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Finite Size Uehling Corrections in Energy Levels of Hydrogen and Muonic Hydrogen Atom

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Abstract: *Corrections in energy levels of hydrogen and muonic hydrogen atom are calculated using Uehling potential with point and finite size proton. The finite size protonis used by introducing the charge density of the proton. The derivative expansion theory is used to obtain approximate finite size potentials by taking two forms of the proton charge densities (Gaussian and the exponential). The three potentials (point charge and approximated Gaussian and exponential potentials) give approximately the same results. These calculations are performed with Schrödinger and Dirac coulomb wave functions using perturbation theory. For point proton there is a very small difference (in the second decimal) in the Lamb shiftbetween the results calculated bySchrödinger wave functions and those with Diracwave functions. The finite size of proton gives values of Lamb shifthigher than that of point charge. The fine structure correction is very small compared to the Lamb shift values.*

Key words*: Uehling potential – Point proton – Finite size proton – Hydrogen atom – Muonic hydrogen atom – Energy levels corrections – Lamb shift.*

I. Introduction

The electronic vacuum polarization effects and in particular the Uehling potential plays an important role in the calculations of the energy levels and wave functions in muonic atoms. It is responsible for thedominantquantum electrodynamics (QED) effects in atoms with heavy orbiting particle (such as muon [1]). The Uehling potential is able to calculate relativistic corrections for a variety of levels in atoms [2]. One of the important effects is the finite nuclear size. This effect depends on nuclear charge Ze and principle and orbital quantum numbers, n and l, respectively. The low l states and mostly, the $1s$ and $2s$ states are sensitive to the finite nuclear size effects. They have been used to determine the charge radius of nuclei starting fromhydrogen [3] to Uranium [4]. In this paper we calculated the Uehling corrections in the energy levels ($1s$, $2s$, $3s$,$4s$, $2p$, $3p$ and $3d$) of hydrogen and muonic hydrogen atom for the pointand finite size proton using Schrodinger wave functions. The Lamb shift($\Delta E_{2p} - \Delta E_{2s}$)is calculated in case of nonrelativistic [5] and relativistic wave functions [6] usingboth point and finite size proton by applying perturbation theory.

II. One photon exchange Uehling Potential

A simple example of the effective Lagrangian formalism and the validity of the derivative expansion (DE) we consider the vacuum polarization process in QED. Abundant evidence exists which supports the idea that QED is the fundamental theory of electromagnetic interactions below 100 GeV. As well, it is usually considered to be the most well understood physical field theory. The simplest form is that of a theory of spin-1/2 charged fermions with fieldψ and mass$m$, and charge e, withinteractions mediated by the spin-1 massless gauge field for photons, A_μ. The QED Lagrangian in the Feynman gauge is

$$\mathcal{L}_{QED} = \bar{\psi}\left[\gamma_\mu (i\partial^\mu - eA^\mu) - M\right]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\left(\partial_\mu A^\mu\right)^2 + \delta\mathcal{L} \qquad (1)$$

$$F_{\mu\nu} = \partial_\mu A_\mu - \partial_\nu A_\nu$$

One treatsthis case perturbatively about the free particle solution. For this setting, we will clearly be able to see the effect of the shape of the source density in a calculation that has the same flavor as theDE approximation. The analysis is simplified by treating the interaction as a perturbation in the coupling and comparing quantities only to$\mathcal{O}(\alpha)$,where $\alpha = e^2/4\pi$ is the usual fine structure constant. This is accomplished by considering the modification of the free photon propagator by the $\mathcal{O}(\alpha)$ vacuum polarization insertion. In momentum-space, the propagator $iD_{\alpha\beta}(q)$ modified by

$$iD_{\alpha\beta}(q) = iD_{0\alpha\beta}(q) + iD_{0\alpha\mu}(q)i\Pi^{\mu\nu}(q)iD_{0\nu\beta}(q) \qquad (2)$$

Note that from gauge invariance$q^\mu \Pi_{\mu\nu}(q) = 0$, which dictates the Lorentz invariant form

$$\Pi^{\mu\nu} = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\Pi(q^2) \qquad (3)$$

So that

$$iD_{\alpha\beta}(q) = -\frac{ig_{\alpha\beta}}{q^2 + i\epsilon} - \frac{ig_{\alpha\beta}}{(q^2 + i\epsilon)^2}\Pi(q^2) \quad (4)$$

From the Feynman rules of QED with the usual charge renormalization, the propagator polarization insertion is found to be [7]

$$\Pi(q^2) = \frac{2\alpha}{\pi} q^2 \int_0^1 dz\, z(1-z) \ln\left(1 - z(1-z)\frac{q^2}{M^2}\right) \quad (5)$$

This integral can be evaluated, and for the case of a stationary source, the momentum is space-like, $q^2 = -\vec{q}^2$, and

$$\Pi(q^2) = -\frac{\alpha\vec{q}^2}{3\pi}\left(-\frac{5}{3} + \frac{4M^2}{\vec{q}^2} + \left(1 - \frac{2M^2}{\vec{q}^2}\right)\sqrt{1 + \frac{4M^2}{\vec{q}^2}} \ln\left(\frac{\sqrt{1 + \frac{4M^2}{\vec{q}^2}} + 1}{\sqrt{1 + \frac{4M^2}{\vec{q}^2}} - 1}\right)\right) \quad (6)$$

To obtain an expression for the potential we fold the background spherically symmetric charge density source, $\rho_{ch}(r)$, over the new part of the propagator. For a time independent source

$$V_{vac}^E = \int_0^\infty \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \frac{\Pi_R(-q^2)}{|q|^4} \rho_{ch}(q) \quad (7)$$

The angular part is integrated, leaving

$$V_{vac}^E = \frac{2\alpha}{\pi} \int_0^\infty dq \frac{\sin(qr)}{qr} \frac{\Pi_R(-q^2)}{q^2} \rho_{ch}(q) \quad (8)$$

Where

$$\rho_{ch}(q) = \int_0^\infty d^3x e^{i\vec{q}\cdot\vec{x}} \rho_{ch}(x) = 4\pi \int_0^\infty dr r^2 \frac{\sin(qr)}{qr} \rho_{ch}(r) \quad (9)$$

This expression gives the exact effect of the vacuum polarization in theDE theory (to$\mathcal{O}(\alpha^2)$). Since the DE coefficients has the form

$$Z_1 = -\frac{\partial\Pi_s(q^2)}{\partial(q^2)}\bigg|_{q^2=0} \quad Z_2 = -\frac{1}{2}\frac{\partial^2\Pi_s(q^2)}{\partial(q^2)^2}\bigg|_{q^2=0} \quad (10)$$

And

$$\Pi_s(q^2) \cong -\frac{\alpha}{15\pi M^2} q^4 \quad (11)$$

One can evaluate these coefficientsusing equation (10) and (11), giving

$$Z_1 = 0, \qquad Z_2 = \frac{\alpha}{15\pi M^2} \quad (12)$$

The result $Z_1 = 0$ is a manifestation of charge conversation, which implies that corrections to the charge density are total derivatives that vanish under a spatial integration. This allows us to write an effective Lagrangian for low energy photons that takes into account the vacuum polarization loop in an additional derivative term. The full effective one-loop Lagrangian will contain contributions for the photon-electron vertex correction that are of the order α^2. It is reffered to as the Euler and Heisenberg Effective Lagrangian [8]. Here we are interested in the order α part (only the vacuum polarization)

$$\mathcal{L}_{eff} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\alpha}{30\pi M^2} \partial_\mu F^{\mu\lambda} \partial^\nu F_{\nu\lambda} - j^\mu A_\mu \quad (13)$$

The fermion part of the Lagrangian is dropped here, and an external source current j^μ is included. The gauge fixing term can be droppedbecause we will restrict ourselves to the time like part of the vector potential in the case where it is independent of time. The suitable form for the Euler-Lagrange equation is

$$\frac{\partial\mathcal{L}}{\partial A_\mu} - \partial_\lambda\left(\frac{\partial\mathcal{L}}{\partial(\partial_\lambda A_\mu)}\right) + \partial^2\left(\frac{\partial\mathcal{L}}{\partial(\partial^2 A_\mu)}\right) = 0 \quad (14)$$

Considering the time-like part of the potential $A_0$and a source current$j^\mu = \delta^{\mu 0} j_0$, we obtain a modified form of Maxwell's equation

$$\partial^2 A_0 = j_0 + \frac{\alpha}{15\pi M^2} \partial^4 A_0, \quad (15)$$

Which for a time independent potential becomes

$$\nabla^2 A_0 = -j_0 - \frac{\alpha}{15\pi M^2} \nabla^4 A_0 \quad (16)$$

Making use of the identity $\nabla^2(1/|x|) = -4\pi\delta(\vec{x})$ this may be written as an integrodifferential equation:

$$A_0(\vec{x}) = \frac{1}{4\pi}\int d^3x' \frac{1}{|\vec{x}' - \vec{x}|}\left(j_0(\vec{x}') + \frac{\alpha}{15\pi M^2}\nabla^4 A_0(\vec{x}')\right)$$
$$= \frac{1}{4\pi}\int d^3x' \frac{1}{|\vec{x}' - \vec{x}|} j_0(\vec{x}') + \frac{\alpha}{60\pi M^2}\int d^3x' \left(\nabla^2 \frac{1}{|\vec{x}' - \vec{x}|_0}\right)\nabla^2 A_0(\vec{x}')$$
$$= \frac{1}{4\pi}\int d^3x' \frac{1}{|\vec{x}' - \vec{x}|} j_0(\vec{x}') - \frac{\alpha}{15\pi M^2}\nabla^2 A_0(\vec{x}) \quad (17)$$

As the electromagnetic coupling α is small, we can solve this equation iteratively by substituting for the RHS A_0 with the LHSA_0 in an iterative manner. For example, with a point-like source withcharge $-Ze$ we have $j_0(\vec{x}) = -Ze\delta^{(3)}(\vec{x})$, so

$$A_0(\vec{x}) = -\frac{Ze^2}{4\pi|\vec{x}|} - \frac{\alpha}{15\pi M^2}\nabla^2 A_0(\vec{x})$$
$$= -\frac{Z\alpha}{|\vec{x}|} - \alpha Z\alpha \frac{4\delta^{(3)}(\vec{x})}{15M^2} \quad (18)$$

This is the familiar term, which contributes to the Lamb shift in hydrogen [9]. To understand how useful the effective Lagrangian is, here we consider the spherically symmetric charge density, $j_0 = \rho_{ch}(r)$. Solving (17) iteratively, we have for the vacuum polarization contribution to the potential

$$V_{vac}^{D} = \frac{4\alpha^2}{15M^2}\rho_{ch}(r) \quad (19)$$

We made a comparison between the results obtained by using two densities. One in the Gaussianform, equation (21), while the second is in the exponential form, equation (22), and those obtained using Uehlingpotential $U_o(r)$, [10], where

$$U_o(r) = -\frac{Z\alpha^2}{3\pi r}\int_1^\infty dt \frac{(2t^2+1)}{t^4}\sqrt{(t^2-1)}e^{-2mtr} \quad (20)$$

$$\rho_{ch}(r)|_G = \frac{1}{\pi^{3/2}a^3}e^{-(r/a)^2} \quad ; \quad a = \sqrt{\frac{2}{3}\langle r_p^2\rangle} \quad (21)$$

$$\rho_{ch}(r)|_E = \frac{\eta^3}{8\pi}e^{-\eta r} \quad ; \quad \eta = \sqrt{12/\langle r_p^2\rangle} \quad (22)$$

$\langle r_p^2\rangle$is the mean square radius of the proton. The parameters a and η control the shape of the potential.

III. Energy Levels Corrections

To obtain the energy levels correction we apply perturbation theory

$$\Delta E_{nlj}^{VP} = \int |R_{nl}(r)|^2 \Delta A_o^{VP}(r) r^2 dr \quad (23)$$

Where $R_{nl}(r)$ is the radial unperturbed Coulomb wave functions of the orbiting particle, electron in hydrogen and muon in muonic hydrogen atom [11].

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\theta,\phi) \quad (24)$$

Where

$$R_{nl}(r) = (2k)^{3/2}A_{nl}\rho^l e^{-\rho/2}F_{nl}(\rho) \quad (25)$$

$$\rho = 2kr$$

$$k = \frac{Z}{a_o n}$$

$$A_{nl} = \sqrt{\frac{(n-l-1)!}{2n\big((n+1)!\big)^3}}$$

$$a_o = \frac{1}{\mu\alpha}$$

$\mu$is the reduced mass of electron in case of hydrogen atom and reduced mass of muon in case of muonic hydrogen. $n, l$are the principle and orbital quantum numbers respectively.And $Y_{lm}(\theta,\phi)$ are the spherical harmonics.For relativistic calculations, we take the wave function in the form [6]

$$\left.\begin{matrix} g(r) \\ f(r) \end{matrix}\right\} = \frac{\pm(2\lambda)^{3/2}}{\Gamma(2\gamma+1)} \sqrt{\frac{(m_o c^2 \pm E)\Gamma(2\gamma+n'+1)}{4m_o c^2 \frac{(n'+\gamma)m_o c^2}{E}\left(\frac{(n'+\gamma)m_o c^2}{E}-k\right)n'!}} (2\lambda r)^{\gamma-1} e^{-\lambda r} \times$$

$$\times \left\{ \left(\frac{(n'+\gamma)m_o c^2}{E} - k \right) F(-n', 2\gamma+1; 2\lambda r) \mp n' F(1-n', 2\gamma+1; 2\lambda r) \right\} \qquad (26)$$

Which explicitly implies $\int_0^\infty (f^2+g^2) r^2 dr = 1$, and m_o is the reduced mass of the corresponding particle. And,

$$k = \begin{cases} -(l+1) = -\left(j+\frac{1}{2}\right) & for j = l+\frac{1}{2} \\ l = \left(j+\frac{1}{2}\right) & for j = l-\frac{1}{2} \end{cases}$$

$$\gamma = \pm\sqrt{k^2 - (Z\alpha)^2}$$

$$E = m_o c^2 \left[1 + \frac{(Z\alpha)^2}{\left[n - j - \frac{1}{2} + \left[\left(j+\frac{1}{2}\right)^2 - (Z\alpha)^2 \right]^{1/2} \right]^2} \right]^{-1/2}$$

$$\lambda = \frac{(m_o^2 c^4 - E^2)^{1/2}}{\hbar c}$$

$$n' = n - j - \frac{1}{2} \quad n = 1,2,3, \ldots$$

$$F(a,c;x) = 1 + \frac{a}{c}x + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \cdots$$

IV. Results and Discussion

In these calculations we use the relativistic units $\hbar = c = 1$ and the electron mass $m = 0.5109989\ MeV$ and the muon mass $m_\mu = 105.658357\ MeV$ and $\alpha = 1/137.0359998$ is the fine structure constant and$< r_p^2 >^{1/2} = 0.9295\ (fm)$ for Gaussian potential and $< r_p^2 >^{1/2} = 0.9553\ (fm)$ for exponential potential.Figure 1 shows the spherically symmetric charge distribution of the proton, equations (21) and (22). Figure 2 is the point proton Uehling potential, equation (20). It is clear that this potential is a short range potential. Figures 3-a and 3-b show the comparison between the exact potential and its approximate shape. Figure 3-c represents the comparison between the two approximate potentials

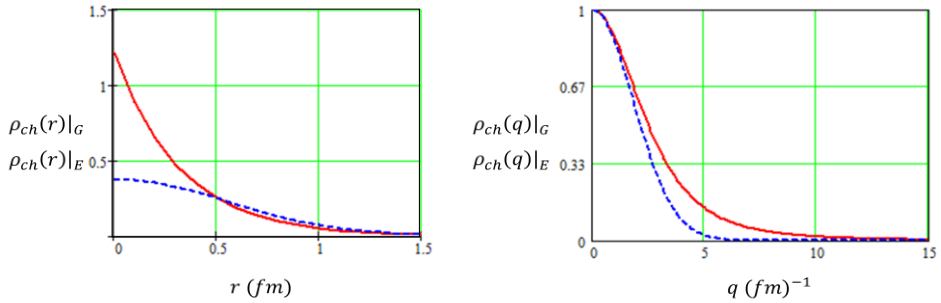

Fig. 1.A Comparison between the exponential and Gaussian proton charge densities in configuration, r, and momentum, q, space.

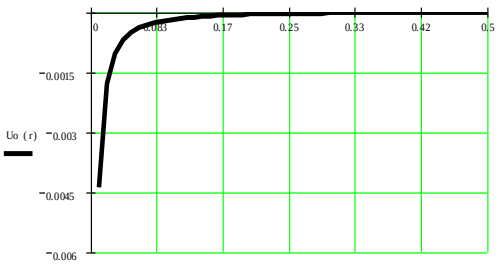

Fig.2. Is the electronic Uehling potential for thepoint charge proton, $U_o(r)$.

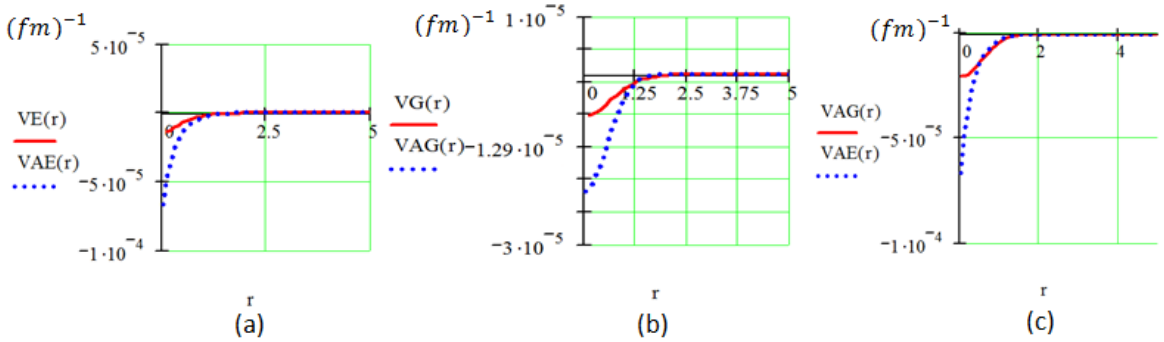

Fig. 3.a. Comparison between exact exponential potential, VE,and its approximatedexponentialpotential, VAE.
b. Comparison between exact Gaussian potential, VG,and approximated Gaussian potential, VAG.
c. Comparison between approximated Gaussian, VAG, and approximated exponential potential, VAE.

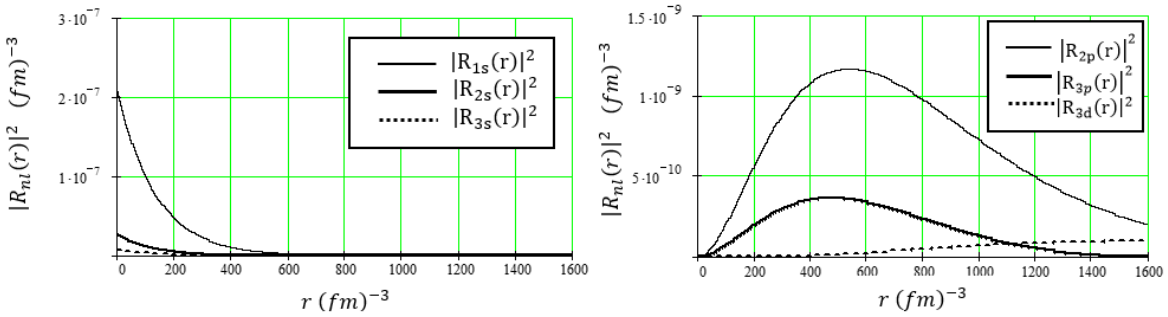

Fig. 4. Shows the distributions of the electron states (1s, 2s, 3s, 2p, 3p, 3d) in case of Schrödinger's wave functions.

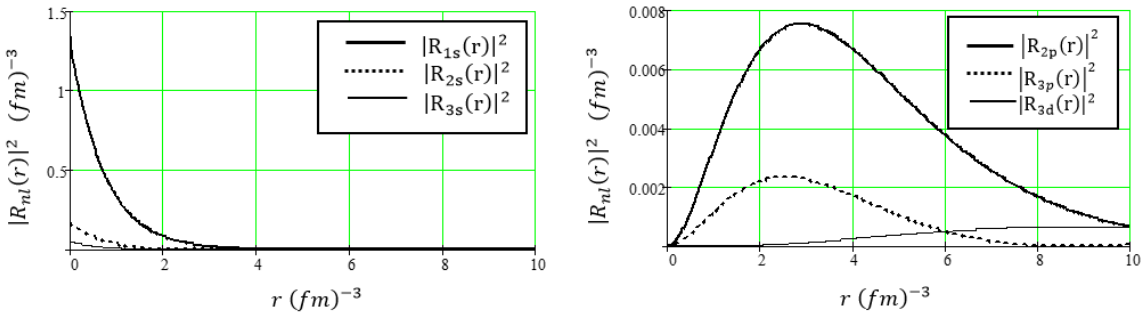

Fig. 5.Shows the distributions of the muonic hydrogen states (1s, 2s, 3s, 2p, 3p, 3d) in caseof Schrödinger's wave functions.

Figure 4. Shows the distributions of the electron states (1s, 2s, 3s, 2p, 3p, 3d) in case of Schrödinger's wave functions.The corresponding distributions for muonic hydrogen are shown in figure 5 for the same states.In these distributions, we take the reduced mass of muon in place of its mass.Looking at figures 4 and 5, the shapeare the same except that, in case of muon the distributions are more closely to the proton center, and the overlap with the potential is more than that of the electron,which explains the higher values of the vacuum polarization corrections in the energy levels in case of muon than in case of electron. See table 1 and table 2.To study the relativistic effect we take the wave functions from equation (26). Tables 1 and 2 show that the corrections decreases with the increase in the principle quantum number of the state. These corrections in case of point charge potential are approximately the same as those calculated by the two approximated (Gaussian and exponential) potentials for the s-states.The corrections in energy levels calculated with approximated Gaussian potential and exponential one agree to the second decimal for s-states as shown in table 1.The corrections as a hole for muonic atom are much higher than the corresponding corrections for hydrogen atom as shown in tables 1 and 2. This comes as a result of the more overlap of muon states with proton than that of electron states.In case of muonic hydrogen atom, the corrections agree to the first decimal in case of approximated Gaussian and exponential potentials in the s-states. For the 2p state corrections of muonicatom the results are nearly the same.

Which explains the approximately same values of the Lamb shift. Table 3 shows a comparison for the Lamb shift in case of muonic hydrogen ($\Delta E_{2p} - \Delta E_{2s}$). In general, the value of this Lamb shift is between 205-206 meV. The relativistic values of this shift are higher than the non-relativistic in the second decimal. The Lamb shift calculated with the approximated Gaussian potential has the highest values compared to the values calculated by Uehling potential and the approximated exponential potential. The fine structure correction values are very small compared to the values of the Lamb shift, the higher value is obtained in case of approximated Gaussian potential while the lowest value is in case of the point charge Uehling potential.

Table1.Vacuum polarization corrections for energy levels of thehydrogen atom calculated with Schrödingerwave functionsin (eV)

Hydrogen atom			
State	Uehling potential (Point Charge)	Approximated Gaussian potential (AGP)	Approximated exponential potential (AEP)
1s	$-8.8959033 \times 10^{-7}$	$-8.9169474 \times 10^{-7}$	$-8.9117327 \times 10^{-7}$
2s	$-1.1119785 \times 10^{-7}$	$-1.1146149 \times 10^{-7}$	$-1.1139618 \times 10^{-7}$
3s	$-3.2947459 \times 10^{-8}$	$-3.3025607 \times 10^{-8}$	$-3.3006251 \times 10^{-8}$
4s	$-1.3899702 \times 10^{-8}$	$-1.3932675 \times 10^{-8}$	$-1.3924508 \times 10^{-8}$
2p	$-3.1660475 \times 10^{-13}$	$-1.1812519 \times 10^{-13}$	$-1.5856255 \times 10^{-13}$
3p	$-1.1118033 \times 10^{-13}$	$-4.1481523 \times 10^{-14}$	$-5.5681707 \times 10^{-14}$
3d	$-9.7389533 \times 10^{-20}$	$-4.8782316 \times 10^{-21}$	$-1.3199193 \times 10^{-19}$

Table2. Vacuum polarization corrections for energy levels of themuonic atom calculated with Schrödingerwave functionsin (eV)

Muonic atom			
State	Uehling Potential (Point Charge)	Approximated Gaussian Potential (AGP)	Approximated Exponential Potential (AEP)
1s	-1.8988523	-1.9214441	-1.91659
2s	-2.195864×10^{-1}	-2.188657×10^{-1}	$-2.1905266 \times 10^{-1}$
3s	$-6.4277311 \times 10^{-2}$	$-6.3776909 \times 10^{-2}$	$-6.3915769 \times 10^{-2}$
4s	$-2.7005198 \times 10^{-2}$	$-2.6750391 \times 10^{-2}$	$-2.6822146 \times 10^{-2}$
2p	$-1.4576756 \times 10^{-2}$	$-1.2058891 \times 10^{-2}$	$-1.2997757 \times 10^{-2}$
3p	$-4.7694767 \times 10^{-3}$	$-4.1543179 \times 10^{-3}$	$-4.4221641 \times 10^{-3}$
3d	$-1.1998811 \times 10^{-4}$	$-1.9855728 \times 10^{-5}$	$-3.8733148 \times 10^{-5}$

Table3.Lamb shift for different potentials ($\Delta E_{2p} - \Delta E_{2s}$) in meV in mionic atom

	state	Point Charge	(AGP)	(AEP)
Schrodinger functions	$\Delta E_{2p} - \Delta E_{2s}$	205.009644	206.806809	206.054903
Dirac functions	$\Delta E_{2p_{(1/2)}} - \Delta E_{2s}$	205.03049	206.82023	206.07044
	$\Delta E_{2p_{(3/2)}} - \Delta E_{2s}$	205.03551	206.8257	206.07579
Fine structure contribution (L-S coupling) $\left(\Delta E_{2p_{(3/2)}} - \Delta E_{2p_{(1/2)}}\right)$		0.00502	0.00547	0.00535

V. Conclusion

From these results, we can conclude that the corrections in energies in case of muon are much higher than in case of hydrogen atom. These corrections decrease with the increase in the principle quantumnumber. There is no great difference between the results obtained for the three studied potentials. The corrections in energy levels obtained by taking the proton density in Gaussian form and exponential shape are approximatelyequal. In case of non-relativistic and relativistic calculations the finite size proton gives Lamb shift values very near from that using point charge proton. The fine structure contribution (L-S coupling$\left(\Delta E_{2p_{(3/2)}} - \Delta E_{2p_{(1/2)}}\right)$) is very small compared to the Lamb shift.

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