

Entropy in quantum gates and qubit-based information

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Abstract:

In this work, we apply the expressions that we have obtained for the calculation of the entropy for operators acting on a probability space, to the goal of obtaining the entropy of the so called quantum gates that indeed are operators acting on the space of qubit-states and change them following the process of quantum computation. The qubit-states are the basic unity of information on quantum computation and we treat them like points in a probabilistic space. We remember also the correct summation rules we developed in previous works for the operators for the purpose to create a superposition of states that is one of the most powerful tools on quantum computation. Also we establish the connection of the matrix representation of the quantum gates and the qubits with operators that have a defined entropy that can be evaluated in this work. We underline that we are not interested in any specific programming feature but only in the application of the entropy formula for operators.

Key Words: Projection Operators, Qubits, Quantum Gates, Group of Transformations, Matrix Representations, Quantum Computation, Pauli Matrices.

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I. Introduction

On Communication Theory we call bit to the basic information unity, this concept is shared with conventional computation, but when one gets to quantum computing, the latter is based on the entanglement of states and their superposition, so we first treat the qubits as quantum states that is as vectors, and show how we can act over them with operators that can change a qubit with another one or by a superposition of states. It is important to establish the appropriate language for putting in correspondence the specialized with the more usual for operators. For example some operations over a qubit will be called quantum gates and transform a qubit over the so called Bloch Sphere into other qubit or by a superposition of states, In this work we do not review the mechanism for quantum computation but only about the primary family of operations over the qubits and quantum gates, and underline the equivalence with the operators on quantum mechanics and communication theory. Then we recall the rules for adding operators that we have developed in previous works to the end of also apply the appropriate expression for the entropy. So as a first step we review the notation and representation of the action of the gates over the qubits

II. Representing the qubits as quantum vector states

We will use the quantum mechanics notation for the vector states, that is we name the vector state q

$$|q\rangle \text{ (bra) and their dual vector} \quad (1)$$

$$\langle q| \text{ (ket)} \quad (2)$$

So we can define the inner product of two states $|a\rangle$ and $|b\rangle$ as

$$\langle a|b\rangle \quad (3)$$

And for the outer product

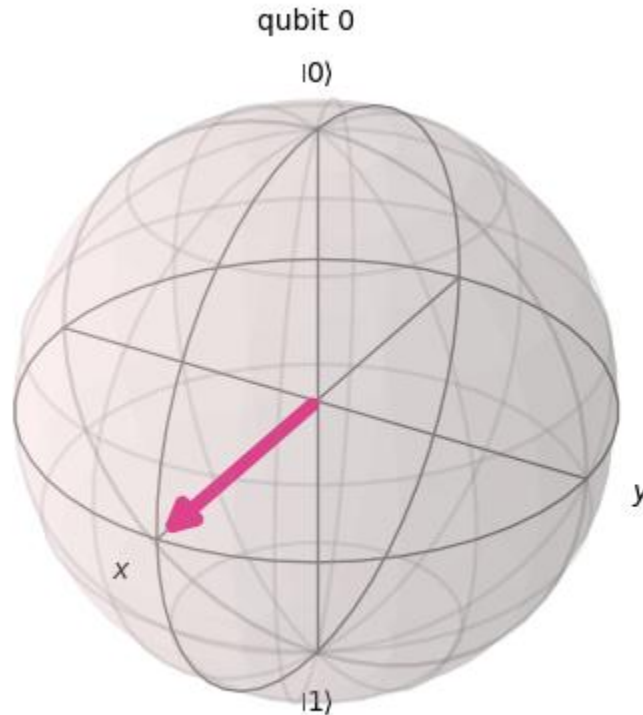


Figure 1.
The Bloch Sphere

$$|a\rangle\langle b| = \begin{bmatrix} a_0 b_0^* & a_0 b_1^* & \cdots & a_0 b_n^* \\ a_1 b_0^* & a_1 b_1^* & \cdots & a_1 b_n^* \\ \vdots & \vdots & \ddots & \vdots \\ a_n b_0^* & \cdots & \cdots & a_n b_n^* \end{bmatrix} \quad (4)$$

On figure 1, it is represented the Bloch Sphere which allow us to visualize the possible states or qubits, all of them can exist only on the surface of the sphere, the states $|0\rangle$ and $|1\rangle$ are at the poles just over the z axis. The two later states can be used as the basis for the representation of any other qubit over the Bloch Sphere as

$$|q\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi} \sin\left(\frac{\theta}{2}\right)|1\rangle \quad (5)$$

Because of the requirements of quantum computation, we also have the superposing of states that is we must add pure states with some probability to exist, some examples are the following states with a particular notation:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad (6)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \quad (7)$$

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \quad (8)$$

$$|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) \quad (9)$$

Any qubit of information can be taken and transformed by a particular process which indeed is an operator that properly is a projection operator named as a quantum gate. Quantum gates represents the distinct operations like measurements or initializations of the qubits on a quantum circuit and can be built from the proper qubits through the outer product (4).

Some Quantum Gates

As we said above, quantum gates are the tools for measure the probability for get some specific state or qubit and in this section we review some of them. We can think that thanks to all the gates, we can reach any point on the Bloch sphere by applying the appropriate gate to a chosen qubit.

The first gate is named the X gate and is represented by the Pauli matrix and the qubits (6) and (7) are eigenstates of this operator.

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0| \quad (10)$$

Expression (10) was obtained also from the outer product (4), the effect of this operator over $|0\rangle$ is:

$$X|0\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|)|0\rangle \quad (11)$$

$$X|0\rangle = |0\rangle\langle 1|0\rangle + |1\rangle\langle 0|0\rangle \quad (12)$$

$$X|0\rangle = |1\rangle \quad (13)$$

That is X transform $|0\rangle$ into $|1\rangle$, we can obtain equation (13) directly from multiplication of the Pauli matrix in (10)

with the vector representation of $|0\rangle$:

$$X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle \quad (14)$$

The effect of X over $|1\rangle$ is:

$$X|1\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|)|1\rangle \quad (15)$$

$$X|1\rangle = |0\rangle\langle 1|1\rangle + |1\rangle\langle 0|1\rangle \quad (16)$$

$$X|1\rangle = |0\rangle \quad (17)$$

The next gates we review are named Y and Z :

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad (18)$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (19)$$

$$Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0| \quad (20)$$

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1| \quad (21)$$

The effect of Y on $|0\rangle$ and $|1\rangle$ is:

$$Y|0\rangle = (-i|0\rangle\langle 1| + i|1\rangle\langle 0|)|0\rangle \quad (22)$$

$$Y|0\rangle = -i|0\rangle\langle 1|0\rangle + i|1\rangle\langle 0|0\rangle \quad (23)$$

$$Y|0\rangle = i|1\rangle \quad (24)$$

And

$$Y|1\rangle = (-i|0\rangle\langle 1| + i|1\rangle\langle 0|)|1\rangle \quad (25)$$

$$Y|1\rangle = -i|0\rangle\langle 1|1\rangle + i|1\rangle\langle 0|1\rangle \quad (26)$$

$$Y|1\rangle = -i|0\rangle \quad (27)$$

The result of applying the so called Hadamard gate H to the qubits $|0\rangle$ and $|1\rangle$ is:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (28)$$

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (29)$$

$$H|0\rangle = |+\rangle \quad (30)$$

$$H|1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (31)$$

$$H|1\rangle = |-\rangle \quad (32)$$

Also the qubits (8) and (9) are eigenstates for the gate Y , that is:

$$Y|i\rangle = (-i|0\rangle\langle 1| + i|1\rangle\langle 0|)|i\rangle \quad (33)$$

$$Y|i\rangle = (-i|0\rangle\langle 1| + i|1\rangle\langle 0|) \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \quad (34)$$

$$Y|i\rangle = -i|0\rangle \frac{1}{\sqrt{2}}\langle 1|0\rangle - i|0\rangle \frac{i}{\sqrt{2}}\langle 1|1\rangle + i|1\rangle \frac{1}{\sqrt{2}}\langle 0|0\rangle + i|1\rangle \frac{i}{\sqrt{2}}\langle 0|1\rangle \quad (35)$$

$$Y|i\rangle = -i|0\rangle \frac{i}{\sqrt{2}}\langle 1|1\rangle + i|1\rangle \frac{1}{\sqrt{2}}\langle 0|0\rangle \quad (36)$$

$$Y|i\rangle = -i|0\rangle \frac{i}{\sqrt{2}}\langle 1|1\rangle + i|1\rangle \frac{1}{\sqrt{2}}\langle 0|0\rangle \quad (37)$$

$$Y|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \quad (38)$$

That is

$$Y|i\rangle = |i\rangle \quad (39)$$

And also

$$Y|-i\rangle = (-i|0\rangle\langle 1| + i|1\rangle\langle 0|)|-i\rangle \quad (40)$$

$$Y|-i\rangle = (-i|0\rangle\langle 1| + i|1\rangle\langle 0|)\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) \quad (41)$$

$$Y|-i\rangle = -i|0\rangle \frac{1}{\sqrt{2}}\langle 1|0\rangle - \frac{1}{\sqrt{2}}|0\rangle\langle 1|1\rangle + i|1\rangle \frac{1}{\sqrt{2}}\langle 0|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\langle 0|1\rangle \quad (42)$$

$$Y|-i\rangle = -\frac{1}{\sqrt{2}}|0\rangle\langle 1|1\rangle + i|1\rangle \frac{1}{\sqrt{2}}\langle 0|0\rangle \quad (43)$$

$$Y|-i\rangle = -\frac{1}{\sqrt{2}}|0\rangle + i|1\rangle \frac{1}{\sqrt{2}} \quad (44)$$

$$Y|-i\rangle = -\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) \quad (45)$$

$$Y|-i\rangle = -|-i\rangle \quad (46)$$

That is in agreement with equations (39) and (46) the eigenvalues are 1 and -1.

The more general gate for one qubit is the U gate defined as:

$$U(\theta, \phi, \Lambda) = \begin{bmatrix} \cos(\frac{\theta}{2}) & -e^{i\Lambda} \sin(\frac{\theta}{2}) \\ e^{i\phi} \sin(\frac{\theta}{2}) & e^{i(\phi+\Lambda)} \cos(\frac{\theta}{2}) \end{bmatrix} \quad (47)$$

The gates I , Z , S , T , H , and P are special cases of (47):

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (48)$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (49)$$

$$P(\phi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix} \quad (50)$$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{bmatrix} \quad (51)$$

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix}$$

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (52)$$

When we go to a greater number of qubits, we use the tensor product between operators to build the appropriate gates as we will see in the next chapter.

IV. Vector representation for more than one qubit

Starting for the operators A and B each of them acting over separate, independent qubits, we describe the multiqubit states as the tensor product of operators:

$$A \otimes B = \begin{bmatrix} A_{00} \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} & A_{01} \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} \\ A_{10} \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} & A_{11} \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} \end{bmatrix} \quad (53)$$

The notation for the bi-qubits is:

$$|00\rangle = \begin{bmatrix} 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{bmatrix} \quad (54) \quad \Rightarrow \quad |00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (55)$$

$$|01\rangle = \begin{bmatrix} 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix} \quad (56) \quad \Rightarrow \quad |01\rangle = \begin{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix} \quad (57)$$

$$|10\rangle = \begin{bmatrix} 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{bmatrix} \quad (58) \quad \Rightarrow \quad |10\rangle = \begin{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{bmatrix} \quad (59)$$

$$|11\rangle = \begin{bmatrix} 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix} \quad (60) \quad \Rightarrow \quad |11\rangle = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix} \quad (61)$$

In all last 4-vectors the two first places are reserved for the first qubit and the two second places for the second qubit.

For three qubits we have:

$$A \otimes B \otimes C = \begin{bmatrix} A_{00} \begin{pmatrix} B_{00} \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix} & B_{01} \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix} \\ B_{10} \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix} & B_{11} \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix} \end{pmatrix} & A_{01} \begin{pmatrix} B_{00} \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix} & B_{01} \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix} \\ B_{10} \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix} & B_{11} \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix} \end{pmatrix} \\ A_{10} \begin{pmatrix} B_{00} \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix} & B_{01} \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix} \\ B_{10} \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix} & B_{11} \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix} \end{pmatrix} & A_{11} \begin{pmatrix} B_{00} \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix} & B_{01} \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix} \\ B_{10} \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix} & B_{11} \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix} \end{pmatrix} \end{bmatrix} \quad (62)$$

And the generic three-qubits components are:

$$|\overline{pqr}\rangle = \begin{pmatrix} p_1 q_1 r_1 \\ p_1 q_1 r_2 \\ p_1 q_2 r_1 \\ p_1 q_2 r_2 \\ p_2 q_1 r_1 \\ p_2 q_1 r_2 \\ p_2 q_2 r_1 \\ p_2 q_2 r_2 \end{pmatrix} \quad (63)$$

In equation (63) p_i, q_i, r_i have the values 0 or 1 for example:

$$|000\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (64)$$

And following the rule (65) for three-qubits:

$$|100\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (65)$$

Also we have:

$$|001\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (66)$$

Also:

$$|011\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (67)$$

And as a final example:

$$|111\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (68)$$

V. Recalling the rules for operator addition

At this point, it is important to remember how to add operators. So in this chapter we review the rules we have obtained in another paper. If we imagine a space constituted by the probabilities that a system is in a specific state, that is, each point is the probability that the system be in a specific state. For an example, we can think in a system that can be found in only two different states, say state 1 and state 2, and that the respective probabilities for each state are p and q , so $p + q$ must sum 1. The space is the set of the two p and q probabilities. Suppose that $p = 0.3$ and $q = 0.7$. Then the projection of the space in the event p can be found by taking the complete space 1 and subtracting $q = 0.7$, that is 0.3. Also the corresponding projection of the space in the event q can be found subtracting $p = 0.3$ from the entire space 1 that is $q = 0.7$. But what about the projection over the entire space $p + q$ over himself? Let us define the operators:

$$\Omega_p \equiv -q \quad \text{And} \quad (69)$$

$$\text{And } \Omega_q \equiv -p, \quad (70)$$

And because $p + q = 1$ then

$$\Omega_p = p - 1. \quad (71)$$

If we try to operate

$$(p + q)1 \quad (72)$$

We must build

$$\Omega_p + \Omega_q = (p - 1) + (-p) = -1 \quad (73)$$

Which is obviously incorrect. Now, we can try in other way:

$$\begin{aligned} (\Omega_p + \Omega_q)1 &= \Omega_p 1 + \Omega_q 1 = (1 - q) + (1 - p) \\ &= [p + q - (1 - p)] + [p + q - p] = 1 \end{aligned} \quad (74)$$

That is: take 1 (the entire space) and subtract q so we have p ; then we sum to this the result of now subtracting p to the entire space obtaining finally 1. So we must adjust the procedure to finally obtain the desired

result 1. Before we propose even in a major dimension the convenient procedure we must note that the preceding two represents the fact that if we do not chose the appropriate rule then the conventional property of the complex and real numbers, vectors and matrices that is the associativity does not works for this kind of operators that we will call simply projection operators. Also we underline the possible application of the present work to the study of the Fredholm integral equations for which we have defined some integral projection operators in previous works [1].

VI. The correct rule for adding projection operators

In this section we begin with a generalization of the rule for adding operators when we need to sum a higher number of them but first we will associate operators for the new probabilities and we denote the entire space as ξ that is:

$$\xi = \{p_i\} \quad (75)$$

Where p_i is the probability to find the system in the i -th state and at the same time is a point of the set ξ and frequently we replace ξ for the complete probability to have the system in any state that is for the number 1:

$$\xi = 1 \quad (76)$$

Let us denote:

$$\Omega_{p_i} \quad (77)$$

For the corresponding projector operator over the state (or probability) p_i then, the sum of m projectors when the dimension of the space is n must be defined like:

$$\left(\sum_{i=1}^m \Omega_{p_i}\right)\xi = \xi - \sum_{j \neq i}^n p_j = \xi - 0 = \xi \quad \left(\sum_{i=1}^m \Omega_{p_i}\right)\xi \equiv \xi - \sum_{j \neq i}^n p_j \quad (78)$$

Because by the moment it is not necessary to show the case $m = n$ we take for example the case $m = n = 3$:

$$\left(\sum_{i=1}^3 \Omega_{p_i}\right)\xi = \xi - \sum_{j \neq i}^3 p_j = \xi - 0 = \xi \quad (11) \quad (79)$$

For $m=2, n=3$ that is for the sum over only two projectors is:

$$\left(\sum_{i=1}^2 \Omega_{p_i}\right)\xi = \xi - \sum_{j \neq i}^3 p_j = p_1 + p_2 + p_3 - p_3 = p_1 + p_2 \quad (80)$$

It is necessary to eliminate a restriction that appears in definition (78). If we try to sum:

$$p_3 + p_1 \quad (81)$$

That is we want to sum from $i = 3$ to $i = 1$ or symbolically:

$$\sum_{\substack{i=3 \\ i \neq 2}}^1 \Omega_{p_i} \equiv p_3 + p_1 \quad (82)$$

This can be also written as a special case (with $m=1, s=3$) of:

$$\sum_{\substack{i=s \\ m < s}}^m \Omega_{p_i} \equiv \sum_{[i=1]}^{[m]} \Omega_{[p_i]} \quad (83)$$

Where the parenthesis $[]$ signifies that we change the conventional order $i = 1, 2, \dots, m, \dots, n$ by a permutation $i = [1], [2], \dots, [m], \dots, [n]$ which in the case $m=2, n=3$ is: $i = [1], [2], [3]$ goes to $i = 3, 1, 2$. So in the general case equation (78) can be rewritten as:

$$\left(\sum_{[i=1]}^{[m]} \Omega_{[p_i]}\right)\xi \equiv \xi - \sum_{[j \neq i]}^{[n]} p_{[j]} \quad (84)$$

Now, remembering the discussion of a previous work [1], we will obtain an expression for the Entropy for the projection operators acting over a probabilistic space by using a common tool in quantum mechanics, to this end, we define the following function:

$$J \equiv \sum_{[i=1]}^{[m]} [e^{\Omega_i}] \log[e^{\Omega_i}] \quad (85)$$

$$J = \sum_{[i=1]}^{[m]} [e^{\Omega_i}] [\Omega_i] \quad (86)$$

$$J = \sum_{[i=1]}^{[m]} [e^{\Omega_i}] [\Omega_i] \quad (87)$$

But we can develop the exponential and then:

$$J = \sum_{[i=1]}^{[m]} \sum_{n=0}^{\infty} \left[\frac{\Omega_i}{n!} \right] [\Omega_i] \quad (88)$$

$$J = \sum_{[i=1]}^{[m]} \sum_{n=0}^{\infty} \left[\frac{\Omega_i^n}{n!} \right] [\Omega_i] \quad (89)$$

Now acting on the entire space ξ :

$$J\xi = \sum_{[i=1]}^{[m]} \sum_{n=0}^{\infty} \left[\frac{\Omega_i^{n+1}}{n!} \right] \xi \quad (90)$$

$$J\xi = \sum_{[i=1]}^{[m]} \sum_{n=0}^{\infty} \left[\frac{p_i^{n+1}}{n!} \right] \quad (91)$$

$$J\xi = \sum_{[i=1]}^{[m]} [e^{p_i}] \log[e^{p_i}] \quad (92)$$

Now we can make

$$X_i = e^{p_i} \quad (93)$$

From this definition we can write:

$$J\xi = \sum_{[i=1]}^{[m]} [X_i] \log[X_i] \quad (94)$$

From this expression we recognize that we can propose as a first attempt for the entropy of the projection operators $\{\Omega_i\}$ [2, 4, 5, 6, 7, 8]:

$$H = -\log \sum_{[i=1]}^{[m]} [X_i] \log[X_i] = -\log \sum_{[i=1]}^{[m]} [e^{\Omega_i}] \log[e^{\Omega_i}] \xi \quad (95)$$

VII Matrix representation for the projector operators

The preceding rule expressed in equation (84) is useful for the purpose of give an understanding for the projection operators but we can go farther and find matrix representations that may also give an augmented vision of the possible implications to more complex systems than those former taken into account. In the process we will find additional properties like group properties and a dual space for this representations that must be interpreted but indeed have not known existence at this moment.

Let us associate to the space of probabilities for the case $n=2$ a vector notation:

$$\xi = \begin{pmatrix} p \\ q \end{pmatrix} \quad (96)$$

Now, we define de four following projection operators:

$$\Omega_p = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad (97)$$

$$\Omega_q = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \quad (98)$$

$$\Omega_\phi = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (99)$$

$$\Omega_I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (\text{Identity}) \quad (100)$$

The multiplication table for the projectors (97-100) is:

	Ω_I	Ω_p	Ω_q	Ω_ϕ	
Ω_I	Ω_I	Ω_p	Ω_q	Ω_ϕ	
Ω_p	Ω_p	$-i\Omega_\phi$	$i\Omega_I$	Ω_q	
Ω_q	Ω_q	$i\Omega_I$	$i\Omega_\phi$	$-\Omega_p$	
Ω_ϕ	Ω_ϕ	Ω_q	$-\Omega_p$	$-\Omega_I$	

(101)

We can see that also the following rule is accomplished [3]:

$$(\Omega_i)^4 = \Omega_I \quad \text{With } i = I, p, q, \phi \quad (102)$$

So the set of four matrices:

$$\chi = \{\Omega_p, \Omega_q, \Omega_I, \Omega_\phi\} \quad (103)$$

Form an abelian cyclic group of fourth degree [2].

But we can verify that if we make the following associations:

$$p \Rightarrow \begin{pmatrix} p \\ iq \end{pmatrix} \equiv (1+i)p \quad (104)$$

$$q \Rightarrow \begin{pmatrix} ip \\ q \end{pmatrix} \equiv (1+i)q \quad (105)$$

Then, p, q , and ξ also they stick to the same rule (84):

$$(\Omega_p + \Omega_q) \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} - (0) = \begin{pmatrix} p \\ q \end{pmatrix} = \xi \quad (106)$$

$$\Omega_p \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ iq \end{pmatrix} = (1+i)p \quad (107)$$

$$\Omega_q \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} ip \\ q \end{pmatrix} = (1+i)q \quad (108)$$

Also, we can verify that we can express the complete space ξ as the following sum:

$$p + q = \xi \quad (109)$$

We can see that other possibilities exist:

$$\Omega_\phi \begin{pmatrix} p \\ q \end{pmatrix} = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = i \begin{pmatrix} p \\ -q \end{pmatrix} \equiv \phi \quad (110)$$

And for example:

$$\Omega_\phi \phi = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} i \begin{pmatrix} p \\ -q \end{pmatrix} = - \begin{pmatrix} p \\ q \end{pmatrix} = -\xi \quad (111)$$

From equations (110) and (111) we can interpret ϕ as a dual space of probabilities and Ω_ϕ as the projection operator over that dual space.

Now, we calculate:

$$(\Omega_p + \Omega_q + \Omega_\phi)\xi = \xi + \Omega_\phi \xi = \xi + \phi \quad (112)$$

It is a time to calculate the Entropy using definition (95), but first we note that

$$\{\Omega_k\}\xi = p_k \quad (113)$$

Then we can see that a better definition must be taken (95) but with

$$e^{\Omega_k} \equiv e^{\{\Omega_k\}\xi} = e^{p_k} \quad (114)$$

That is:

$$H = -\log \sum_{[i=1]}^{[m]} [X_i] \log[X_i] = -\log \sum_{[i=1]}^{[m]} [e^{p_i}] \log[e^{p_i}] \quad (115)$$

So we can avoid the exponential forms and directly define for the Entropy of operators acting in a probabilistic space:

$$H = -\sum_{[i=1]}^{[m]} [\Omega_i] \log[\Omega_i] = -\sum_{[i=1]}^{[m]} [p_i] \log[p_i] \quad (116)$$

Remember that the symbol $[]$ represents a specific permutation between the desired $\{\Omega_i\}$

Let's evaluate H for the operator Ω_p for $p = 0.3$ and $q = 0.7$:

$$H = -(p)\log(p) \quad (49) \quad (117)$$

$$H = -0.3(\log(0.3)) = 0.521 \quad (118)$$

And for the operator Ω_q

$$H = -(q)\log(q) \quad (119)$$

$$H = -0.7(\log(0.7)) = 0.36 \quad (120)$$

For the sum $\Omega_p + \Omega_q$:

$$H = -0.3(\log(0.3)) - 0.7(\log(0.7)) = 0.521 + 0.36 = 0.881 \quad (121)$$

We obtain the maximum Entropy for $p = 0.5$ and $q = 0.5$:

$$H = -0.5(\log(0.5)) - 0.5(\log(0.5)) = 0.5 + 0.5 = 1 \quad (122)$$

But we can obtain the same result if we use new representations for the projection operators:

$$\Omega_\varphi = i \begin{pmatrix} p & 0 \\ 0 & -q \end{pmatrix} \quad (123)$$

$$\Omega_I = \frac{1}{2} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \quad (124)$$

$$\Omega_p = \begin{pmatrix} p & 0 \\ 0 & iq \end{pmatrix} \quad (125)$$

$$\Omega_q = \begin{pmatrix} ip & 0 \\ 0 & q \end{pmatrix} \quad (126)$$

And we can write a new version for Entropy appropriate for more general operators:

$$H = - \sum_{[i=1]}^{[m]} [\Omega_i] \log[\Omega_i] = - \sum_{[i=1]}^{[m]} [\text{Re}\{\text{Trace}\{\Omega_i\}\}] \log[\text{Re}\{\text{Trace}\{\Omega_i\}\}] \quad (127)$$

Let us apply eq. (127) first to operator (125):

$$H = -\Omega_p \log \Omega_p = -[\text{Re}\{\text{Trace}\{\Omega_p\}\}] \log[\text{Re}\{\text{Trace}\{\Omega_p\}\}] \quad (128)$$

$$H = -[\text{Re}\{p + iq\}] \log[\text{Re}\{p + iq\}] \quad (129)$$

$$H = -[p] \log[p] = -(.3) \log(.3) = 0.521 \quad (130)$$

That is the same as eq. (118)

Then we take operator (126) and apply definition (127):

$$H = -\Omega_q \log \Omega_q = -[\text{Re}\{\text{Trace}\{\Omega_q\}\}] \log[\text{Re}\{\text{Trace}\{\Omega_q\}\}] \quad (131)$$

$$H = -[\text{Re}\{ip + q\}] \log[\text{Re}\{ip + q\}] \quad (132)$$

$$H = -[q] \log[q] = -(.7) \log(.) = 0.36 \quad (133)$$

That is the same as eq. (120).

Up to this point, the expressions involved in the calculation of Entropy require that the trace of the matrices was different from zero, but if we wish to include Hermitian operators we must modify these expressions because the trace of a Hermitian matrix is zero. This problem can be saved thanks to the use of Pauli matrices.

Because Pauli matrices form a base for the description of every 2×2 Hermitian matrix, it is possible to include Hermitian operators whose trace is zero with the aid of them. The following are Pauli matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (134)$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (135)$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (136)$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (137)$$

If $\{O_i\}$ is a set of Hermitian 2×2 operators we change slightly (127) to the form:

$$H = - \sum_{[l=0]}^{[s]} \sum_{[i=1]}^{[m]} [\lambda \sigma_l^\dagger O_i] \log[\lambda \sigma_l^\dagger O_i] = - \sum_{[l=0]}^{[s]} \sum_{[i=1]}^{[m]} [\text{Re}\{\text{Trace}\{\lambda \sigma_l^\dagger O_i\}\}] \log[\text{Re}\{\text{Trace}\{\lambda \sigma_l^\dagger O_i\}\}] \quad (138)$$

Provided that:

$$\sum_{[l=0]}^{[s]} \sum_{[i=1]}^{[m]} [\text{Re}\{\text{Trace}\{\lambda \sigma_l^\dagger O_i\}\}] \leq 1 \quad (139)$$

The symbol $[]$ again represents permutation in the order of the indices (see equation (84)) and λ is a normalization factor.

We have for example with $O_1 = \sigma_1$:

$$H = -\lambda \sigma_1^\dagger O_1 \log \lambda \sigma_1^\dagger O_1 = -\text{Re}\{\text{Trace}\{\lambda \sigma_1^\dagger \sigma_1\}\} \log \text{Re}\{\text{Trace}\{\lambda \sigma_1^\dagger \sigma_1\}\} \quad (140)$$

The Entropy will be:

$$H = -\text{Re}\{2\lambda\} \log \text{Re}\{2\lambda\} = -2\lambda \log 2\lambda = 0.2497 \quad (141)$$

With a normalization factor of:

$$\lambda = 0.0312 \quad (142)$$

So we have for the four Pauli matrices:

$$H(\sigma_0) = 0.2497 \quad (143)$$

$$H(\sigma_1) = 0.2497 \quad (144)$$

$$H(\sigma_2) = 0.0.2497 \quad (145)$$

$$H(\sigma_3) = 0.2497 \quad (146)$$

And for the four matrices (the entire space of Pauli matrices):

$$H(\sigma_0 + \sigma_1 + \sigma_2 + \sigma_3) = 0.9988 \quad (147)$$

That is, the Pauli matrices space behaves like a probabilistic space.

Now, if the Hermitian operator O_1 is:

$$O_1 = \begin{pmatrix} \alpha_0 + \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_0 - \alpha_3 \end{pmatrix} \quad (148)$$

$$= \alpha_0 I + \alpha_1 \sigma_1 + \alpha_2 \sigma_2 + \alpha_3 \sigma_3 \quad (149)$$

The Entropy will be:

$$H = -\sum_{l=0}^3 [\text{Re}\{\text{Trace}\{\lambda \sigma_l^\dagger O_1\}\}] \log [\text{Re}\{\text{Trace}\{\lambda \sigma_l^\dagger O_1\}\}] \quad (150)$$

$$H = -[\text{Re}\{\text{Trace}\{\lambda \sigma_0^\dagger O_1\}\}] \log [\text{Re}\{\text{Trace}\{\lambda \sigma_0^\dagger O_1\}\}] \quad (151)$$

$$-[\operatorname{Re}\{\operatorname{Trace}\{\lambda\sigma_1^\dagger O_1\}\}]\log[\operatorname{Re}\{\operatorname{Trace}\{\lambda\sigma_1^\dagger O_1\}\}] \quad (152)$$

$$-[\operatorname{Re}\{\operatorname{Trace}\{\lambda\sigma_2^\dagger O_1\}\}]\log[\operatorname{Re}\{\operatorname{Trace}\{\lambda\sigma_2^\dagger O_1\}\}] \quad (153)$$

$$-[\operatorname{Re}\{\operatorname{Trace}\{\lambda\sigma_3^\dagger O_1\}\}]\log[\operatorname{Re}\{\operatorname{Trace}\{\lambda\sigma_3^\dagger O_1\}\}] \quad (154)$$

$$H = -2\lambda\alpha_0[\log[2\lambda\alpha_0]] - 2\lambda\alpha_1[\log[2\lambda\alpha_1]] - 2\lambda\alpha_2[\log[2\lambda\alpha_2]] - 2\lambda\alpha_3[\log[2\lambda\alpha_3]] \quad (155)$$

If we choose an arbitrary Hermitian matrix like:

$$O_1 = \begin{pmatrix} \alpha_0 + \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_0 - \alpha_3 \end{pmatrix} = \begin{pmatrix} 2 & 1-i \\ 1+i & 0 \end{pmatrix} \quad (156)$$

Then $\alpha_i = 1$ and thus putting $\lambda = 0.0312$ we have:

$$H = 0.998977 \quad (157)$$

We can think now in the operators $\{O_i\}$ like projectors acting on a probabilistic space as occurred in the case of abelian operators.

It is a very interesting fact that while the former projector operators defined over the probabilistic space ξ has been represented by means of members of abelian groups, the Hermitian operators $\{\sigma_i\}$ are the base of the $SU(2)$ (non-abelian) group.

Nevertheless, the expression for the Entropy it's very similar.

VIII. Entropy for quantum gates

Now, we calculate the entropy of very special operators like the different quantum gates by using

$$H = -\sum_{[l=0]}^{[s]} \sum_{[i=1]}^{[m]} [\lambda\sigma_l^\dagger O_i] \log[\lambda\sigma_l^\dagger O_i] = -\sum_{[l=0]}^{[s]} \sum_{[i=1]}^{[m]} [\operatorname{Re}\{\operatorname{Trace}\{\lambda\sigma_l^\dagger O_i\}\}] \log[\operatorname{Re}\{\operatorname{Trace}\{\lambda\sigma_l^\dagger O_i\}\}] \quad (158)$$

Let's choose the X gate:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0| \quad (159)$$

By comparing with equation (88)

$$O_1 = \begin{pmatrix} \alpha_0 + \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_0 - \alpha_3 \end{pmatrix} = \begin{pmatrix} 2 & 1-i \\ 1+i & 0 \end{pmatrix} \quad (160)$$

We have that:

$$\alpha_0 + \alpha_3 = 0 \quad (161)$$

$$\alpha_1 - i\alpha_2 = 1 \quad (162)$$

$$\alpha_1 + i\alpha_2 = 1 \quad (163)$$

$$\alpha_0 - \alpha_3 = 0 \quad (164)$$

And then:

$$\alpha_0 = \alpha_2 = \alpha_3 = 0 \quad (165)$$

With:

$$\alpha_1 = 1 \quad (166)$$

So we have that:

$$O_1 = \sigma_1 \quad (167)$$

The entropy of the gate X will be

$$H = -\lambda \sigma_1^\dagger O_1 \log \lambda \sigma_1^\dagger O_1 = -\text{Re}\{\text{Trace}\{\lambda \sigma_1^\dagger \sigma_1\} \log \text{Re}\{\text{Trace}\{\lambda \sigma_1^\dagger \sigma_1\}\} \quad (168)$$

$$H(X) = -\text{Re}\{2\lambda\} \log \text{Re}\{2\lambda\} = -2\lambda \log 2\lambda = 0.2497 \quad (169)$$

On (98) the normalization constant is

$$\lambda = 0.0312 \quad (170)$$

For the generic gate

$$U(\theta, \phi, \Lambda) = \begin{bmatrix} \cos(\frac{\theta}{2}) & -e^{i\Lambda} \sin(\frac{\theta}{2}) \\ e^{i\phi} \sin(\frac{\theta}{2}) & e^{i(\phi+\Lambda)} \cos(\frac{\theta}{2}) \end{bmatrix} \quad (171)$$

We can identify when comparing with equation (156):

$$O_1 = \begin{pmatrix} \alpha_0 + \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_0 - \alpha_3 \end{pmatrix} \quad (156)$$

$$\alpha_0 + \alpha_3 = \cos(\frac{\theta}{2}) \quad (172)$$

$$\alpha_1 - i\alpha_2 = -e^{i\Lambda} \sin(\frac{\theta}{2}) \quad (173)$$

$$\alpha_1 + i\alpha_2 = e^{i\phi} \sin(\frac{\theta}{2}) \quad (174)$$

$$\alpha_0 - \alpha_3 = e^{i(\phi+\Lambda)} \cos(\frac{\theta}{2}) \quad (175)$$

With:

$$O_1 = \alpha_0 I + \alpha_1 \sigma_1 + \alpha_2 \sigma_2 + \alpha_3 \sigma_3 \quad (176)$$

The corresponding values for the α_i are:

$$\alpha_0 = \cos(\frac{\theta}{2}) \quad (177)$$

$$\alpha_1 = i \sin\left(\frac{\theta}{2}\right) \sin(\phi) \quad (178)$$

$$\alpha_2 = -i \sin\left(\frac{\theta}{2}\right) \cos(\phi) \quad (179)$$

$$\alpha_3 = 0 \quad (180)$$

So we have that:

$$O_1 = \cos\left(\frac{\theta}{2}\right) I + i \sin\left(\frac{\theta}{2}\right) \sin(\phi) \sigma_1 - i \sin\left(\frac{\theta}{2}\right) \cos(\phi) \sigma_2 \quad (181)$$

And then the entropy will be:

$$H = - \sum_{[l=0]}^{[s]} \sum_{[i=1]}^{[m]} [\text{Re}\{\text{Trace}\{\lambda \sigma_l^\dagger O_i\}\}] \log[\text{Re}\{\text{Trace}\{\lambda \sigma_l^\dagger O_i\}\}] \quad (182)$$

$$= - \sum_{[l=0]}^3 [\text{Re}\{\text{Trace}\{\lambda \sigma_l^\dagger O_1\}\}] \log[\text{Re}\{\text{Trace}\{\lambda \sigma_l^\dagger O_1\}\}] \quad (183)$$

Substituting (110) in equation (112):

$$H = -2\lambda \cos\left(\frac{\theta}{2}\right) [\log[2\lambda \cos\left(\frac{\theta}{2}\right)]] \quad (184)$$

With $\lambda = 0.0312$:

$$H(U) = -2(0.0312) \cos\left(\frac{\theta}{2}\right) [\log[2(0.0312) \cos\left(\frac{\theta}{2}\right)]] \quad (185)$$

So, for the Hadamard gate H with $\theta = \frac{\pi}{2}$ the entropy is:

$$H = -2(0.0312) \cos\left(\frac{\pi}{2}\right) [\log[2(0.0312) \cos\left(\frac{\pi}{2}\right)]] \quad (186)$$

$$H(H) = 0.1987 \quad (187)$$

We remember that gate H takes the pure state $|0\rangle$ toward the mixed state $|+\rangle$:

$$H|0\rangle = |+\rangle \quad (188)$$

IX. CONCLUSION

The principal goal of the present paper is to review the tools we have developed in previous works for calculate the Entropy of a very general class of projection operators, hermitian and non hermitian which, as we have seen includes the quantum gates for qubits and multi-qubits that can be applied for quantum computation. In this way we have giving explicit expressions for the Entropy in the cases of projection operators defined over a space of probabilities that shows a hidden symmetry when are represented in abelian matrix form but also we give an alternative Entropy for operators represented by Hermitian operators, as we know, Hermitian matrices are traceless so we must modify slightly our former expressions to include these important class of operators. Then we have taken our proposed expressions and calculate explicitly Entropy values for distinct situations and compare. We observe that it is possible to normalize the probabilistic space and compare very different projectors thanks to the use of appropriate expression for the Entropy as we have recognize from eqs. (130), (133) and (157) that represents Entropy values for two different types of operators: abelian and Hermitian. After we remember how we can represent the qubits over the Bloch sphere and the notation of quantum mechanics for them, we present the named quantum gates which can be considered as projection operators that changes the status of each

single or mixed state and that are very crucial tools for quantum computation, also we recall some properties of these. Finally we calculate the Entropy for these quantum gates in equations (169), and from (182) to (187), all these are original and exclusive for this paper, and which includes the quantum gates: I , Z , S , T , H , and P as we can see from definition (47).

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