\alpha \sigma_3$$
$$= \sigma_3 \cos\beta + \sigma_1 \sin\beta$$

Hence the corresponding matrix for the tracking transformation is

$$\begin{vmatrix} \cos\alpha\cos\beta & -\sin\alpha\cos\beta & \sin\beta \\ \sin\alpha & \cos\alpha & o \\ -\cos\alpha\sin\beta & \sin\alpha\sin\beta & \cos\beta \end{vmatrix}$$

3.3 Euler angles

Rotations transform one coordinate frame XYZ into another coordinate frame xyz preserving the angle between them. Hence it preserves the orthogonality property of the basis vectors. There is another widely used system to represent rotations is the system of Euler angles. Euler stated that every rotation can be expressed as a product of two or three rotations about fixed axes of a standard basis in such a way that no two successive rotations have the same axis of rotation. This theorem is known as 'Euler's theorem'. Thus every rotation can be divided further into two or three rotations about the fixed axes of the standard basis.

3.4 Theorem Every rotation can be expressed as a sequence of Euler angles.

Proof: We shall prove this by establishing the relation between spinor sequence of Euler angles and the angles of any arbitrary spinor sequence that represent the same rotation. As an example for an arbitrary rotation, let us choose the symmetric 1-2-1 sequence of Euler angles obtained above. Equating the matrix representations of both we get the Euler angles in terms of α and β.

$$R_\beta R_\alpha = R_\phi Q_\theta R_\psi$$

$$\begin{vmatrix} \cos\theta & (\sin\theta\sin\psi) & (\sin\theta\cos\psi) \\ \sin\theta\sin\phi & (\cos\phi\cos\psi \quad -\sin\phi\cos\theta\sin\psi) & (\quad -\sin\phi\cos\theta\cos\psi \quad -\cos\phi\sin\psi) \\ -\sin\theta\cos\phi & (\sin\phi\cos\psi+\cos\phi\cos\theta\sin\psi) & (\cos\phi\cos\theta\cos\psi \quad -\sin\phi\sin\psi) \end{vmatrix}$$

$$\begin{vmatrix} \cos\alpha\cos\beta & -\sin\alpha\cos\beta & \sin\beta \\ \sin\alpha & \cos\alpha & o \\ -\cos\alpha\sin\beta & \sin\alpha\sin\beta & \cos\beta \end{vmatrix}$$

$$\cos\theta = \cos\alpha\cos\beta$$

$$\tan\phi = \frac{\tan\alpha}{\sin\beta}$$

$$\tan\psi = -\frac{\sin\alpha}{\tan\beta}$$

These relations establish the existence of Euler angles and these relations can also be obtained using other methods also. We shall prove this by using Spinor half angle method.

3.5 Spinor half angle method

$R_\beta R_\alpha = R_\phi Q_\theta R_\psi \Rightarrow$ Let $p=\frac{\alpha}{2}, q=\frac{\beta}{2}, r=\frac{\psi}{2}, s=\frac{\theta}{2}, t=\frac{\phi}{2}$ to avoid half angles

$$(\cos q+\sigma_2\sigma_3\sin q)\,(\cos p+\sigma_1\sigma_2\sin p) = (\cos t+\sigma_2\sigma_3\sin t)\,(\cos s+\sigma_3\sigma_1\sin s)\,(\cos r+\sigma_2\sigma_3\sin r)$$

$$\therefore \cos q\cos p+\sigma_1\sigma_2\sin p\cos q+\sigma_3\sigma_1\sin q\cos p \quad -\sigma_2\sigma_3\sin q\sin p$$

$$=(\cos t+\sigma_2\sigma_3\sin t)(\cos s\cos r+\sigma_2\sigma_3\cos s\sin r+\sigma_3\sigma_1\sin s\cos r+\sigma_1\sigma_2\sin r\sin s)$$

$$=\cos t\cos s\cos r+\sigma_2\sigma_3\cos t\cos s\sin r+\sigma_3\sigma\cos t_1\sin s\cos r+\sigma_1\sigma_2\cos t\sin r\sin s$$

$$+\sigma_2\sigma_3\sin t\cos s\cos r-\sin t\cos s\sin r+\sigma_2\sigma_1\sin t\sin s\cos r+\sigma_3\sigma_1\sin t\sin r\sin s$$

$$=\cos t\cos s\cos r-\sin t\cos s\sin r+\sigma_2\sigma_3\cos t\cos s\sin r+\sigma_2\sigma_3\sin t\cos s\cos r$$

$$+\sigma_2\sigma_1\sin t\sin s\cos r+\sigma_1\sigma_2\cos t\sin r\sin s+\sigma_3\sigma_1\sin t\sin r\sin s+\sigma_3\sigma_1\cos t_1\sin s\cos r$$

$$=\cos s(\cos t\cos r-\sin t\sin r)+\sigma_2\sigma_3\cos s(\cos t\sin r+\sin t\cos r)$$

$$+\sigma_1\sigma_2\sin s(\cos t\sin r-\sin t\cos r)+\sigma_3\sigma_1\sin s(\sin t\sin r+\cos t\cos r)$$

$$=\cos s\cos(r+t)+\sigma_2\sigma_3\cos s\sin(r+t)+\sigma_3\sigma_1\sin s\cos(r-t)+\sigma_1\sigma_2\sin s\sin(r-t)$$

$$\therefore \cos q\cos p+\sigma_1\sigma_2\sin p\cos q+\sigma_3\sigma_1\sin q\cos p-\sigma_2\sigma_3\sin q\sin p$$

$$=\cos s\cos(r+t)+\sigma_1\sigma_2\sin s\sin(r-t)+\sigma_2\sigma_3\cos s\sin(r+t)+\sigma_3\sigma_1\sin s\cos(r-t)$$

Equating the coefficients of like terms we get

$$\cos q\cos p=\cos s\cos(r+t) \quad \text{.......... (3)}$$

$$\sin p\cos q=\sin s\sin(r-t) \quad \text{.......... (4)}$$

$$\sin q\cos p=\sin s\cos(r-t) \quad \text{..........(5)}$$

$$-\sin q\sin p=\cos s\sin(r+t) \quad \text{..........(6)}$$

Squaring (3) and (6) and adding

$$(\cos q\cos p)^2+(\sin q\sin p)^2=(\cos s\cos(r+t))^2+(\cos s\sin(r+t))^2$$

$$\Rightarrow \cos^2 q\cos^2 p+\sin^2 q\sin^2 p=\cos^2 s$$

$$\Rightarrow \frac{(1+\cos 2q)}{2}\frac{(1+\cos 2p)}{2}+\frac{(1-\cos 2q)}{2}\frac{(1-\cos 2q)}{2}=\frac{(1+\cos 2s)}{2}$$

$$\Rightarrow \frac{1}{4}(1+\cos 2p+\cos 2q+\cos 2q\cos 2p+1-\cos 2q-\cos 2q+\cos 2q\cos 2q)=\frac{(1+\cos 2s)}{2}$$

$$\Rightarrow \frac{1}{4}(2+2\cos 2q\cos 2p)=\frac{(1+\cos 2s)}{2}$$

$$\Rightarrow 1+\cos 2q\cos 2p = 1+\cos 2s$$
$$\Rightarrow \cos 2q\cos 2p = \cos 2s$$
$$\Rightarrow \cos\theta = \cos\alpha\cos\beta$$

Let $r+t=a$ and $r-t=b$

$\Rightarrow a+b=2r=\psi$ and $a-b=2t=\phi$

Dividing (6) by (3) we get $\tan p\tan q = -\tan(r+t) = -\tan a$

Dividing (4) by (5) $\dfrac{\tan p}{\tan q} = \tan(r-t) = \tan b$

$$\tan\psi = \tan(a+b) = \frac{\tan a+\tan b}{1-\tan a\tan b}$$
$$= \frac{-\tan p\tan q+\dfrac{\tan p}{\tan q}}{1-(-\tan p\tan q)\left(\dfrac{\tan p}{\tan q}\right)}$$
$$= \frac{-\tan p\tan^2 q+\tan p}{\left(1+\tan^2 p\right)\tan q}$$
$$= \frac{\tan p\left(1-\tan^2 q\right)}{\left(1+\tan^2 p\right)\tan q}$$
$$= \frac{\sin 2p}{\tan 2q} = \frac{\sin\alpha}{\tan\beta}$$
$$\therefore \tan\psi = \frac{\sin\alpha}{\tan\beta}$$
$$\tan\phi = \tan(p-q) = \frac{\tan p-\tan q}{1+\tan p\tan q}$$
$$= \frac{-\tan p\tan q-\dfrac{\tan p}{\tan q}}{1+(-\tan p\tan q)\left(\dfrac{\tan p}{\tan q}\right)}$$
$$= -\frac{\tan p\tan^2 q+\tan p}{\left(1-\tan^2 p\right)\tan q}$$
$$= -\frac{\tan p\left(1+\tan^2 q\right)}{\left(1-\tan^2 p\right)\tan q}$$
$$= -\frac{\tan 2p}{\sin 2q} = -\frac{\tan\alpha}{\sin\beta}$$
$$\therefore \tan\phi = -\frac{\tan\alpha}{\sin\beta}$$

IV. Discussion

There is a difference in sign of the 'sine' function in the conventional matrix method and the one used by us that is the Spinor method due to the difference in the handedness of the basis. Quaternions form a left handed coordinate system where as Spinors form a right handed coordinate system.

And also the matrix obtained for a frame rotation is different to that of vector rotation. For example $\begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$ is the matrix used to rotate a vector about σ_3 through an angle θ where as $\begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$ is the matrix used to rotate the coordinate frame about σ_3 through an angle θ .

V. Conclusions:

We conclude that Spinor methods can replace the conventional methods and it is better formalism as they can be converted into any other convenient form as per the available data. When compared to the other methods, the number of parameters in the Spinor notation $R = \alpha + \beta\ \mathbf{i}$ reduce further as α and β are not independent of each other.

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