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Bifurcations and Attractors in Synchronization Dynamics of Coupled Duffing-Van der Pol Oscillators

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Abstract: *We demonstrate the synchronization dynamics of coupled Duffing - Van der Pol (DVP) equation. The stability and sufficient criteria for synchronization alongside some pairs of period doubling (route to chaos) are discovered and periodic windows with the range of angular velocities for which the system is non-chaotic are experienced using linear matrix inequality (LMI). Numerical simulation tools such as bifurcation diagrams, Poincare' maps and phase projections are used to exhibit some new complex dynamical behaviors of the systems. We obtained the full synchronization using a forth order Runge kutta routine as well as the software dynamics when the coupling parameter (κ_c) reaches a certain threshold $\kappa_c^{th} \approx 0.2587$. The numerical value of κ_c^{th} obtained from the average energies of the systems is in good agreement with the theoretical analysis.*
Keywords: *Duffing-Van Der Pol Oscillator, Chaos, Sychronization, Bifurcation, Periodic window.*

I. Introduction

Duffing and Van der Pol are paradigm of a deterministic mathematical model for studying nonlinear dynamic and chaos in mechanical systems. Since the discovery of chaos in the Lorenz equation, the study of chaotic phenomena in deterministic nonlinear dynamical systems has been a subject of prominent research [1]. Sequel to this, enormous progress has been made in understanding various types of synchronization, bifurcations, period doubling (route to chaos), e.t.c. Synchronization has been of intense interest date back from the earlier discovery of synchronization of two pendula clock by Huygens in 1673 [2], also the research work of Pecora and Carroll on the synchronization of identical chaotic systems [3,4,5].

Synchronization can be defined as a state in which two or more systems (with dynamics that can either be periodic or chaotic) adjust to each other given rise to a common dynamical behavior. The behavior can be induced either by coupling the systems locally, (that is, the synchronized state is stable, and that once synchronized, it will be difficult to desynchronized the system with small perturbation); or globally (that is, the system will surely synchronized regardless of the starting point of the systems in relation to one another) or by forcing them [6]. Due to the practical applications of synchronization phenomena in the study of physical, biological and technological problems; a lot of research has been carried out both experimentally, numerically and theoretically about synchronization for many systems in the last decade. [7,8,9,10,11]. Among these are complete synchronization [3,12,13]; chaotic synchronization [14]; generalized synchronization [15]; phase synchronization [5,16]; lag synchronization [5]; adaptive synchronization [14,15]; anticipated synchronization [17]; measure synchronization [18,19,20] and quasi-synchronization [21].

Coupled dynamical systems from the view of its practical applications are interesting in that they permit the study of the properties of chaos, synchronization, multi-stability of attractors and can be use to make a system whose dynamics is more complex than that of its constituents. They are usually obtained from simpler, low dimensional systems to produce new and more complex orderliness. This is realized with the view of modeling spatially extended systems, retaining the dominant features of the constituent systems [6]. Intensive studies of coupled systems in a wide range of disciplines such as condensed matter, Optical systems, biological systems, physical systems, neural networks and a lot more has facilitated coupled systems [22].

Synchronized dynamics in coupled or driven nonlinear oscillators have wide technical applications and are fundamentally important in many areas like information control, monitoring of the dynamical systems and control, Chemical reactions, modeling brain and cardiac rhythm activity and earthquake dynamics [4,10,12,14]. Some of the extensively studied phenomena associated with synchronization in coupled or driven nonlinear oscillators are: bifurcation diagram, poincare' map, period doubling, phenomena crises, transient chaos, quasi-periodicity, intermittencies [11,18] , boundary crises [13,17], interior crises [23], Multi-stability attractors and basin crises [10]. In this paper, we examine the synchronization behavior in coupled Duffing-Van Der Pol oscillator (DVP) with the giving potential using bifurcation diagrams, Poincare map, period doubling leading to chaos and Multi-stability attractors . This is because the properties of chaotic systems are basically depend on the coupling strength and mechanism of the system. One of such mechanisms is the coupling procedure between

Duffing and Van der Pol. This oscillator has been studied by many researchers but to obtain some necessary and sufficient conditions for the occurrence of full synchronization. We examined in detail the synchronization behavior of linearly coupled chaotic systems and show that the backward sweep in the bifurcation diagram gives rise to a more chaotic system during the transition to synchronization and make the periodic window narrower.

The remainder of this paper is structured as follows: Section 2 describes the methodology. The results and discussions of the computational experiments are presented in section 3. Finally, the conclusion is given in Section 4.

II. Methodology

The most widely investigated system in coupled or driven nonlinear oscillators that has provided fundamental models of the dynamical problems in physical sciences, medical sciences, biological sciences, engineering, electronics, and many other disciplines are: the Duffing Oscillator, the Van der Pol Oscillator and the coupled Duffing-Van der Pol Oscillator. The classical Duffing-Van der Pol Oscillator (which appears in many physical problems) in dimensionless form is governed by the nonlinear Eq. (1).

$$\ddot{x} - \varsigma(1 - x^2)\dot{x} + \frac{d\Theta(x)}{dx} = \Psi(t) \tag{1}$$

Where $\Psi(t)$ is an external force ($\Psi(t) = \upsilon \cos \Omega$), x represents the displacement, $\varsigma > 0$, f and ω are constant parameters. Physically, ς is regarded as the damping factor which removes the energy conservation to obtain chaos and $\Theta(x)$ is an anharmonic potential function. The dots denote the derivative with respect to time t.

Eq. (1) is generally referred in this case as Duffing-Van der Pol Oscillator, which evolve from the combination of Duffing and Van der Pol Oscillators' equations. Each one has a wide view in dynamical systems and they are employed as models of various physical and engineering problems such as Josephson Junctions, electrical circuit and plasma Oscillators [24]. A lot of researches are focused on the case where $\Theta(x) = ax^2 + bx^4$. In this paper, we consider an extension of this important model corresponds to the case where the potential is of U^6 type defined by equation 2. Because of its universal nonlinear differential equation, it has attracted works in biological and physical problems but only few works has been carried out with this potential.

$$\Theta(x) = \frac{\alpha}{2}x^2 + \frac{\beta}{4}x^4 + \frac{\delta}{6}x^6 + \kappa_c[x(1) - x(2)] \tag{2}$$

Where α, β, and δ are constant parameters and the last term is the coupling term, k is the coupling parameter which determines the strength of the coupling. Through $\kappa_c = 0$ Eq. (2) illustrates the potential for two uncoupled systems exhibiting both regular and chaotic dynamics. Through $\kappa_c \neq 0$ the change in symmetry and the equilibrium make the dynamics richer and interesting. We can consider at least three physically situations based on the sets of these parameters were the potential is single, double and triple well.

When two of such systems (1) interact with each other through a specific coupling, the potential (2) is perturbed.

Ueda and Akamatsu, 1981 [25] gave the first report on the chaotic motion of system (1) with single-well-type restoring force as a model of the negative resistance oscillator. Momeni et al., 2007, [26] used the Van der Pol equation to model the dynamical behavior of the dust grain charge near parametric resonance. The resonance and off-resonance oscillations of an extendend DVP oscillator was analysed by Siewe et al., 2004, [27] using multiple time scale method. The nonintegrability of a family of DVP oscillators was investigated by Lu et al., 2002 [28] using the analytic properties of the solution in the complex time plane. It was also discovered that the DVP oscillator with a double- well potential possesses a rich dynamical behavior with a vast number of state. It also exhibits smale horseshoe chaos when transverse intersections of the homoclinic orbits occur. Another research also simulated the DVP oscillator using analog with a double-hump potential (see ref [28] and the references therein).

2.1 The Model and dynamics of Duffing-Van der Pol Oscillators

In this work we used some numerical tools like: bifurcation diagram, Poncare map and Period doubling route leading to chaos to consider the different routes to chaos in order to have a better insight about the dynamics of the system under investigation. In this case from Eq. (2), Eq. (1) becomes

$$\ddot{x} - \varsigma(1 - x^2)\dot{x} + \alpha x + \beta x^3 + \delta x^5 + \kappa_c[x(1) - x(2)] = \upsilon \cos \Omega t \qquad (3)$$

Where $\varsigma, \alpha, \beta, \upsilon$ and Ω are all constant parameters. Physically, ς is regarded as damping or dissipation factor, β is the strength of nonlinearity and υ and Ω are the amplitude and frequency of the external force respectively. Eq. (3) is essentially equivalent to a combination of DVP oscillators and has wide applications in the modelling of nonlinear oscillation processes. Eq. (3) is Duffing (if $\varsigma = 0$) and Van der Pol (if $\beta = 0$) these have been studied from analytical and numerical investigations with rich structures of the bifurcation set, bifurcation routes chaotic dynamics and phase-locking phenomenon [29].
The equation that govern the coupled system is given as,

$$\ddot{x}(1) = \varsigma(1 - x^2)\dot{x}(1) - \alpha x(1) - \beta x^3(1) - \delta x^5(1) + \upsilon(1)\cos \Omega t - \kappa_c[x(1) - x(2)] \qquad (4)$$

$$\ddot{x}(2) = \varsigma(1 - x^2)\dot{x}(2) - \alpha x(2) - \beta x^3(2) - \delta x^5(2) + \upsilon(2)\cos \Omega t - \kappa_c[x(1) - x(2)] \qquad (5)$$

These equations can be written in autonomous form using the transformation $\dot{x}(1) = y(1)$ and $\dot{x}(2) = y(2)$ by doing this, the second order differential equations (4) and (5) are expressed as systems of first order differential equations with variables $(x(1), y(1))$ and $(x(2), y(2))$ respectively.
In a compact vector form, we can write Eq. (4) and (5) as

$$\dot{X}_1 = A_1(X_1)_1 - B_1(X_2)_1 + C_1(X_3)_1 + D_1(X_4)_1 + \vartheta_1 f(x_1) - q_1 t_1 \qquad (6)$$

and

$$\dot{X}_2 = A_2(X_1)_2 - B_2(X_2)_2 + C_2(X_3)_2 + D_2(X_4)_2 + \vartheta_2 f(x_2) - q_2 t_2 \qquad (7)$$

Where,

$$A_1 = \begin{pmatrix} 0 & 0 \\ \varsigma(1-x^2) & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 \\ 0 & \varsigma(1-x^2) \end{pmatrix}, (X_1)_i = (y(1), y(2))^T \in \Re^2,$$

$$B_1 = \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 0 \\ 0 & -\alpha \end{pmatrix}, (X_2)_i = (x(1), x(2))^T \in \Re^2$$

$$C_1 = \begin{pmatrix} 0 & 0 \\ -\beta & 0 \end{pmatrix}, C_2 = \begin{pmatrix} 0 & 0 \\ 0 & -\beta \end{pmatrix}, (X_3)_i = (x^3(1), x^3(2))^T \in \Re^2,$$

$$D_1 = \begin{pmatrix} 0 & 0 \\ -\delta & 0 \end{pmatrix}, D_2 = \begin{pmatrix} 0 & 0 \\ 0 & -\delta \end{pmatrix}, (X_4)_i = (x^5(1), x^5(2))^T \in \Re^2$$

$$\vartheta_{1,2} = \upsilon_{1,2} \text{ and } f(x_{1,2}) = \cos \Omega t,$$

$$q_{1,2} = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}, t_{1,2} = (x_1, x_2)^T \in \Re^2$$

Where $i = 1$ & 2 , $q = k$ (the coupling parameter) which determines the strength of the coupling. The over dots represent the derivative of x with respect to time.

Basically, chaos synchronization problem can be formulated as follows. Given a chaotic system, which is considered as the master (or driven) system (4) phase synchronization of Duffing-Van der Pol oscillators and another identical system, which is considered as the slave (or response) system (5), the aim is to force the response of the slave system to synchronize the master system in such a way that the dynamical behavior of these two system be identical after a transient time. One particular characteristic in DVP model is that its phase depends on initial conditions. Therefore, if two DVP oscillators run with different initial condition with two phase trajectories $\varphi(1)$ and $\varphi(2)$; The objective of the synchronization in this paper is to phase-lock the oscillators (phase synchronization) so that $\varphi(2) - \varphi(1) = 0$. That is coupling the slave to the master system in such a way that

$$\lim_{t \to \infty} \| \varphi(2) - \varphi(1) \| = 0 \qquad (8)$$

The justification of the condition in equation (8) implies full synchronization between $\varphi(1)$ and $\varphi(2)$. Empirically, the limit does not always approach zero asymptotically but a constant value $\in$ according to the inequality, $\lim_{t \to \infty} \| \varphi(2) - \varphi(1) \| < 0$, suggesting imperfect full synchronization which according to Vincent et al., 2015 [21] in most cases arises from parameter mismatches between the two coupled systems.
To simplify the analysis, all the parameters except the coupling parameter (κ_c) are kept constant.
Taking the system parameters for $\varsigma = 0.40$, $\kappa = 0.35$, $\beta = -0.65$, $\delta = 0.19$, $f(1) = f(2) = 1.20$, $\Delta T = 0.001$ and $\alpha = 0.36$. The dynamical response of Eq. (4) and (5) as κ_c is progressively increases is described in the figures below.

III. Results And Discussions

To achieve a long-term behavior or stability, the first one hundred values were discarded and the results obtained are presented as follows. The bifurcation diagram of the coupled system (1) in $(\omega - X)$ plane is given in Fig. 1 and 2.

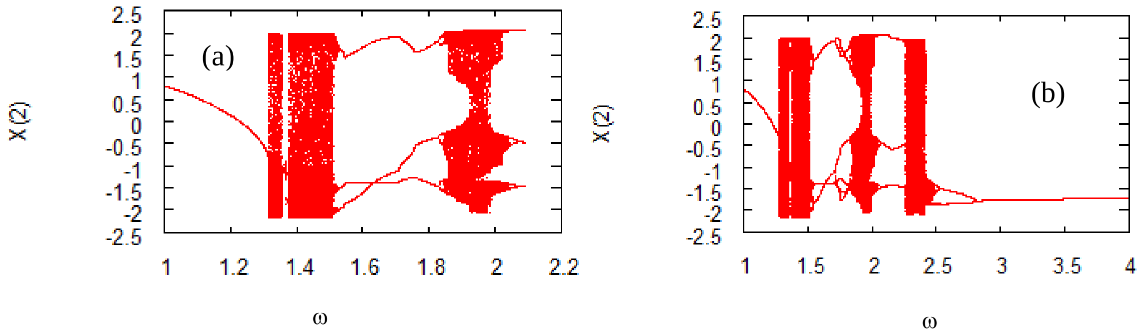


Fig. 1 Bifurcation diagram of system (1) in $(\omega - X)$ plane for $\varsigma = 0.40$, $\kappa = 0.35$, $\beta = -0.65$, $\delta = 0.19$, $f(1) = f(2) = 1.20$, $\Delta T = 0.001$ and $\alpha = 0.36$ when (a) $\kappa_c = 0$ (b) $\kappa_c = 0.25$

Fig.1(a) is the bifurcation diagram when the coupling parameter $(\kappa_c) = 0$ (i.e the system is uncoupled) and the set of control parameter which contain periodic and chaotic dynamics is $1 < \omega < 2.3$. As ω is increased from 1.3rad/s, the dynamic of system (1) gets into chaos from period-1 orbit through intermittent bifurcation. In the bifurcation diagram, the map iterated several hundred times at each of many intervening values of control parameter ω. However, there is instability in this bifurcation. The control parameter ω varies smoothly from 1.310 to 2.090rad/s and the angular displacement is between -2.0 to 2.0m a well pronounced chaotic situation is experienced. A pair of period-doubling (route to chaos) begins at 1.550rad/s. Periodic windows experienced from 1.00 to 1.33rad/s (period-1), 1.520 to 1.850rad/s (period-3), and 2.070 to 2.10rad/s (period-3) show the range of angular velocities for which the system is non-chaotic. The choice of this range of angular velocity is favourable when a normal, predictable and non-chaotic behavior of the system is desired. It is advisable that the control parameter ω range between 1.330 to 1.520rad/s and 1.850 to 2.070rad/s angular velocities should be avoided when working with this system because of its unpredictable chaotic manner. Fig. 1(b) is the bifurcation diagram when $(\kappa_c) = 0.25$. $1 < \omega < 4.0$ gives the set of control parameter which contain chaotic dynamics. The control parameter ω varies smoothly from 1.310 to 2.750rad/s, and the angular displacement is between -2.0 to 2.0. A well pronounced chaotic situation was also experienced. A pair of period-doubling (route to chaos) begins at 1.850rad/s. Periodic windows experienced from 1.00 to 1.33, 1.520 to 1.850, 2.070 to 2.30 and 2.500 to 4.000 show the range of angular velocities for which the system is non-chaotic. The choice of this range of angular velocity is favourable when a normal, predictable and non-chaotic behavior of the system is desired; it is advisable that the control parameter ω range between 1.330 to 1.520rad/s, 1.850 to 2.009rad/s and 2.300 to 2.500rad/s. The angular velocities should be avoided when working with this system because of its unpredictable chaotic manner.

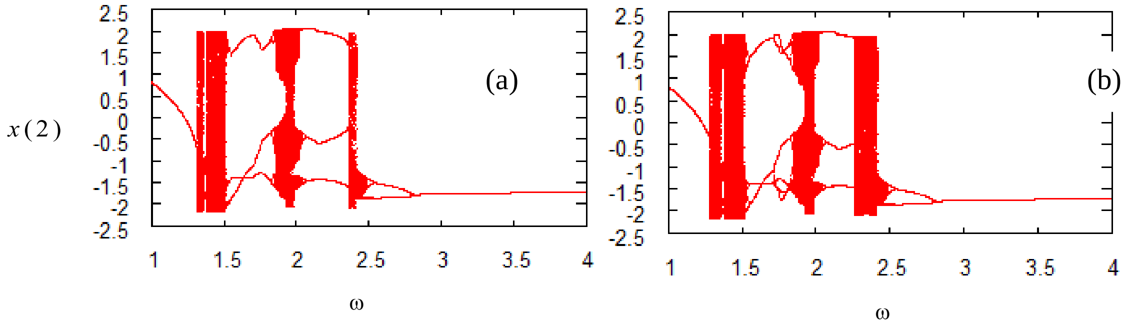


Fig. 2 Bifurcation diagram of system (1) in $(\omega - X)$ plane for $\varsigma = 0.40$, $\kappa = 0.35$, $\beta = -0.65$, $\delta = 0.19$, $f(1) = f(2) = 1.20$, $\Delta T = 0.001$ and $\alpha = 0.36$ when $\kappa_c = 0.35$ (a) Forward sweep (b) backward sweep.

Fig. 2 is the bifurcation diagram when the coupling parameter ($\kappa_c = 0.35$). Skipping some analogous analysis, it is obvious from Fig. 2(a), that there are reductions in the chaotic range for forward sweep from period-1via period-6 to period-1. The control parameter which contains chaotic dynamics for the forward sweep is $1 < \omega < 4.0$. Also, the backward sweep produces more period doubling cascade from period-6 attractors as shown in Fig. 2 (a) to period-12 attractors as shown in Fig. 2 (b), hence the system becomes more chaotic at backward sweep.

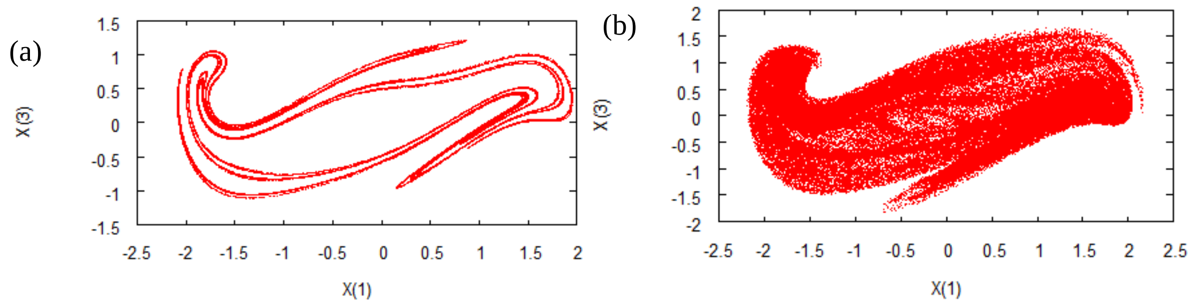
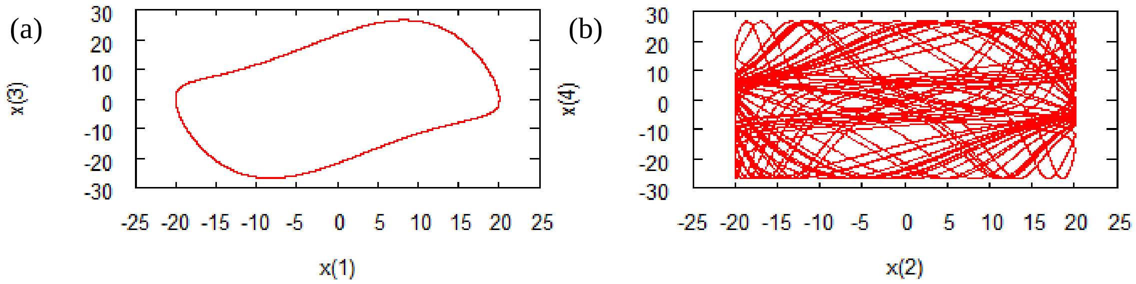


Fig.3 Poincaré section of some chaotic states for $\kappa = 3.05$, $\delta = 1.5$, $\dot{\omega} = 1$, $F = 22$ (a) $\kappa_c = 0$ (uncoupled system) (b) $\kappa_c > 0$ (coupled system)

In order to gain better insight into chaos features and dynamics; various kinds of strange attractors are also shown in the Poincare' map (as shown in Fig. 3). Transition to synchronization gives the structural changes associated with the transition to stable synchronous behavior. To illustrate this, we consider the strange attractor exhibited by the drive Duffing-Van der Pol oscillator. When the oscillators become synchronized, the attractor for the response system would be precisely superimposed, point-to-point with that of the slave attractor.

To understand the structural changes that took place in the system, we obtained the Poincare' section of the phase portraits within and somewhere outside the synchronization region for the response system. Fig. 4(a & c) show some phase plane plots of an imbedded period doubling cascade when $\kappa_c = 0$ and > 0 respectively. Fig. 4 (b & d) are used to demonstrate the phenomenon of crises.



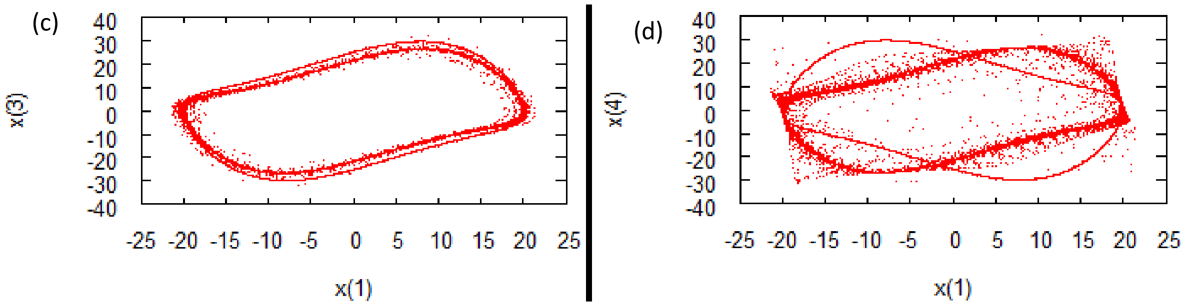


Fig.4 Attractors of some chaotic states for $\kappa = 3.05$, $\delta = 1.5$, $\dot{\omega} = 1$ and $F = 22$ at (a), (b) $\kappa_c = 0$ (uncoupled) and (c), (d) $\kappa_c = 0.2$ (coupled).

Basically, when the oscillators are synchronized, any microscopic property of the systems are synchronized and equal or nearly equal. One of such microscopic quantity that we consider is the average bare energies $(\zeta_{1,2})$ written as

$$\zeta_{1,2} = \frac{1}{2}\int_0^T E_{1,2}(t)\,dt \tag{9}$$

Where, $$E_{1,2}(t) = \frac{P_{1,2}^2}{2} + \Omega(x_{1,2}),\ p_{1,2} \tag{10}$$

$E_{1,2}(t)$ is the associated momentum and $\Omega(x_{1,2})$ is the potential . The measurement is carried out after discarding a sufficient initial transient when allowed to run for a sufficient time. We calculate $\zeta_{1,2}$ as functions of the coupling strength κ as shown in Fig.6. When κ_c reaches a certain threshold $\kappa_c^{th} \approx 0.2578$ full synchronization is realized. Above the κ_c^{th}, the synchronization is stable, which indicates a strong correlation between the oscillators that are asymptotically approach identical trajectories.

To obtain the quality of synchronization, we adopt the method defined by Vincent et al., 2007 [30], by examining the behavior of the average error E_{av} given by

$$E_{av} = \frac{1}{T}\int_0^T E(t)\,dt \tag{11}$$

where

$$E(t) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

With this result, the error dynamics asymptotically becomes zero when the coupled oscillators are synchronized. But when the system is uncoupled (i.e $\kappa_c = 0$) no noticable synchronization in the system (as shown in Fig. 5). Vincent et al., 2007 reported the case of irregular pattern of dynamic with average error against time for the uncoupled case. In this paper, we look at the case when the coupling parameter is nonzero to achieve a complete synchronization as shown in Fig.7.

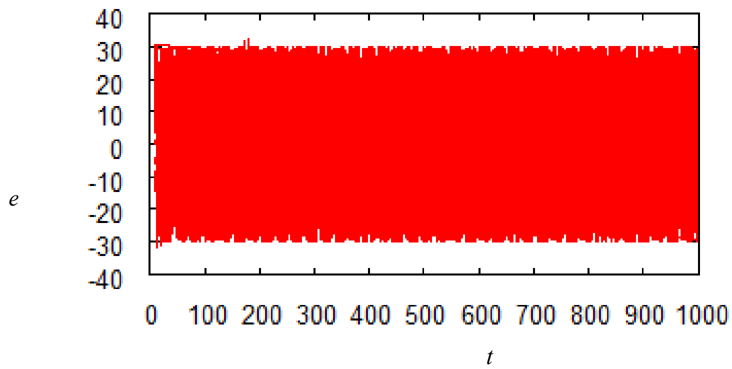


Fig.5 Time series of the error state of system (4) and (5) for $\kappa = 3.05$, $\delta = 1.5$, $\dot{\omega} = 1$, $F = 22$ when the coupling parameter is 0 (no synchronization).

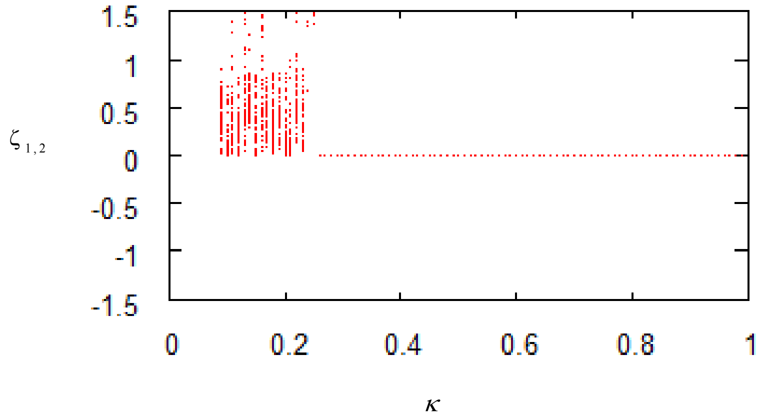


Fig. 6 The average bare energies, $\zeta_{1,2}$ V_s κ for the coupled system

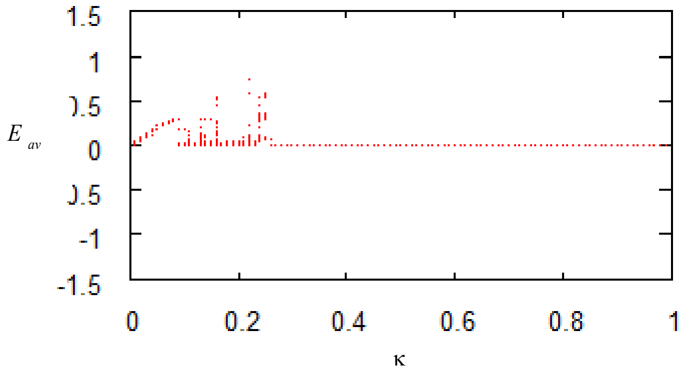


Fig. 7 The average synchronization error E_{av} Vs κ

IV. Conclusion

We have examined the synchronization behavior of a coupled master-slave system using the regular and chaotic motion of the Duffing-Van der Pol oscillator with some useful numerical simulation tools like bifurcation diagram, Poincare' map and phase projections. Some pairs of period-doubling route leading to chaos were discovered and periodic windows were experienced at the same time the range of angular velocities for which the system is non-chaotic was observed in figure 1 and 2.

We also observed the backward sweep in Fig. 2, this produces more period doubling cascade from period-6 to period-12 attractors as shown in figure 2 (b), hence the system becomes more chaotic at backward sweep. Some interesting dynamical phenomena such as Poincare' map and crises phenomena were also used to observe the state of chaos in the system. It is advisable that the control parameter Ω for the chaotic range should be avoided when working with this system because of its unpredictable chaotic manner. A numerical solution were obtained using a forth order Ruge-Kutta routine as well as the software dynamics to obtained full synchronization when the coupling parameter k reaches a certain threshold $k_{th} \approx 0.2587$.

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