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Efimov Effect in a Three body System

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Abstract: *The quantum three body problem generally has its own complexities and is difficult to solve. It requires rigorous steps and iterations to find the solution. We often encounter three body system in various branches of physics including Nuclear Physics, Particle, Atomic, Molecular physics, and condensed matter. The underlying physics involved is quite complicated. Faddeev in 1961 made a successful attempt and formulated theory of three particle scattering in a rather rigorous mathematical way. Efimov developed a method that enabled one to obtain the universal properties of three particle system and is quite independent of the two body potentials. This article attempts to introduce Efimov effect. The formalism and the salient features are also discussed.*

Key Word*: Scattering length; Efimov states; three body system.*

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I. Introduction

Efimov effect [1] is a quantum mechanical effect resulting in emergence of infinite bound states for three particle system provided that the two body interactions barely bind the two body binary system. The number of bound states for three body system is governed by relation $N = \frac{1}{\pi}\ln(|a|/r_0)$, where a the scattering length of binary system and r_0 is the range parameter. The infinite number of three bound states occurring only for the critical value two body coupling constant. Interestingly when the two body coupling constant is increased the three body bound states start disappearing. The most important feature of this effect is its 'universality' i.e. this effect is independent of details of two body forces involved and can occur in any three body system provided the necessary conditions are met. The main reason for the universal property is that the inter-particle distances are much larger than the two body force rangerr_0, only the asymptotic behavior of the wave function is important which is quite independent of nature of inter-particle forces involved. This effect is therefore studied in various three body quantum systems like Helium trimer, triton, 2n halo nuclei like Be^{14}, C^{20}, C^{22}.

II. Formalism

Efimov [2] considered three identical bosons interacting via short range forces with r_0 as two body range parameter. Using Jacobian coordinates, the Schrodinger equation for system of three-particle is obtained

$$[-\nabla^2_{x_1} - \nabla^2_{y_1} + V(x_1) + V(x_2) + V(x_3) - E)\psi = 0 \quad (1)$$

Three body wave function is then split into three components:

$$\psi = \chi(x_1, y_1) + \chi(x_2, y_2) + \chi(x_{3,}y_3) \quad (2)$$

Since the particles are identical χ will have similar functional forms one can obtain solution for any one of the set.

It is worth pointing here that two body potentials are short ranged so dividing the configuration space in the two regions:

$$V(x) = \begin{cases} V(x); \quad x < r_0 \quad (interior\ region) \\ 0 \quad ; \quad x > r_0 \quad (exterior\ region) \end{cases} \quad (3)$$

For exterior region ($x > r_0$) the equation reduces to

$$\left[-\nabla_{x_1}^2 - \nabla_{y_1}^2 - E\right] \chi(x_1, y_1) = 0 \quad (4)$$

The above equation reduces to free three-particle Schrodinger equation. Rewriting equation in terms $u(x_1, y_1)$ where u is

$$u(x_1, y_1) = x_1\ y_1\ \chi(x_1, y_1) \quad (5)$$

It will be useful to introduce hyper-spherical coordinates R and ϕ defined as:

$$x_1 = R\sin\phi \quad y_1 = R\cos\phi \quad (6)$$

$$\left(-\frac{\partial^2}{\partial R^2} - \frac{\partial}{R\partial R} - \frac{\partial^2}{R^2\partial\phi^2} - E\right) u(R,\phi) = 0 \quad (7)$$

The solution of Eqn. (7) is obtained using method of separation of variables is given as

$$u(R,\phi) = \sum_{n_i} F_{n_i}(R)\ G_{n_i}(\phi) \quad (8)$$

Where n_i represents the number of Eigen states of the three body system. The radial part of Eqn. (7) is:

$$\left(-\frac{d^2}{dR^2} - \frac{d}{RdR} + \frac{n_i^2}{R^2} - E\right) F_{n_i}(R) = 0 \quad (9)$$

While the angular part of Eqn. (7) is given by

$$\left(\frac{d^2}{d\phi^2} + n_i^2\right) G_{n_i}(\phi) = 0 \quad (10)$$

We seek solution for the above equations by applying suitable boundary conditions. Using the condition for the occurrence of Efimov state for $R \ll |a|$ but large compared to r_0, we obtain

$$-n_i \cos\left(n_i \frac{\pi}{2}\right) + \frac{8}{\sqrt{3}} \sin\left(n_i \frac{\pi}{6}\right) = 0 \quad (11)$$

This is the transcendental equation in n_i which has one imaginary root $n_i = 1.0002i$ and has infinite number of real roots.

Inserting imaginary root in the radial part of the Schrodinger equation (refer to Eqn. (9)) leads to attractive long range effective interaction ($\frac{1}{R^2}$) of the three body system. Substituting $R' = E^{\frac{1}{2}} R$

Radial part of Schrodinger equation becomes:

$$\frac{d^2F_{n_i}(R')}{dR'^2} + \frac{1}{R'}\frac{dF_{n_i}(R')}{dR'} + \left(1 - \frac{{n_i}^2}{R'^2}\right) F_{n_i}(R') = 0 \quad (12)$$

It can be seen that this is Bessel's differential equation and hence solution is the Bessel's function

$$F_{n_i} = \sin(\ln(|E|^{\frac{1}{2}} R) + \Delta) \quad (13)$$

Here Δ is the initial phase of exterior solution (choosing $\Delta = 0$).

We are now in a position to find number of bound states from the node of wave function. We have

$$\sin\left(\ln\left(|E|^{\frac{1}{2}} R\right) + \Delta\right) = 0 \quad (14)$$

Or

$$\ln\left(|E|^{\frac{1}{2}} R\right) = N\pi \quad (15)$$

According to Efimov condition $E \ll \frac{1}{{r_0}^2}$ or $E^{\frac{1}{2}} \sim \frac{1}{r_0}$, and also R is of the order of a for large distances ($R \sim a$)

We have $N = \frac{1}{\pi} \ln\frac{|a|}{r_0}$ (number of Efimov states).

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III. Conclusion

The solution of Schrodinger equation for three body system (identical bosons) subjected to condition that the two body scattering length approaching infinity, results in infinite number of bound states in the three body system. The possibility of infinite states in Efimov effect is attributed to emergence of $1/R^2$ long range attractive interaction. It is universal in nature i.e. the form of long range attraction does not depend on the shape of inter-particle forces the only thing that matters is that the forces are capable of supporting the resonances whose extent is a two body scattering length. The necessary conditions for the occurrence of Efimov effect are firstly the scattering length 'a' is much greater than the force range r_0 ($a \gg r_0$). The second condition is low energy requirement i.e. two body and three body energies should be small. The number of Efimov states decreases with the increase of two body binding energy. The size of the N^{th} Efimov state is $R_n = r_0\, e^{N\pi}$ where r_0 is the range of two body potential. Thus spatial sizes of Efimov states are exponentially large. The energies of Efimov states are exponentially small. The energies E_N and E_{N+1} of adjacent levels are related by relation:

$$\frac{E_N}{E_{N+1}} = e^{2\pi} \qquad (16)$$

This relation is called as the scaling property. The Efimov states have been observed experimentally in the ultra-cold thermal gas of Caesium atoms [3].While earlier studies were devoted to search the Efimov states in helium trimer [4], its occurrence in 2n halo nuclei were investigated (theoretically) in various halo nuclei like Be^{14}, C^{20} [5,6] but its experimental observation in halo nuclei is still awaited.

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