IOSR Journal of Applied Physics (IOSR-JAP)
e-ISSN: 2278-4861.Volume 7, Issue 2 Ver. II (Mar. - Apr. 2015), PP 82-87
www.iosrjournals.org

Derivation of Schrodinger and Einstein Energy Equations from Maxwell's Electric Wave Equation

Mohammed Ismail Adam'1, Mubarak Dirar Abd Allah'2
1'Department of Physics and Mathematics, College of Education, University of Al-Butana, Sudan
2'Department of Physics, College of Science, Sudan University of Science and Technology, Sudan

Abstract: *In these work Maxwell's equations are used to derive Schrödinger quantum equation beside Einstein relativistic energy-momentum relation. The derivations are made by considering particles as oscillators and by using Plank quantum hypothesis. The electric field intensity vector is replaced by the wave function since the two terms are related to the photon number density.*
Keywords: *Schrödinger equation, Maxwell's equation, electric field*

I. Introduction

Maxwell's equations are one of the biggest achievements that describe the behavior of electromagnetic waves (e.m.w) they describe interference, diffraction of light, as well as generation, reflection, transmittance and interaction of electromagnetic waves with matter[1,2,3].

The light was accepted as having a wave nature for long time. But, unfortunately, this nature was unable to describe black body radiation phenomenon. This forces Max Plank to propose that light and electromagnetic waves behave as discrete particles known later as photons. This particle nature succeeded in describing a number of physical phenomena, like atomic radiation, photoelectric, Compton and pair production effects. The pair production effect needs particle nature of light as well as special relativity (SR) to be explained [4, 5, 6].

This dual nature of light encourages De Broglie to propose that particles like electrons can behave some times as waves. The experimental confirmation of this hypothesis leads to formation of new physical laws known as quantum mechanics. Quantum mechanics (QM) is formulated by Heisenberg first and independently by Schrödinger, to describe the dual nature of the atomic world [7, 8].

Despite the fact that the De Broglie hypothesis is based on Max Plank energy expression beside special relativity, there is no link made with Maxwell's equation. Some attempts were made by K.Algeilani [9] to derive Klein-Gordon and special relativity energy relation.

This paper devoted to make further links, by deriving Schrödinger equation as well as SR energy-momentum relation from Maxwell's equation. This is done in section2 and 3 respectively. Section 4 and 5 are devoted for discussion and conclusion.

II. Derivation of Schrödinger Equation from Maxwell's Equations

Maxwell's electric wave Equation can be rewritten as:

$$-\hbar^2c^2\nabla^2E+\hbar^2c^2\mu\sigma\frac{\partial E}{\partial t}+\hbar^2c^2\mu_0\varepsilon_0\frac{\partial^2E}{\partial t^2}+\hbar^2c^2\mu\frac{\partial^2P}{\partial t^2}+\frac{m^2c^4}{\hbar^2}\hbar^2E=0$$

$$-\hbar^2c^2\nabla^2E+\hbar^2c^2\mu\sigma\frac{\partial E}{\partial t}+\hbar^2\frac{\partial^2E}{\partial t^2}+\hbar^2\mu c^2\frac{\partial^2P}{\partial t^2}+m^2c^4E=0 \qquad (1)$$

Where

$$\mu_0\varepsilon_0=\frac{1}{c^2}$$

Neglecting the dipole moment contribution and taking into account the fact that:

$c>>1$

Thus the terms that do not consist of c can be neglected to get:

$$-\hbar^2c^2\nabla^2E+\hbar^2c^2\mu\sigma\frac{\partial E}{\partial t}+zero+zero+m^2c^4E=0 \qquad (2)$$

Dividing both sides of equation (2) by $2mc^2$, yields

$$\frac{-\hbar^2 c^2 \nabla^2 E}{2mc^2} + \frac{\hbar^2 c^2 \mu\sigma}{2mc^2}\frac{\partial E}{\partial t} + \frac{m^2c^4}{2mc^2}E = 0$$

$$\frac{-\hbar^2}{2m}\nabla^2 E + \frac{\hbar^2}{2m}\mu\sigma\frac{\partial E}{\partial t} + \frac{1}{2}mc^2 E = 0 \qquad (3)$$

To find conductivity consider the electron equation for oscillatory system, where the electron velocity is given by:

$$v = v_0 e^{i\omega t} \qquad (4)$$

And its equation of motion takes the form

$$m\frac{dv}{dt} = eE \qquad (5)$$

Differentiate equation (4) over dt , one gets

$$\frac{dv}{dt} = i\ \omega\ v_0 e^{iwt}$$

$$\frac{dv}{dt} = i\ \omega\ v \qquad (6)$$

Inserting equation (6) in (5)

$$i\omega\, m\, v = eE$$

$$v = \frac{e}{i\omega m}E \qquad (7)$$

But for electron, the current J is given by

$$J = nev$$

$$J = \frac{ne^2E}{im\omega} = \frac{ine^2E}{i^2\omega m}$$

$$J = \frac{-ine^2E}{\omega m} \qquad (8)$$

Also we know that

$$J = \sigma E \qquad (9)$$

Comparing equation (8) and (9), one get

$$\sigma = \frac{-ine^2}{m\omega} \qquad (10)$$

The coefficient of the first order differentiation of E with respect to time is given with the aid of equations (3) and (10)

$$\frac{\hbar^2\mu\sigma}{2m} = \frac{-i\hbar^2\mu n e^2}{2m^2\omega} \qquad (11)$$

Using Gauss law

$$\varepsilon E A = Q = neAx \qquad (12)$$

Where x is the average distance of oscillator and is related to the maximum displacement according to the relation

$$x = \frac{1}{2}x_0 \qquad (13)$$

$$\frac{\hbar^2\mu\sigma}{2m} = \frac{-i\hbar^2\mu n e^2}{2m^2\omega} = \frac{-i\hbar(\hbar\omega)\mu(ne^2Ax_0{}^2)}{4(\frac{1}{2}m\omega^2 x_0{}^2)mA} \qquad (14)$$

By using equation (13)

$$x_0{}^2 = 4x^2$$

Thus equation (14) becomes

$$\frac{\hbar^2\mu\sigma}{2m} = \frac{-i\hbar(\hbar\omega)\mu(ne^2A4x^2)}{4(\frac{1}{2}m\omega^2 x_0{}^2)mA}$$

$$= \frac{-i\hbar(\hbar\omega)\ (ne^2A\ x)(x)\mu}{(\frac{1}{2}m\omega^2 x_0{}^2)mA}$$

$$= \frac{-i\hbar(\hbar\omega)\ (neA\ x)e(x)\mu}{(\frac{1}{2}m\omega^2 x_0{}^2)mA} \qquad (15)$$

By using equation (12), one gets

$$= \frac{-i\hbar(\hbar\omega)\ (\varepsilon EA)e(x)\mu}{(\frac{1}{2}m\omega^2 x_0{}^2)mA} = \frac{-i\hbar(\hbar\omega)c^2(\mu\varepsilon)eEAx}{(mc^2)A(\frac{1}{2}m\omega^2 x_0{}^2)}$$

$$= \frac{-i\hbar(\hbar\omega)c^2(\frac{1}{c^2})(F\,x)}{(mc^2)\ (\frac{1}{2}m\omega x_0{}^2)} \qquad (16)$$

But according to quantum mechanical and classical energy formula

$$\hbar\omega = mc^2 = E \qquad (quantum)$$

$$F\,x = \frac{1}{2}m\omega^2 x_0{}^2 = E \qquad (classical)$$

Therefore equation (16) reduce to

$$\frac{\hbar^2\ \mu\sigma}{2m} = -i\hbar \qquad (17)$$

As a result equation (3) becomes

$$\frac{-\hbar^2}{2m}\nabla^2 E - i\hbar\frac{\partial E}{\partial t} + \frac{1}{2}mc^2E = 0 \qquad (18)$$

We have

$$m = m_0\,(1+\frac{2\varphi}{c^2}-\frac{v^2}{c^2})^{\frac{1}{2}}$$

Since Schrödinger deals with low speed therefore

$$\frac{v}{c} << 1$$

Thus one can neglect the speed term to get

$$m = m_0\,(1+\frac{2\varphi}{c^2})^{\frac{1}{2}} \qquad (19)$$

Taking c as a maximum value of light speed, such that the average light speed c_e is given by

$$c_e = \frac{1}{\sqrt{2}}c$$

$$\Rightarrow \qquad c_e = \frac{c^2}{2} \qquad (20)$$

And assuming

$$m \;=\; m_0\,(1+\frac{2\varphi}{2c_e^{\,2}})^{\frac{1}{2}}$$

$$\cong \; m_0\,(1+\frac{\varphi}{c_e^{\,2}}) \qquad (21)$$

Thus

$$\frac{1}{2}mc^2 \;=\; mc_e^2 = \; m_0 c_e^{\,2}\,(1+\frac{\varphi}{c_e^{\,2}})$$

$$= \; m_0 \;+\; m_0\,\varphi$$

$$= \; m_0 \;+\; V \qquad (22)$$

Since atomic particles which are describes by quantum laws are very small, thus one can neglect m_0 compared to the potential V to get

$$m_0 \;+\; V \quad \cong \quad V \qquad (23)$$

Hence from equations (22) and (23)

$$\frac{1}{2}mc^2 \;=\; m_{\,0} \;+\; V \quad \cong \; V \qquad (24)$$

Thus equation (18) reduce to

$$\frac{-\hbar^2}{2m}\nabla^2 E - i\hbar\frac{\partial E}{\partial t} + V\,E = 0$$

Taking into account that the electromagnetic energy density is proportional to E^2 , and since $|\psi|^2$ is also reflects photon density. Thus one cans easily Replace E by ψ , in the above equation, to get

$$\frac{-\hbar^2}{2m}\nabla^2\psi - i\hbar\frac{\partial\psi}{\partial t} + V\,\psi = 0$$

$$\frac{-\hbar^2}{2m}\nabla^2\psi + V\,\psi = i\,\hbar\,\frac{\partial\psi}{\partial t} \qquad (25)$$

This is Schrödinger equation

III. The Electric Polarization And Special Relativity

We have

$$P = -n\,e\,x \qquad (26)$$

$$\mu\frac{\partial^2 P}{\partial t^2} \;=\; -n\;\mu e\ddot{x} \qquad (27)$$

Where $\qquad x \;=\; x_0 e^{i\omega_0 t}$

$$\dot{x} = i\omega_0\; x_0 e^{i\omega_0 t}$$

$$\ddot{x} = i^2\omega_0^{\;2}\; x_0 e^{i\omega_0 t} \qquad = \; -\,\omega_0^{\;2}\; x \qquad (28)$$

$$\mu\frac{\partial^2 P}{\partial t^2} \;=\; +\mu\,\omega_0^{\;2} n \;\; e\,x \qquad (29)$$

From equation (12) we have

$$\varepsilon\,E\,A \;=\; Q \;=\; n\,e\,A\,x$$

$$\varepsilon\,E \;=\; n\,e\,x \qquad (30)$$

Inserting equation (30) in equation (29), one gets

$$\mu \frac{\partial^2 P}{\partial t^2} = \mu \omega_0{}^2 \varepsilon E = \mu \varepsilon \omega_0{}^2 E$$

$$= \frac{\omega_0{}^2}{c^2} E \qquad (31)$$

Equation (1) can be written as

$$-\nabla^2 E + \mu\varepsilon \frac{\partial^2 E}{\partial t^2} + \mu \frac{\partial^2 P}{\partial t^2} = 0 \qquad (32)$$

From equations (31) and (32), one gets

$$-\nabla^2 E + \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} + \frac{\omega_0{}^2}{c^2} E = 0 \qquad (33)$$

Consider

$$E = E_0 e^{i(kx-\omega t)} \qquad (34)$$

$$\nabla^2 E = -k^2 E \qquad (35)$$

$$\frac{\partial E}{\partial t} = -i\omega E_0 e^{i(kx-\omega t)}$$

$$\frac{\partial^2 E}{\partial t^2} = -i \times -i\,\omega^2 E_0 e^{i(kx-\omega t)}$$

$$\frac{\partial^2 E}{\partial t^2} = -\omega^2 E \qquad (36)$$

Substituting equations (35) and (36) in (33)

$$+k^2 E - \frac{1}{c^2} \omega^2 E + \frac{\omega_0{}^2}{c^2} E = 0$$

Multiplying both sides of above equation by $\frac{c^2}{E}$

$$\frac{k^2 E c^2}{E} - \frac{1}{c^2} \omega^2 \frac{E c^2}{E} + \frac{\omega_0{}^2}{c^2} \frac{E c^2}{E} = \frac{0 c^2}{E}$$

$$k^2 c^2 - \omega^2 + \omega_0{}^2 = 0 \qquad (37)$$

Multiplying both sides of equation (37) by $\hbar^2$

$$\hbar^2 k^2 c^2 - \hbar^2 \omega^2 + \hbar^2 \omega_0{}^2 = 0\hbar^2$$

$$\hbar^2 k^2 c^2 + \hbar^2 \omega_0{}^2 = \hbar^2 \omega^2 \qquad (38)$$

For a photon the energy and momentum are given by Plank and De Broglie hypothesis to be

$$p = \frac{h}{\lambda} \qquad , \qquad E = h f$$

$$E = h \frac{c}{\lambda} = p c \qquad (39)$$

But the photon momentum is given by

$$p = mc$$

Thus

$$E = pc = mc^2 \qquad (40)$$

Therefore

$$\left.\begin{array}{l} \hbar\omega = h f = E = m c^2 \\ \hbar\omega_0 = h f_0 = E_0 = m_0 c^2 \end{array}\right] \qquad (41)$$

Substituting equations (39), (d40) and (41) in equation (38), one gets

$$p^2 c^2 + m_0^{\;2} c^4 = m_0^{\;2} c^4 \qquad (42)$$

IV. Discussion

Since Schrödinger equation is first order in time, thus the second order time term should disappear in equation (1). This is achieved by taking into account that all terms that consist of c are larger compared to terms free of c. this is since the speed of light is very large ($c \approx 10^8$). The dipole term in(1) is neglected, which is also natural as well as Schrödinger equation deals only with particles moving in a field potential through the term v which is embedded in the mass term according to GSR [see equation (1)].

In deriving the conductivity term the effect on the particle is only the electric field, while the effect of friction is neglected. This is also compatible with Schrödinger hypothesis which considers the effect of the medium is only through the potential according to the energy wave equation.

$$E\psi = \frac{p^2}{2m}\psi + V\psi \qquad (43)$$

$$\psi = A\, e^{\frac{i}{\hbar}(px - Et)} \quad , \quad i\hbar \frac{\partial \psi}{\partial t} = E\psi$$

$$, \quad -\hbar^2 \nabla^2 \psi = p^2 \psi \qquad (44)$$

The fact that the velocity in equation (4) represents oscillating reflects the wave nature of particles, on which one of the main quantum hypotheses is based. By using this hypothesis together with plank expression of energy, beside classical energy of an oscillating system, the coefficient of the first time derivative of E is found to be equal to $(-i\hbar)$.

In view of equation (18) and the GSR expression of mass (19) the potential term in Schrödinger equation is clearly stems from the mass term. Again the wave nature of particles relates the maximum light speed to its average speed according to equation (20). Neglecting the rest mass, in the third term in equation (18) the coefficient of E is equal to the potential. The final Schrödinger equation was found by the replacing E by ψ. This is not surprising since number of photons $\propto |\psi|^2 \propto E^2$.

The relation of energy and momentum in SR, by assuming oscillating atoms in the media with frequency ω_0 as representing the background rest energy as shown by equations (28-31). The energy gained by the system is the electromagnetic energy of frequency ω [see equations34-36]. Using Plank hypothesis for a photon, beside momentum mass relation in equations (39, 40, 41) the special relativity momentum energy relation was found.

V. Conclusion

The derivation of Schrödinger quantum equation and SR energy-momentum relation from Maxwell electric equation shows the possibility of unifying the wave and particle nature of electromagnetic waves. It shows also of unifying Maxwell's equations, SR and quantum equations.

Acknowledgements

The authors are thanks to all those who encouraged or assisted them in this work.

References

[1]. Bans Lal, mathematical theory of electromagnetism, Asia publishing house(1965).

[2]. Clark MJ, A dynamical Theory of Electrodynamics Field, Royal Society, Transactions 155: 454- 512. (1965).

[3]. Griffiths DJ, Introduction to Electrodynamics, Third Edition, Prentice Hall New Jersey (1999).

[4]. Fleisch D, A student's Guide to Maxwell's Equations, Cambridge University Press, Cambridge (2008).

[5]. Lal B, Mathematical Theory of Electromagnetism, Asia PublishingHouse, Delhi(1965).

[6]. Salih BEA, Teich MC, Fundamentals of Photonics, Second Edition, John Wiley and sons, New York.(2007).

[7]. Taylor JR, Classical Mechanics, University Science Books(2005).

[8]. Hand LN, Finch JD, Analytical Mechanics, Cambridge University press, Cambridge(1998).

[9]. K.Algeilani, Derivation of Klein-Gordon equation from Maxwell's electric wave equation, International Journal of Physical Sciences,USA(2014).