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Klauder-Daubechies Deformation and Landau's Problem

Daddy Balondo Iyela [a;c], Prosper Lufungula Nkwambiaya [a], Didier Kileba Matondo [a], Nestor Anzola Kibamba [b].
[a]*Department of Physics, University of Kinshasa (UNIKIN), B.P.190, Kinshasa XI, Democratic Republic of Congo*
[b]*Department of Mathematics, Statistics and Science Computer, University of Kinshasa (UNIKIN), B.P.190, Kinshasa XI, Democratic Republic of Congo*
[c]*International Chair in Mathematical Physics and Applications (ICMPA{UNESCO Chair), University of Abomey{Calavi, 072 B. P. 50, Cotonou, Republic of Benin*

Abstract
In the current work, we have considered an approach based on the canonical operator formulation corresponding to the Klauder-Daubechies construction of a deformation of quantum mechanics. We have applied this formulation to the Landau harmonic problem in order to construct the quantisation of its extended formulation, and its quantum solution, allowing for an explicit analysis of the limits $\tau_0 = 0$ and $m = 0$ corresponding to the effective projection onto the lowest Landau sector of the system. We have solved the physics spectrum of this system by use to the Fock algebra in the Hilbert space. We have obtained the physical spectrum as well as the diverse limits possible, leading to the algebraic structure of the non commutative geometry of the Moyal space.

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1 Introduction

In this investigation, we focus on the classical Landau problem, which is generalised by the inclusion of a harmonic potential, both in terms of its classical dynamics and its quantum resolution. It concerns a non-relativistic charged particle moving in a plane, subjected both to a perpendicular magnetic field and to a spherically symmetric harmonic potential, characterised by its own angular frequency. The correspondence between classical mechanics and quantum mechanics is most evident within the framework of the Hamiltonian formalism, which is developed below for this system. In the classical approach, a canonical transformation was performed, whereby the Cartesian coordinates were altered whilst ensuring the formulation dynamics of the system remained consistent within the same phase space. Following the implementation of the canonical transformation, an expression for the action is obtained in terms of $Q_\pm$ and $P_\pm$ coordinates defining a canonical transformation of the Hamiltonian formulation of the system not yet deformed as Klauder-Daubechies which then allows the two helicity sectors of the Landau problem to be factored. Furthermore, under this factorisation, each helicity sector, with coordinates (Q_+, P_+) or (Q_-, P_-), has a dynamic similar with that of a harmonic oscillator characterided by an angular frequency ω_- or ω_+.

The Klauder-Daubechies deformation of the system is not easy to build directly in terms of the initial coordinates and the functional integral approach on coherent states because of the external magnetic field. This is a situation for which the Klauder-Daubechies approach is not designed in its original form. It corresponds to the Klauder-Daubechies deformation of each of these decoupled or factorised harmonic oscillators. This Klauder-Daubechies deformation, which facilitates a rigorous and accurate definition of the functional integral, involves a parameter that ultimately takes a zero value, analogous to a regularisation parameter. It is possible to choose this parameter to be associated, for example, with a new fundamental time scale (see, for example, [1], [7], [8], [9], [10], [11] and [12]), denoted below τ_0. Initially, such a time scale would be linked to the Planck mass or Planck time scale, in a context of relativistic quantum gravity, and the implications of this correlation would need to be examined. Since the two oscillators associated with the two helicity modes of motion in the plane of Landau's problem are coupled, this means that their deformation, and then their quantization by deformation, amounts to taking the tensor product for the Hilbert spaces concerned, results for a single oscillator for which the Klauder-Daubechies approach is perfectly suitable.

From a quantum perspective, the preferred methodological approach is purely algebraic. This approach is based on the use of an abstract Hilbert space, which serves as a theoretical framework. The Hamiltonian of the system, as an abstract operator, acts on this Hilbert space, thereby enabling the modelling and understanding of the underlying physical phenomena. In order to quantify the system under consideration, it is essential to first identify its physical spectrum. This process involves a thorough analysis of the spectrum of the energy eigenstates and their respective degeneracies. In addition, it is imperative to examine the evolution of this spectrum as the limit $\tau_0 \to 0$. These considerations are developed with the help of a detailed analysis of the Lagrangian and Hamiltonian formulations of the system. Furthermore, an exhaustive exploration of the various conceivable limits has been undertaken, leading to the algebraic structure of the non-commutative geometry. These are either zero mass limits for the original Landau problem or zero time scale τ_0 limits for the Klauder-Daubechies deformation. It is well known that the zero mass limit for the Landau problem leads to the structure of the non-commutative plane. The following questions arise: firstly, whether, in the approach of quantum dynamics proposed by the Klauder-Daubechies deformation, the two limits commute or not; secondly, whether it is possible for another non-commutative plane structure to exist in the presence of the deformation parameter τ_0 of the Klauder-Daubechies formulation.

The original results presented in this work can be found with full details in [2], including the construction of the functional integral representing the dynamics of a quantum system in the Klauder-Daubechies deformation formulation. For this last point, we refer to the discussion on this subject in reference [1], [3], [4], [5] and [6]. The structure of the present paper is delineated as follows: Sect. 2 offers a brief highlight on The Landau problem and its deformation. The quantisation of the harmonic Landau problem is discussed in Sect. 3. The quantum non-commutative plane is described in Sect. 4. The canonical Hamiltonian formulation of the system is developed in Sect. 5. A brief overview of the quantified system is given in Sect. 6. In Sect. 7, we have presented the Klauder-Daubechies deformation analysis. The rules of canonical quantisation and the construction of the algebra and diagonalisation of the Hamiltonian are described in Sects. 8 and 9 respectively. In Sect. 10, we discuss on the limit $\tau_0 \to 0$. The non-commutative geometry in the Moyal plane is treated in Sect. 11. Finally, the Klauder-Daubechies quantum non-commutative plane is illustrated in Sect. 12 and we discuss and conclude our results in Sect. 13.

2 Landau problem and its deformation

In this study, we focused on the Landau problem, which is defined in a plane parametrised by x_i $(i = 1, 2)$, in the presence of a spherically symmetric harmonic potential in the Euclidean plane with stiffness constant k_0. In the symmetric gauge for the potential vector describing the homogeneous and transverse magnetic field, the system is defined by the following Lagrangian function:

$$L = \frac{1}{2}m\dot{x_i}^2 - \frac{1}{2}B\epsilon_{ij}\dot{x_i}x_j - \frac{1}{2}k_0x_i^2 - {}^cE, \quad B > 0. \tag{1}$$

It is obvious that the parameter B, which can always be assumed to be positive without loss of generality by a judicious choice of plane orientation, plays the crucial role of the product of the particle's electric charge and the magnetic field. While $\omega_0 = \sqrt{k_0/m} > 0$ represents the angular frequency of the spherically symmetric harmonic potential. The quantity ${}^cE$ denotes an arbitrary choice of zero energy for the classical system. In the context of this study, it is important to note that the minimum energy value is subject to quantum corrections in the quantum system. Indeed, it has been observed that, in the limit $\hbar = 0$, the value of ${}^cE$ is reproduced. For the purposes of this study, it is necessary to introduce the following parameters,

$$\omega = \sqrt{\omega_0^2 + \frac{1}{4}\omega_c^2}, \quad \omega_c = \frac{|B|}{m}, \quad \omega_\pm = \omega \pm \frac{1}{2}\omega_c, \quad m\omega = \sqrt{m^2\omega_0^2 + \frac{1}{4}B^2} = \sqrt{mk_0 + \frac{1}{4}B^2}. \tag{2}$$

In the Heisenberg representation for quantum dynamics, the classical solutions are transformed into quantum expressions for their time dependence. By taking the limit $\hbar = 0$ in the expressions of these latter dependencies, given later, the classical solutions for these dynamics are also obtained. Consequently, these classical solutions are not explicitly provided here.

If the mass is zero, i.e. $m \to 0$, the Lagrange function previously referenced becomes,

$$L_{NC} = -\frac{1}{2}B\epsilon\dot{x_i}x_j - \frac{1}{2}k_0x_i^2 - {}^cE_0, \quad {}^cE_0 = \lim_{m\to 0} {}^cE. \tag{3}$$

Particularly, the index "NC" is an abbreviation that designates the limit of the system. When this limit is reached, the system undergoes a transformation, resulting in a system whose phase space is of two dimension and corresponds to the Euclidean plane. In the quantum context, however, the structure of the space of quantum states is the non-commutative Euclidean plane.

We consider First, the action in its original form, with $m \neq 0$, and secondly in its Hamiltonian formulation. The conjugate momentum at x_i coordinates are then defined by,

$$p_i = m\dot{x}_i - \frac{1}{2}B\epsilon_{ij}x_j, \qquad \dot{x}_i = \frac{1}{m}\left(p_i + \frac{1}{2}B\epsilon_{ij}x_j\right), \tag{4}$$

leading to the canonical Hamiltonian for the system defined by the construction,

$$H = \dot{x}_i p_i - L. \tag{5}$$

Following substitution, the expression below is obtained for the Hamiltonian. This expression serves as the initial point from which the quantification of the system in Cartesian coordinates can be conducted. The subsequent procedure is well established and is based on the correspondence principle. The phase space is generated by the two pairs of conjugate variables (x_i, p_j). The Hamiltonian function is expressed as follows,

$$\begin{aligned} H &= \frac{1}{m}\left(p_i + \frac{1}{2}B\epsilon_{ij}x_j\right)p_i - \frac{1}{2m}\left(p_i + \frac{1}{2}B\epsilon_{ij}x_j\right)^2 + \frac{1}{2}B\epsilon_{ij}\frac{1}{m}\left(p_i + \frac{1}{2}B\epsilon_{ij}x_j\right)x_j + \\ &\quad + \frac{1}{2}k_0 x_i^2 + {}^cE \\ &= \frac{1}{2m}\left(p_i + \frac{1}{2}B\epsilon_{ij}\right)^2 + \frac{1}{2}k_0 x_i^2 + {}^cE \\ &= \frac{1}{2m}p_i^2 + \frac{1}{2}m\omega^2 x_i - \frac{1}{2}\omega_c\epsilon_{ij}x_i p_j + {}^cE. \end{aligned} \tag{6}$$

The Hamiltonian action associated is then

$$L_1 = \frac{1}{2}\left(\dot{x}_i p_i - x_i\dot{p}_i\right) - \frac{1}{2m}p_i^2 - \frac{1}{2}m\omega^2 x_i^2 + \frac{1}{2}\omega_c\epsilon_{ij}x_i p_j - {}^cE. \tag{7}$$

In the deformed Klauder–Daubechies formulation, the deformed action take the form [1],

$$\begin{aligned} L_1 &= \frac{1}{2}\left(\dot{x}_i p_i - x_i\dot{p}_i\right) - \frac{1}{2m}p_i^2 - \frac{1}{2}m\omega^2 x_i^2 + \frac{1}{2}\omega_c\epsilon_{ij}x_i p_j + \frac{1}{2}i\tau_0\left(\frac{1}{m\omega}\dot{p}_i^2 + m\omega\dot{x}_i^2\right) - {}^cE_{KD}, \\ &\lim_{\tau_0\to 0} {}^cE_{KD} = {}^cE. \end{aligned} \tag{8}$$

It is important to note that, in addition to the introduction of a fundamental time scale τ_0, whose limit $\tau_0 = 0$ is considered in order to reproduce the initial system for both its classical and quantum dynamics, this deformation confers on the coordinates of the phase space of the initial system, noted (x_i, p_i), the nature of coordinates of a configuration space of four dimension, whose phase space is itself of eight dimension. It is also noteworthy that the additional term proportional to the time scale τ_0 is, in effect, equivalent to a kinetic energy term in the extended configuration space (x_i, p_i). However, it is imperative to emphasise that the mass factor, proportional to τ_0, is an imaginary pure positive mass factor. This feature is pivotal to the rigorous definition of the functional integral for the corresponding quantum system [1]. This last remark also illustrates that, within the context of temporal evolution, initial conditions corresponding to purely real values of (x_i, p_i) inevitably result in trajectories in the complexified version of the configuration space variables (x_i, p_i). However, this complexity is only apparent, because the formulation of the dynamics, once deformed, only makes sense when considered in the context of the quantised system and from the point of view of the functional integral. The changes of variables examined in the context of the deformed system, with the help of the conjugate variables and the extended phase space, which is characterised by a dimension of eight, need to be considered in the functional integral, a domain in which they are applicable and have been substantiated [1]. It should be noted that these considerations have already been explained in the literature [1]. This discussion shows how the deformed system leads to a specific quantum spectrum. To this end, the results obtained by considering two limits are compared: $m = 0$ and $\tau_0 = 0$.

We consider now the zero-mass limit of the system, staring by the following terms which contribute to the above action,

$$\frac{1}{2} i\tau_0 \left(\frac{1}{m\omega}\dot{p}_i^2 + m\omega\dot{x}_i^2\right) - \frac{1}{2m}\left(p_i^2 + m^2\omega^2 x_i^2 - B\epsilon_{ij}x_i p_j\right). \tag{9}$$

This expression can be rewritten with a factor of $m\omega$ in the first term and a square in the second term as,

$$\frac{1}{2}\frac{i\tau_0}{m\omega}\left(\dot{p}_i^2 + m^2\omega^2\dot{x}_i^2\right) - \frac{1}{2m}\left[\left(p_i + \frac{1}{2}B\epsilon_{ij}x_j\right)^2 + mk_0 x_i^2\right]. \tag{10}$$

In the zero mass limit, the consideration of configurations is constrained to the space defined by the following condition,

$$p_i = -\frac{1}{2}B\epsilon_{ij}x_j, \tag{11}$$

leading to the following expression,

$$\dot{p}_i^2 + m^2\omega^2\dot{x}_i^2 = \frac{1}{4}B^2\dot{x}_i^2 + \left(mk_0 + \frac{1}{4}B^2\right)\dot{x}_i^2 = \frac{1}{2}B^2\dot{x}_i^2 + mk_0\dot{x}_i^2. \tag{12}$$

With this restriction, we have in limit $m \to 0$,

$$\begin{aligned} L_{KD}^{NC} &= \frac{1}{2}\frac{i\tau_0}{\frac{1}{2}B}\frac{1}{2}B^2\dot{x}_i^2 - \frac{1}{2}k_0 x_i^2 + \frac{1}{2}\left(-\frac{1}{2}B\epsilon_{ij}\dot{x}_i x_j + \frac{1}{2}B\epsilon_{ij}x_i\dot{x}_j\right) - {}^cE_{KD}^{NC} \\ &= \frac{1}{2}i\tau_0 B\dot{x}_i^2 - \frac{1}{2}B\epsilon_{ij}\dot{x}_i x_j - \frac{1}{2}k_0 x_i^2 - {}^cE_{KD}^{NC}, \qquad {}^cE_{KD}^{NC} = \lim_{m\to 0} {}^cE_{KD}. \end{aligned} \tag{13}$$

When zero mass limit is satisfied, that of $\tau_0 \to 0$ gives

$$L_{NC} = -\frac{1}{2}B\epsilon_{ij}\dot{x}_i x_j - \frac{1}{2}k_0 x_i^2 + {}^cE_{NC}, \qquad {}^cE_{NC} = \lim_{\tau_0\to 0} E_{KD}^{NC}. \tag{14}$$

In each case, with regard to the time dependencies of these degrees of freedom, the classical solutions are obtained from the quantum solutions in the Heisenberg image with the help of the classical limit, $\hbar = 0$.

3 Quantification of the harmonic Landau Problem.

In this section, we once again consider the function below, which describes the dynamics of a particle of mass m in a magnetic field transverse to the plane of its motion.

$$L = \frac{1}{2}m\dot{x}_i^2 - \frac{1}{2}B\epsilon_{ij}\dot{x}_i x_j - \frac{1}{2}k_0 x_i^2 - {}^cE. \tag{15}$$

The conjugate momentums and velocities expressed in terms of conjugate momentums and conjugate coordinates are given by the following relationships,

$$p_i = m\dot{x}_i - \frac{1}{2}B\epsilon_{ij}x_j, \qquad \dot{x}_i = \frac{1}{m}\left(p_i + \frac{1}{2}B\epsilon_{ij}x_j\right). \tag{16}$$

The canonic Hamiltonian is

$$\begin{aligned} H &= \frac{1}{2m}\left(p_i + \frac{1}{2}B\epsilon_{ij}x_j\right)^2 + \frac{1}{2}k_0 x_i^2 + {}^cE \\ &= \frac{1}{2m}p_i^2 + \frac{1}{2}m\omega^2 x_i^2 - \frac{1}{2}\omega_c\epsilon_{ij}x_i p_j + {}^cE. \end{aligned} \tag{17}$$

It is important to acknowledge that the quantum system is defined by the Hamiltonian in its quantum version, which corresponds to an abstract operator acting on an abstract Hilbert vector space. This vector space is a representation of the following Heisenberg algebra,

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}\mathbb{I}. \tag{18}$$

Given this algebraic structure of operators, let us first consider a first Euclidean Fock algebra, $i, j = 1, 2,$, defined,

$$a_i = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_i + \frac{i}{m\omega}\hat{p}_i\right), \qquad a_i^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_i - \frac{i}{m\omega}\hat{p}_i\right), \tag{19}$$

$$\left[a_i, a_j^\dagger\right] = \delta_{ij}\mathbb{I}, \tag{20}$$

the inverse relations being,

$$\hat{x}_i = \sqrt{\frac{\hbar}{2m\omega}}\left(a_i + a_i^\dagger\right), \quad \hat{p}_i = -im\omega\sqrt{\frac{\hbar}{2m\omega}}\left(a_i - a_i^\dagger\right), \tag{21}$$

with, in this case,

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}\mathbb{I}. \tag{22}$$

Substituting (21) into (17), we get

$$\hat{H} = \hbar\omega\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right) + \frac{1}{2}i\hbar\omega_c\left(a_1^\dagger a_2 - a_2^\dagger a_1\right) - {}^qE, \tag{23}$$

where ${}^qE$ is a quantum equivalent of the parameter ${}^cE$, namely a value of energy involving in the value of fundamental state energy of the quantum system, enable to differ from the classical value with some corrections in $\hbar$ and such as we get the correspondence $\lim_{\hbar\to 0}{}^qE = {}^cE$.

We now consider the helicity Fock algebra which diagonalise the Hamiltonian (23),

$$a_\pm = \frac{1}{\sqrt{2}}(a_1 \mp ia_2), \qquad a_\pm^\dagger = \frac{1}{\sqrt{2}}(a_1^\dagger \pm ia_2^\dagger), \tag{24}$$

these operators are such as,

$$\left[a_\pm, a_\pm^\dagger\right] = \mathbb{I}, \quad \left[a_\pm, a_\mp^\dagger\right] = 0. \tag{25}$$

Then, we have

$$\begin{aligned} \hat{H} &= \hbar\omega\left(a_+^\dagger a_+ + a_-^\dagger a_- + 1\right) - \frac{1}{2}i\hbar\omega_c\left(a_+^\dagger a_+ - a_-^\dagger a_-\right) - {}^qE \\ &= \hbar\omega_-\left(a_+^\dagger a_+ + \frac{1}{2}\right) + \hbar\omega_+\left(a_-^\dagger a_- + \frac{1}{2}\right) - {}^qE. \end{aligned} \tag{26}$$

When we inverse these relations, the operators $\hat{x}_i$ and $\hat{p}_i$ can be written in terms of the helicity Fock operators such as,

$$\begin{aligned} \hat{x}_1 &= \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\left(a_+ + a_- + a_+^\dagger + a_-^\dagger\right), \quad \hat{x}_2 = \frac{i}{2}\sqrt{\frac{\hbar}{m\omega}}\left(a_+ - a_- - a_+^\dagger + a_-^\dagger\right), \\ \hat{p}_1 &= -\frac{im\omega}{2}\sqrt{\frac{\hbar}{m\omega}}\left(a_+ + a_- - a_+^\dagger - a_-^\dagger\right), \quad \hat{p}_2 = \frac{m\omega}{2}\sqrt{\frac{\hbar}{m\omega}}\left(a_+ - a_- + a_+^\dagger - a_-^\dagger\right). \end{aligned} \tag{27}$$

3.1 Heisenberg representation

In the Heisenberg representation, the time dependence of the quantum system is described by that of the operators. The Schrödinger equation is satisfied by the operators $\hat{A}(t)$ in the following form,

$$i\hbar\frac{d\hat{A}(t)}{dt} = \left[\hat{A}(t), \hat{H}\right], \tag{28}$$

The initial value of this time-dependent first-order differential equation, namely the operator $\hat{A}(t=0)$ (where $t=0$ is the reference time), is the operator whose switching rules are those given by the correspondence principle in the canonical quantisation considered in this chapter. In the limit $\hbar = 0$, it is these operators in the image or the Heisenberg representation that correspond to the classical solutions of the associated Hamiltonian equations.

In the present case, all the degrees of freedom of the system are constructed in terms of the helicity Fock operators whose commutation rules have been specified and established above. Therefore, the time dependence of these Fock operators is sufficient to find that of any composite operator expressed in terms of these Fock operators.

However, The time dependence of these operators is obtained in the following form,

$$\hat{A}(t) = e^{\frac{i}{\hbar}t\hat{H}}\,\hat{A}(t=0)\,e^{-\frac{i}{\hbar}t\hat{H}}. \tag{29}$$

In accordance with the Baker-Campbell-Hausdorff formula, it follows the time dependence for the helicity Fock algebra, which is expressed in terms of the phase factors associated with the angular eigenfrequencies to each helicity mode,

$$a_\pm \longrightarrow a_\pm e^{-i\omega_\mp t}, \qquad a_\pm^\dagger \longrightarrow a_\pm^\dagger e^{-i\omega_\mp t}, \tag{30}$$

which can be established immediately from the expression of the Hamiltonian in (26).

3.2 Zero mass limit

In order to find the zero mass limit of the aforementioned system, it is relevant to take into account the two angular frequencies for the two helicity modes according to the following representation,

$$\begin{aligned}\omega_\pm &= \sqrt{\frac{k_0}{m} + \frac{1}{4}\frac{B^2}{m^2}} \pm \frac{1}{2}\frac{B}{m}\\ &= \frac{1}{2}\frac{B}{m}\left[\sqrt{1 + 4m\frac{k_0}{B^2}} \pm 1\right]\\ &= \frac{1}{2}\frac{B}{m}\frac{\left(1 + 4m\frac{k_0}{B^2}\right) - 1}{\sqrt{1 + 4m\frac{k_0}{B^2}} \mp 1}\\ &= \frac{2\frac{k_0}{B}}{\sqrt{1 + 4m\frac{k_0}{B^2}} \mp 1}.\end{aligned} \tag{31}$$

In view of the development carried out and the final result obtained, we have the following boundaries,

$$\lim_{m\to 0}\omega_+ = \lim_{m\to 0}\frac{B}{m} = \infty, \qquad \lim_{m\to 0}\omega_- = \frac{k_0}{B} = \frac{m}{B}\frac{k_0}{m} = \frac{\omega_0^2}{\omega_c}. \tag{32}$$

It is important to note that the expressions (32) will also be utilised in the remainder of this work; however, the results will only be provided without a demonstration. ${}^{q}E$ should be chosen so that the Landau sector of the lowest energy remains up to the limit.

$$ {}^{q}E = \frac{1}{2}\hbar\omega_{+} - \Delta E, \quad \lim_{m\to 0} \Delta E \text{ est fini}, \quad \lim_{m\to 0} \Delta E = \Delta E_{NC}. \tag{33}$$

In terms of Fock states, the following set of orthonormal quantum states can be derived as a consequence of the repeated action of creators $a_{\pm}^{\dagger}$,

$$ | \, n_{+}, n_{-} \rangle_{L}, \tag{34}$$

which is the lowest energy Landau sector then corresponds to the subspace generated by the following Fock states,

$$ | \, n_{+}, n_{-} = 0 \rangle_{L}. \tag{35}$$

The corresponding projector is therefore also expressed in the form,

$$ \mathbb{P}_{NC} = \sum_{n_{+}=0}^{\infty} | \, n_{+}, n_{-} = 0 \rangle_{LL} \langle n_{+}, n_{-} = 0 \, |, \tag{36}$$

implying for the projected Hamiltonian,

$$ \bar{H} = \mathbb{P}_{NC} \hat{H} \mathbb{P}_{NC} = \hbar\omega_{-} \left(a_{+}^{\dagger} a_{+} + \frac{1}{2} \right) + \Delta E, \tag{37}$$

so that,

$$ \lim_{m\to 0} \bar{H} = \frac{k_0}{B} \left(a_{+}^{\dagger} a_{+} + \frac{1}{2} \right) + \Delta E_{NC}. \tag{38}$$

3.3 Projected coordinate

Using the projector on the lowest Landau sector, the plan coordinates are also projected in the following form,,

$$ \bar{x}_1 = \frac{1}{2}\sqrt{\frac{2\hbar}{B}} \left[\bar{a}_{+} + \bar{a}_{+} \right], \quad \bar{x}_2 = \frac{i}{2}\sqrt{\frac{2\hbar}{B}} \left[\bar{a}_{+} - \bar{a}_{+} \right]. \tag{39}$$

The expressions (39) can be written as

$$ \bar{x}_1 = \sqrt{\frac{\hbar}{2B}} \left[\bar{a}_{+} + \bar{a}_{+} \right], \quad \bar{x}_2 = i\sqrt{\frac{\hbar}{2B}} \left[\bar{a}_{+} - \bar{a}_{+} \right], \tag{40}$$

but their algebras are,

$$ [\bar{x}_1, \bar{x}_2] = i\frac{\hbar}{2B}(-2) = -\frac{i\hbar}{B}, \tag{41}$$

a result applicable exclusively to the Hilbert subspace corresponding to the first Landau sector, on which the projection has been done (It is imperative to recognise that, in instances where necessary, the PNC projector defined above should be incorporated, thereby encompassing a contribution that is implicit in these subsequent expressions). It is important to note that, once projected onto this subspace, the Cartesian coordinates of the Euclidean plane result in non-commutative Cartesian coordinates of the Euclidean plane.

4 Quantum non-commutative plane.

This result is merely a consequence of the Lagrange function previously considered (15), in the limit $m \to 0$, which can be reduced to the following form

$$L_{NC} = -\frac{1}{2}B\epsilon_{ij}\dot{x}_i x_j - \frac{1}{2}k_0 x_i^2 - {}^cE_0, \quad {}^cE_0 = \lim_{m\to 0} {}^cE. \tag{42}$$

From the previous Lagrange function, the conjugate momentums become

$$p_i = -\frac{1}{2}B\epsilon_{ij}x_j, \qquad x_i = \frac{2}{B}\epsilon_{ij}p_j, \tag{43}$$

showing the existence of primary constraints whose the expressions are given by,

$$\phi_i = p_i + \frac{1}{2}B\epsilon_{ij}x_j \tag{44}$$

Brackets Poisson algebra of this of primary constraints is,

$$\begin{aligned} \{\phi_i, \phi_j\} &= \{p_i + \frac{1}{2}B\epsilon_{ik}x_k, p_j + \frac{1}{2}B\epsilon_{jl}x_l\}, \\ &= \frac{1}{2}B\epsilon_{ij} + \frac{1}{2}B\epsilon_{ij} = B\epsilon_{ij}, \end{aligned} \tag{45}$$

which shows that they are indeed second class, and can therefore be eliminated by the introduction of the corresponding Dirac brackets.

5 Canonical Hamiltonian.

We now propose to develop the canonical Hamiltonian formulation of our system. The canonical Hamiltonian of the system is established as follows,

$$\begin{aligned} H_0 &= \dot{x}_i p_i + \frac{1}{2}B\epsilon_{ij}\dot{x}_i x_j + \frac{1}{2}k_0 x_i^2 + {}^cE_0 \\ &= \frac{1}{2}k_0 x_i^2 + {}^cE_0 \end{aligned} \tag{46}$$

Consequently, the primary Hamiltonian, which now includes the two primary constraints, is defined as follows,

$$H_1 = \frac{1}{2}k_0 x_i^2 + E_0 + u_i\left(p_i + \frac{1}{2}B\epsilon_{ij}x_j\right), \tag{47}$$

where $u_i(t)$ are two arbitrary functions which will be fixed by imposing that the primary constraints are stable under the temporal evolution generated by the Hamiltonian H_1. This temporal dependence is associated to the computed Poisson brackets of these primary constraints with the primary Hamiltonian H_1, leading in this case to equations for the functions u_i which are thus especially determined in the following form,

$$\{\phi, H_1\} = -k_0 x_i + B\epsilon_{ij}u_j, \quad B\epsilon_{ij}u_j = k_0 x_i, \quad u_i = -\frac{k_0}{B}\epsilon_{ij}x_j. \tag{48}$$

Consequently, for the Dirac brackets computation, we have primary constraints for the Poisson bracket matrix,

$$\Delta_{ij} = \{\phi_i, \phi_j\}, \tag{49}$$

$$\Delta = B \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{50}$$

while its inverse is given by

$$\Delta^{-1} = B^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{51}$$

Dirac brackets are written then,

$$\begin{aligned} \{x_i, x_j\}_D &= -\{x_1, \phi_1\} (\Delta)_{12} \{\phi_2, x_2\} \\ &= -1(-B)^{-1}(-1) = -B^{-1}, \end{aligned} \tag{52}$$

soit,

$$\{x_i, x_j\}_D = -B\epsilon_{ij} = -\frac{\epsilon_{ij}}{B}. \tag{53}$$

To summarise, now using the Dirac brackets and solving the second-class primary constraints to eliminate the p_i variables, we obtain,

$$\{x_i, x_j\} = -\frac{\epsilon_{ij}}{B}, \qquad H = \frac{1}{2}k_0 x_i^2 + {}^cE_0. \tag{54}$$

6 Quantified system.

In this section, the canonical quantization of the Hamiltonian is undertaken by introducing the Fock algebra associated with this choice of degrees of freedom of the system. At the quantum level, this is expressed in terms of operators as

$$[\bar{x}_i, \bar{x}_j] = -\frac{i\hbar}{B}\epsilon_{ij}, \qquad \bar{H} = \frac{1}{2}k_0\bar{x}_i^2 + {}^qE_0, \qquad {}^cE_0 = \lim_{m\to 0} {}^cE. \tag{55}$$

The following terms are introduced here,

$$\bar{x}_1 = \sqrt{\frac{\hbar}{2B}}\left(\bar{a}_+ + \bar{a}_+^\dagger\right), \quad \bar{x}_2 = i\sqrt{\frac{\hbar}{2B}}\left(\bar{a}_+ - \bar{a}_+^\dagger\right), \quad \left[\bar{a}_+, \bar{a}_+^\dagger\right] = \mathbb{I}. \tag{56}$$

Expressions corresponding to inverses are as follows,

$$\bar{a}_+ = \sqrt{\frac{B}{2\hbar}}\left[\bar{x}_1 - i\bar{x}_2\right], \quad \bar{a}_+^\dagger = \sqrt{\frac{B}{2\hbar}}\left[\bar{x}_1 + i\bar{x}_2\right], \quad [\bar{x}_1, \bar{x}_2] = -\frac{i\hbar}{B}\mathbb{I} \tag{57}$$

$$\begin{aligned} \bar{x}_i^2 &= \frac{\hbar}{2B}\left[\left(\bar{a}_+ + \bar{a}_+^\dagger\right) - \left(\bar{a}_+ - \bar{a}_+^\dagger\right)\right] \\ &= \frac{\hbar}{B}\left[\bar{a}_+\bar{a}_+^\dagger + \bar{a}_+^\dagger\bar{a}_+\right] \\ &= \frac{2\hbar}{B}\left[\bar{a}_+^\dagger\bar{a}_+ + \frac{1}{2}\right]. \end{aligned} \tag{58}$$

Finallly, the previous Hamiltonian becomes

$$\bar{H} = \hbar\frac{k_0}{B}\left[\bar{a}_+^\dagger\bar{a}_+ + \frac{1}{2}\right] + {}^qE_0. \tag{59}$$

Consequently, the system and the spectrum of the previous system quantized before taking the limit $m = 0$ are recovered, provided that an appropriate choice is made for ${}^qE_0$,

$${}^qE_0 = \Delta E_{NC}. \tag{60}$$

7 Klauder–Daubechies deformation analysis.

In the context of helicity parametrisation, Landau's spherically symmetric harmonic problem consists of the superposition, that means, in quantised system, the tensor product of the two of ordinary quantum harmonic oscillators of angular frequencies ω_- and ω_+. The following Klauder-Daubechies deformed Landau action is to be considered

$$L_{KD} = \frac{1}{2}(\dot{x}_i p_i - x_i \dot{p}_i) - \frac{1}{2m}p_i^2 - \frac{1}{2}m\omega^2 x_i^2 + \frac{1}{2}\omega_c \epsilon_{ij} x_i p_j + \frac{1}{2} i\tau_0 \left(\frac{1}{m\omega}\dot{p}_i^2 + m\omega \dot{x}_i^2\right) - {}^cE_{KD}. \quad (61)$$

We will apply a canonical transformation defined by the following relations,

$$Q_\pm = \sqrt{\frac{\omega}{2\omega_\mp}}\left[x_1 \pm \frac{1}{m\omega}p_2\right], \quad P_\pm = \mp m\omega_\mp \sqrt{\frac{\omega}{2\omega_\mp}}\left[x_2 \mp \frac{1}{m\omega}p_1\right] \quad (62)$$

The inverses resulting from (62) are given by,

$$x_1 = \sqrt{\frac{\omega_-}{2\omega}}Q_+ + \sqrt{\frac{\omega_+}{2\omega}}Q_-, \quad p_2 = m\omega\sqrt{\frac{\omega_-}{2\omega}}Q_+ - m\omega\sqrt{\frac{\omega_+}{2\omega}}Q_-,$$
$$x_2 = -\frac{1}{m\omega_-}\sqrt{\frac{\omega_-}{2\omega}}P_+ + \frac{1}{m\omega_+}\sqrt{\frac{\omega_+}{2\omega}}P_-, \quad p_1 = \frac{\omega}{\omega_-}\sqrt{\frac{\omega_-}{2\omega}}P_+ + \frac{\omega}{\omega_+}\sqrt{\frac{\omega_+}{2\omega}}P_-. \quad (63)$$

Below, we will replace the relations (63) in each term contributing to the deformed Klauder-Daubechies action for the Landau problem given in (61). An immediate computation then allows us to find,

$$\dot{x}_i p_i - x_i \dot{p}_i = \dot{Q}_+ P_+ - Q_+ \dot{P}_+ + \dot{Q}_- P_- - Q_- \dot{P}_-, \quad (64)$$

on the one hand,

$$\frac{1}{m\omega}\dot{p}_i^2 + m\omega \dot{x}_i^2 = \frac{1}{m\omega_-}\dot{P}_+ + \frac{1}{m\omega_+}\dot{P}_-^2 + m\omega_- \dot{Q}_+^2 + m\omega_+ \dot{Q}_-^2 \quad (65)$$

and on the other hand,

$$\frac{1}{2m}p_i^2 + \frac{1}{2}m\omega^2 x_i^2 - \frac{1}{2}\omega_c \epsilon_{ij} x_i p_j = \frac{1}{2m}P_+^2 + \frac{1}{2}m\omega_-^2 Q_+^2 + \frac{1}{2m}P_-^2 + \frac{1}{2}m\omega_+^2 Q_-^2. \quad (66)$$

Consequently, the deformed Klauder-Daubechies action for this Landau problem is therefore expressed as follows,

$$\begin{aligned} L_{KD} &= \frac{1}{2}(\dot{Q}_+ P_+ - Q_+ \dot{P}_+) + \frac{1}{2}(\dot{Q}_- P_- - Q_- \dot{P}_-) - \\ &- \left[\frac{1}{2m}P_+^2 + \frac{1}{2}m\omega_-^2 Q_+^2 + \frac{1}{2m}P_-^2 + \frac{1}{2}m\omega_+^2 Q_-^2\right] + \\ &+ \frac{1}{2} i\tau_0 \left[\frac{1}{m\omega_-}\dot{P}_+^2 + m\omega_- \dot{Q}_+^2 + \frac{1}{m\omega_+}\dot{P}_-^2 + m\omega_+ \dot{Q}_-^2\right] - {}^cE_{KD}. \end{aligned} \quad (67)$$

We can now introduce the following quantities, $\phi_\pm^a$ with $a = 1, 2$ defined by,

$$\phi_\pm^1 = \frac{1}{\sqrt{m\omega_\mp}}P_\pm, \quad \phi_\pm^2 = \sqrt{m\omega_\mp}Q_\pm, \quad (68)$$

or explicitly,

$$\phi_+^1 = \frac{1}{\sqrt{m\omega_-}}P_+, \quad \phi_+^2 = \sqrt{m\omega_-}Q_+, \quad \phi_-^1 = \frac{1}{\sqrt{m\omega_+}}P_-, \quad \phi_-^2 = \sqrt{m\omega_+}Q_-. \quad (69)$$

When we substitute (68) into the Lagrange function (67), we get,

$$\begin{aligned} L_{KD} &= \frac{1}{2}\left(\dot{\phi}_+^2\phi_+^1 - \phi_+^2\dot{\phi}_+^1\right) + \frac{1}{2}\left(\dot{\phi}_-^2\phi_-^1 - \phi_-^2\dot{\phi}_-^1\right) - \\ &- \left(\frac{1}{2}\omega_-\left(\phi_+^1\phi_+^1 + \phi_+^2\phi_+^2\right) + \frac{1}{2}\omega_+\left(\phi_-^1\phi_-^1 + \phi_-^2\phi_-^2\right) + {}^cE_{KD}\right) \\ &+ \frac{1}{2}i\tau_0\left(\dot{\phi}_+^1\dot{\phi}_+^1 + \dot{\phi}_+^2\dot{\phi}_+^2 + \dot{\phi}_-^1\dot{\phi}_-^1 + \dot{\phi}_-^2\dot{\phi}_-^2\right). \end{aligned} \tag{70}$$

Now the invariant symmetric tensor δ_{ab} and the invariant antisymmetric symbols ϵ_{ab} with $\epsilon_{12} = +1$ can be introduced. Then Lagrange function on the configuration space $\phi_\pm^a$ in the Klauder–Daubechies deformation becomes,

$$\begin{aligned} L_{KD} &= \frac{1}{2}i\tau_0\delta_{ab}\left(\dot{\phi}_+^a\dot{\phi}_+^b + \dot{\phi}_-^a\dot{\phi}_-^b\right) - \\ &- \frac{1}{2}\epsilon_{ab}\dot{\phi}_+^a\phi_+^b - \frac{1}{2}\epsilon_{ab}\dot{\phi}_-^a\phi_-^b - \left(\frac{1}{2}\omega_-\delta_{ab}\phi_+^a\phi_+^b + \frac{1}{2}\omega_+\delta_{ab}\phi_-^a\phi_-^b + {}^cE_{KD}\right). \end{aligned} \tag{71}$$

The conjugate momentums of the extended system with $\phi_\pm^a$ variables are then,

$$p_+^a = \frac{\partial L_{KD}}{\partial\dot{\phi}_+^a} = i\tau_0\delta_{ab}\dot{\phi}_+^b - \frac{1}{2}\epsilon_{ab}\phi_+^b, \quad p_-^b = \frac{\partial L_{KD}}{\partial\dot{\phi}_-^b} = i\tau_0\delta_{ab}\dot{\phi}_-^a - \frac{1}{2}\epsilon_{ab}\phi_-^a. \tag{72}$$

Consequently, we have,

$$\dot{\phi_+^a} = \frac{1}{i\tau_0}\delta^{ab}\left(p_+^b + \frac{1}{2}\epsilon_{bc}\phi_+^c\right), \quad \dot{\phi_-^b} = \frac{1}{i\tau_0}\delta^{ab}\left(p_-^a + \frac{1}{2}\epsilon_{ad}\phi_-^d\right), \tag{73}$$

but the canonical variables of phase-space are defined as,

$$\left\{\phi_+^a,\ \phi_+^b\right\} = 0, \qquad \left\{\phi_+^a,\ p_+^b\right\} = \delta^{ab}, \qquad \left\{p_+^a,\ p_+^b\right\} = 0, \tag{74}$$

$$\left\{\phi_-^a,\ \phi_-^b\right\} = 0, \qquad \left\{\phi_-^a,\ p_-^b\right\} = \delta^{ab}, \qquad \left\{p_-^a,\ p_-^b\right\} = 0. \tag{75}$$

The effective canonical Hamiltonian of the extended system in the compactness form can be written in the phase-space as,

$$H = \dot{\phi_+^a}p_+^a + \dot{\phi_-^a}p_-^a - L_{KD}, \tag{76}$$

After substitution, we get a bilinear canonical Hamiltonian in $\phi_\pm^a$ and $p_\pm^a$ below,

$$\begin{aligned} H &= \frac{1}{2i\tau_0}\delta^{ab}\left(p_+^a + \frac{1}{2}\epsilon_{ac}\phi_+^c\right)\left(p_+^b + \frac{1}{2}\epsilon_{bd}\phi_+^d\right) + \frac{1}{2i\tau_0}\delta^{ab}\left(p_-^a + \frac{1}{2}\epsilon_{ac}\phi_-^c\right)\left(p_-^b + \frac{1}{2}\epsilon_{bd}\phi_-^d\right) \\ &+ \left(\frac{1}{2}\omega_-\delta_{ab}\phi_+^a\phi_+^b + \frac{1}{2}\omega_+\delta_{ab}\phi_-^a\phi_-^b + {}^cE_{KD}\right). \end{aligned} \tag{77}$$

In this form, the quantum spectrum of the extended system in the presence of the imaginary mass $i\tau_0$ will be solved. The results (72) will then be utilised to construct explicitly the Hamiltonian.

From the expression (71), we deduce the conjugate momentums

$$p_+^1 = \frac{\partial L_{KD}}{\partial\dot{\phi}_+^1} = i\tau_0\dot{\phi}_+^1 - \frac{1}{2}\phi_+^2, \qquad p_+^2 = \frac{\partial L_{KD}}{\partial\dot{\phi}_+^2} = i\tau_0\dot{\phi}_+^2 + \frac{1}{2}\phi_+^1,$$

$$p_-^1 = \frac{\partial L_{KD}}{\partial \dot{\phi}_-^1} = i\tau_0 \dot{\phi}_-^1 - \frac{1}{2}\phi_-^2, \qquad p_-^2 = \frac{\partial L_{KD}}{\partial \dot{\phi}_-^2} = i\tau_0 \dot{\phi}_-^2 + \frac{1}{2}\phi_-^1, \tag{78}$$

whose the inverse relations give,

$$\begin{aligned} \dot{\phi}_+^1 &= \frac{1}{i\tau_0}\left(p_+^1 + \frac{1}{2}\phi_+^2\right), \qquad \dot{\phi}_+^2 = \frac{1}{i\tau_0}\left(p_+^2 - \frac{1}{2}\phi_+^1\right), \\ \dot{\phi}_-^1 &= \frac{1}{i\tau_0}\left(p_-^1 + \frac{1}{2}\phi_-^2\right), \qquad \dot{\phi}_-^2 = \frac{1}{i\tau_0}\left(p_-^2 - \frac{1}{2}\phi_-^1\right). \end{aligned} \tag{79}$$

Consider the following canonical Hamiltonian under the explicit form

$$H = \dot{\phi}_+^1 p_+^1 + \dot{\phi}_+^2 p_+^2 + \dot{\phi}_-^1 p_-^1 + \dot{\phi}_-^2 p_-^2 - L_{KD}, \tag{80}$$

when we substitute (79) into the equation (80), we get after simplifications

$$\begin{aligned} H &= \frac{1}{2i\tau_0}\left(p_+^1 + \frac{1}{2}\phi_+^2\right)^2 + \frac{1}{2i\tau_0}\left(p_+^2 - \frac{1}{2}\phi_+^1\right)^2 + \frac{1}{2i\tau_0}\left(p_-^1 + \frac{1}{2}\phi_-^2\right)^2 + \frac{1}{2i\tau_0}\left(p_-^2 - \frac{1}{2}\phi_-^1\right)^2 \\ &\quad + \left(\frac{1}{2}\omega_-(\phi_+^1\phi_+^1 + \phi_+^2\phi_+^2) + \frac{1}{2}\omega_+(\phi_-^1\phi_-^1 + \phi_-^2\phi_-^2) + {}^cE_{KD}\right). \end{aligned} \tag{81}$$

This form is analogous to that found in (77), provided that the latter is developed explicitly.

8 Canonical quantification.

In this section, the rules of canonical quantization and the correspondence principle for $\hat{\phi}_\pm^a$ operators and their Hermitian conjugate moments will be verified with ease. Subsequently, the canonical Hamiltonian, defined within phase space, will be employed. This implies that (77) becomes an operator on Hilbert space at the quantum context. This approach has given rise to a very general formulation of the quantisation of classical systems. Let's establish the commutation rules between Hermitian operators for each of the helicity sectors,

$$\left[\hat{\phi}_+^a, \hat{p}_+^b\right] = i\hbar\delta^{ab}, \quad \hat{\phi}_+^{a\dagger} = \hat{\phi}_+^a, \quad \hat{p}_+^{a\dagger} = \hat{p}_+^a, \tag{82}$$

and also the Hamiltonian operator for the extended system,

$$\left[\hat{\phi}_-^a, \hat{p}_-^b\right] = i\hbar\delta^{ab}, \quad \hat{\phi}_-^{a\dagger} = \hat{\phi}_-^a, \quad \hat{p}_-^{a\dagger} = \hat{p}_-^a, \tag{83}$$

$$\begin{aligned} \hat{H} &= \frac{1}{2i\tau_0}\delta^{ab}\left(\hat{p}_+^a + \frac{1}{2}\epsilon_{ac}\hat{\phi}_+^c\right)\left(\hat{p}_+^b + \frac{1}{2}\epsilon_{bd}\hat{\phi}_+^d\right) + \frac{1}{2i\tau_0}\delta^{ab}\left(\hat{p}_-^a + \frac{1}{2}\epsilon_{ac}\hat{\phi}_-^c\right)\left(\hat{p}_-^b + \frac{1}{2}\epsilon_{bd}\hat{\phi}_-^d\right) \\ &\quad + \left(\frac{1}{2}\omega_-\delta_{ab}\hat{\phi}_+^a\hat{\phi}_+^b + \frac{1}{2}\omega_+\delta_{ab}\hat{\phi}_-^a\hat{\phi}_-^b + {}^qE_{KD}\right). \end{aligned} \tag{84}$$

9 Construction of the algebra and diagonalisation of the Hamiltonian.

9.1 Ordinary Fock states of the extended system

Given Hermitian operators $\hat{\phi}_\pm^a$ and $\hat{p}_\pm^a$, we can define the Fock operators,

$$a_+^a = \frac{1}{2\sqrt{\hbar}}\left(\hat{\phi}_+^a + 2i\hat{p}_+^a\right), \qquad a_+^{a\dagger} = \frac{1}{2\sqrt{\hbar}}\left(\hat{\phi}_+^a - 2i\hat{p}_+^a\right), \tag{85}$$

$$a_-^a = \frac{1}{2\sqrt{\hbar}}\left(\hat{\phi}_-^a + 2i\hat{p}_-^a\right), \qquad a_-^{a\dagger} = \frac{1}{2\sqrt{\hbar}}\left(\hat{\phi}_-^a - 2i\hat{p}_-^a\right), \tag{86}$$

which express the tensor product of two Fock algebras of right and left sector, namely,

$$\left[a_+^a, a_+^{b\dagger}\right] = \delta^{ab}\mathbb{I}, \tag{87}$$

$$\left[a_-^a, a_-^{b\dagger}\right] = \delta^{ab}\mathbb{I}. \tag{88}$$

Next, we define the chiral Fock operators of the Fock algebra of the right and left sectors as follows,

$$a_{+,\pm} = \frac{1}{\sqrt{2}}\left(a_+^1 \mp ia_+^2\right), \qquad a_{+,\pm}^\dagger = \frac{1}{\sqrt{2}}\left(a_+^{1\dagger} \pm ia_+^{2\dagger}\right), \tag{89}$$

$$a_{-,\pm} = \frac{1}{\sqrt{2}}\left(a_-^1 \mp ia_-^2\right), \qquad a_{-,\pm}^\dagger = \frac{1}{\sqrt{2}}\left(a_-^{1\dagger} \pm ia_-^{2\dagger}\right), \tag{90}$$

which are such that,

$$\left[a_{+,\pm}, a_{+,\pm}^\dagger\right] = \mathbb{I} \quad , \qquad \left[a_{+,\pm}, a_{+,\mp}^\dagger\right] = 0, \tag{91}$$

$$\left[a_{-,\pm}, a_{-,\pm}^\dagger\right] = \mathbb{I} \quad , \qquad \left[a_{-,\pm}, a_{-,\mp}^\dagger\right] = 0. \tag{92}$$

Inverting these various relations, we obtain the inverse relations of the phase space operators for each right and left helicity sector, given by

$$\begin{aligned}\hat{\phi}_+^1 &= \sqrt{\frac{\hbar}{2}}\left(a_{+,+} + a_{+,-} + a_{+,+}^\dagger + a_{+,-}^\dagger\right), \quad \hat{p}_+^1 = -\frac{i}{2}\sqrt{\frac{\hbar}{2}}\left(a_{+,+} + a_{+,-} - a_{+,+}^\dagger - a_{+,-}^\dagger\right), \\ \hat{\phi}_+^2 &= i\sqrt{\frac{\hbar}{2}}\left(a_{+,+} - a_{+,-} - a_{+,+}^\dagger + a_{+,-}^\dagger\right), \quad \hat{p}_+^2 = \frac{1}{2}\sqrt{\frac{\hbar}{2}}\left(a_{+,+} - a_{+,-} + a_{+,+}^\dagger - a_{+,-}^\dagger\right), \end{aligned} \tag{93}$$

as well as,

$$\begin{aligned}\hat{\phi}_-^1 &= \sqrt{\frac{\hbar}{2}}\left(a_{-,+} + a_{-,-} + a_{-,+}^\dagger + a_{-,-}^\dagger\right), \quad \hat{p}_-^1 = -\frac{i}{2}\sqrt{\frac{\hbar}{2}}\left(a_{-,+} + a_{-,-} - a_{-,+}^\dagger - a_{-,-}^\dagger\right), \\ \hat{\phi}_-^2 &= i\sqrt{\frac{\hbar}{2}}\left(a_{-,+} - a_{-,-} - a_{-,+}^\dagger + a_{-,-}^\dagger\right), \quad \hat{p}_-^2 = \frac{1}{2}\sqrt{\frac{\hbar}{2}}\left(a_{-,+} - a_{-,-} + a_{-,+}^\dagger - a_{-,-}^\dagger\right). \end{aligned} \tag{94}$$

Particularly, we have the following relations,

$$\hat{p}_+^1 + \frac{1}{2}\hat{\phi}_+^2 = -i\sqrt{\frac{\hbar}{2}}\left(a_{+,-} - a_{+,-}^\dagger\right), \qquad \hat{p}_+^2 - \frac{1}{2}\hat{\phi}_+^1 = -\sqrt{\frac{\hbar}{2}}\left(a_{+,-} + a_{+,-}^\dagger\right),$$

$$\hat{p}_-^1 + \frac{1}{2}\hat{\phi}_-^2 = -i\sqrt{\frac{\hbar}{2}}\left(a_{-,-} - a_{-,-}^\dagger\right), \qquad \hat{p}_-^2 - \frac{1}{2}\hat{\phi}_-^1 = -\sqrt{\frac{\hbar}{2}}\left(a_{-,-} + a_{-,-}^\dagger\right). \tag{95}$$

It is evident that each term on the right is accompanied by a normalisation factor. Furthermore, the two pairs of operators plays an analogous role to a Fock or Heisenberg algebra, operating on each of the chiral sectors individually. The abstract construction of these algebraic structures associates a normalised Fock vacuum $|\Omega\rangle$ for the following helicity Fock operators,

$$a_{\pm,\pm}|\Omega\rangle, \qquad a_{\pm,\pm}^\dagger|\Omega\rangle, \qquad \langle\Omega|\Omega\rangle = 1. \tag{96}$$

Next, we define the orthonormalized Fock states of the Hilbert space of the quantum system.

$$\begin{aligned}|n_{+,+},n_{+,-},n_{-,+},n_{-,-};\Omega\rangle &= \frac{1}{\sqrt{n_{+,+}!n_{+,-}!n_{-,+}!n_{-,-}!}}\times \\ &\times\left(a^{\dagger}_{+,+}\right)^{n_{+,+}}\left(a^{\dagger}_{+,-}\right)^{n_{+,-}}\left(a^{\dagger}_{-,+}\right)^{n_{-,+}}\left(a^{\dagger}_{-,-}\right)^{n_{-,-}}|\Omega\rangle,\end{aligned} \tag{97}$$

$$\begin{aligned}&\langle n_{+,+},n_{+,-},n_{-,+},n_{-,-};\Omega|n'_{+,+},n'_{+,-},n'_{-,+}n'_{-,-};\Omega\rangle = \\ =\ &\delta_{n_{+,+},n'_{+,+}}\delta_{n_{+,-},n'_{+,-}}\delta_{n_{-,+},n'_{-,+}}\delta_{n_{-,-},n'_{-,-}},\end{aligned} \tag{98}$$

where $n_{+,+},n_{+,-},n_{-,+},n_{-,-}=0,1,2,\ldots$. Subsequently, the following spectral decomposition is applicable to the unit operator in the chiral Fock basis,

$$\sum_{n_{+,+},n_{+,-},n_{-,+},n_{-,-}=0}^{\infty}|n_{+,+},n_{+,-},n_{-,+},n_{-,-};\Omega\rangle\langle n_{+,+},n_{+,-},n_{-,+},n_{-,-};\Omega|=\mathbb{I}. \tag{99}$$

We will now consider the limit $\tau_0\to 0^+$ (by projections) and observe how projecting the system onto a specific Landau level or sector results in an additional reduction and ultimately leads to the non-commutative Moyal-Voros plane. It is expedient to present the projector in the subspace of the quantum states of the extended system,

$$\begin{aligned}&\mathbb{P}_{0,0}=\sum_{n_{+,+},n_{-,+}=0}^{\infty}|n_{+,+},n_{+,-}=0,n_{-,+},n_{-,-}=0;\Omega\rangle\langle n_{+,+},n_{+,-}=0,n_{-,+},n_{-,-}=0;\Omega| \\ &\mathbb{P}_{0,0}^2=\mathbb{P}_{0,0},\qquad \mathbb{P}_{0,0}^{\dagger}=\mathbb{P}_{0,0}.\end{aligned} \tag{100}$$

The projected operators that yield the Heisenberg algebra in Hilbert space are considered

$$\mathbb{P}_{0,0}\left(\hat{p}^1_+ +\frac{1}{2}\hat{\phi}^2_+\right)\mathbb{P}_{0,0}=0,\qquad \mathbb{P}_{0,0}\left(\hat{p}^2_+ -\frac{1}{2}\hat{\phi}^1_+\right)\mathbb{P}_{0,0}=0,$$

$$\mathbb{P}_{0,0}\left(\hat{p}^1_- +\frac{1}{2}\hat{\phi}^2_-\right)\mathbb{P}_{0,0}=0,\qquad \mathbb{P}_{0,0}\left(\hat{p}^2_- -\frac{1}{2}\hat{\phi}^1_-\right)\mathbb{P}_{0,0}=0. \tag{101}$$

The application of the definitions (100) to the operators of the original phase space $(\phi^a_\pm,p^a_\pm)$ yields the following results,

$$\overline{\phi}^a_\pm=\mathbb{P}_{0,0}\hat{\phi}^a_\pm\mathbb{P}_{0,0},\qquad \overline{p}^a_\pm=\mathbb{P}_{0,0}\hat{p}^a_\pm\mathbb{P}_{0,0}. \tag{102}$$

Explicitly, the expression (102) can be decomposed into a total of eight independent variables, which implies the following results

$$\begin{aligned}\overline{\phi}^1_+ &=\sqrt{\frac{\hbar}{2}}\left(\overline{a}_{+,+}+\overline{a}^{\dagger}_{+,+}\right)\quad,\qquad \overline{p}^1_+=-\frac{i}{2}\sqrt{\frac{\hbar}{2}}\left(\overline{a}_{+,+}-\overline{a}^{\dagger}_{+,+}\right)=-\frac{1}{2}\overline{\phi}^2_+,\\ \overline{\phi}^2_+ &=i\sqrt{\frac{\hbar}{2}}\left(\overline{a}_{+,+}-\overline{a}^{\dagger}_{+,+}\right)\quad,\qquad \overline{p}^2_+=\frac{1}{2}\sqrt{\frac{\hbar}{2}}\left(\overline{a}_{+,+}+\overline{a}^{\dagger}_{+,+}\right)=\frac{1}{2}\overline{\phi}^1_+.\end{aligned} \tag{103}$$

$$\begin{aligned}\overline{\phi}^1_- &=\sqrt{\frac{\hbar}{2}}\left(\overline{a}_{-,+}+\overline{a}^{\dagger}_{-,+}\right)\quad,\qquad \overline{p}^1_-=-\frac{i}{2}\sqrt{\frac{\hbar}{2}}\left(\overline{a}_{-,+}-\overline{a}^{\dagger}_{-,+}\right)=-\frac{1}{2}\overline{\phi}^2_-,\\ \overline{\phi}^2_- &=i\sqrt{\frac{\hbar}{2}}\left(\overline{a}_{-,+}-\overline{a}^{\dagger}_{-,+}\right)\quad,\qquad \overline{p}^2_-=\frac{1}{2}\sqrt{\frac{\hbar}{2}}\left(\overline{a}_{-,+}+\overline{a}^{\dagger}_{-,+}\right)=\frac{1}{2}\overline{\phi}^1_-.\end{aligned} \tag{104}$$

It is observed that, after projection, the only projected coordinates that are independent operators are the following, with commutation relations that are, in fact, those of the Heisenberg algebra,

$$\left[\overline{\phi}_+^a, \overline{\phi}_+^b\right] = -i\hbar\epsilon^{ab}\mathbb{P}_{0,0}, \qquad \left[\overline{\phi}_-^a, \overline{\phi}_-^b\right] = -i\hbar\epsilon^{ab}\mathbb{P}_{0,0}. \tag{105}$$

Indeed, within the projected subspace, we find the quantum Heisenberg algebra of the original system, even though, in the extended Hilbert space corresponding to the phase space, the position operators,$\hat{\phi}_+^a$ and $\hat{\phi}_-^a$, switch between them.

In the two possible limit orders for these two parameters, namely $\tau_0 \to 0$ and $m \to 0$, the operators $\hat{\phi}_\pm^1$ and $\hat{\phi}_\pm^2$ take the following form,

$$\hat{\phi}_\pm^1 = \sqrt{\frac{\hbar}{2}}\left[\alpha_{\pm,+} + \alpha_{\pm,-} + \alpha_{\pm,+}^\dagger + \alpha_{\pm,-}^\dagger\right], \quad \hat{\phi}_\pm^2 = i\sqrt{\frac{\hbar}{2}}\left[\alpha_{\pm,+} - \alpha_{\pm,-} - \alpha_{\pm,+}^\dagger + \alpha_{\pm,-}^\dagger\right]. \tag{106}$$

The projection of (106) into the Hilbert subspace generated by $\mid n_{+,+}, n_{+,-} = 0, n_{-,+} = 0, n_{-,-} = 0; \Omega_{KD}\rangle$, in the two limit orders, we always have,

$$\hat{\phi}_\pm^1 = \sqrt{\frac{\hbar}{2}}\left[\alpha_{\pm,+} + \alpha_{\pm,+}^\dagger\right], \quad \hat{\phi}_\pm^2 = i\sqrt{\frac{\hbar}{2}}\left[\alpha_{\pm,+} - \alpha_{\pm,+}^\dagger\right], \tag{107}$$

with the commutator,

$$\left[\hat{\phi}_\pm^1, \hat{\phi}_\pm^2\right] = -i\hbar. \tag{108}$$

However, it is important to note that the above states do not diagonalise the Hamiltonian $\hat{H}$ in the presence of the spherically symmetric harmonic potential in the case of the Landau problem with two degrees of freedom (84). Consequently, the linear combinations of basic operators, $\hat{\phi}_\pm^a$ et $\hat{p}_\pm^a$, with two specific quantities, namely $R_\pm$ and phases $\varphi_\pm$, are defined as being important for diagonalising the Hamiltonian operator,

$$R_\pm^2 e^{2i\varphi_\pm} = 1 + 4i\omega_\pm\tau_0, \qquad R_\pm > 0, \qquad 0 \le \varphi_\pm < \frac{\pi}{4}. \tag{109}$$

The complex variables $\rho_\pm$ and their conjugates $\bar{\rho}_\pm$ are defined as follows,

$$\rho_\pm = \sqrt{R_\pm}e^{\frac{1}{2}i\varphi_\pm}, \qquad \bar{\rho}_\pm = \sqrt{R_\pm}e^{-\frac{1}{2}i\varphi_\pm}. \tag{110}$$

In the limit where $\tau_0 \to 0^+$ or in the absence of coupling $\omega_\pm$, $R_\pm$ and $\rho_\pm$ are two factors equal to unity, while $\varphi_\pm$ vanishes.

Consider the following quantum operators for each helicity sector, right and left,

$$A_+^a = \frac{1}{2\sqrt{\hbar}}\left(\rho_-\hat{\phi}_+^a + \frac{2i}{\rho_-}\hat{p}_+^a\right), \qquad B_+^a = \frac{1}{2\sqrt{\hbar}}\left(\rho_-\hat{\phi}_+^a - \frac{2i}{\rho_-}\hat{p}_+^a\right), \tag{111}$$

$$A_-^a = \frac{1}{2\sqrt{\hbar}}\left(\rho_+\hat{\phi}_-^a + \frac{2i}{\rho_+}\hat{p}_-^a\right), \qquad B_-^a = \frac{1}{2\sqrt{\hbar}}\left(\rho_+\hat{\phi}_-^a - \frac{2i}{\rho_+}\hat{p}_-^a\right), \tag{112}$$

and their adjoints,

$$A_+^{a\dagger} = \frac{1}{2\sqrt{\hbar}}\left(\overline{\rho}_-\hat{\phi}_+^a - \frac{2i}{\overline{\rho}_-}\hat{p}_+^a\right) \quad , \qquad B_+^{a\dagger} = \frac{1}{2\sqrt{\hbar}}\left(\overline{\rho}_-\hat{\phi}_+^a + \frac{2i}{\overline{\rho}_-}\hat{p}_+^a\right), \tag{113}$$

$$A_-^{a\dagger} = \frac{1}{2\sqrt{\hbar}}\left(\overline{\rho}_+\hat{\phi}_-^a - \frac{2i}{\overline{\rho}_+}\hat{p}_-^a\right) \quad , \qquad B_-^{a\dagger} = \frac{1}{2\sqrt{\hbar}}\left(\overline{\rho}_+\hat{\phi}_-^a + \frac{2i}{\overline{\rho}_+}\hat{p}_-^a\right), \tag{114}$$

such as,

$$\left[A_+^a, B_+^b\right] = \delta^{ab}\mathbb{I} = \left[B_+^{a\dagger}, A_+^{b\dagger}\right], \tag{115}$$

$$\left[A_-^a, B_-^b\right] = \delta^{ab}\mathbb{I} = \left[B_-^{a\dagger}, A_-^{b\dagger}\right]. \tag{116}$$

It is evident that the nature of $\rho_\pm$ as a complex quantity enables the derivation of the operators $\hat{\phi}_+^a$ and $\hat{p}_+^a$ in each of the four relations (111) and (113). Through the subsequent substitution of one operator into the other, the operators A_+^a and B_+^a are expressed, which are deemed to be adjoint operators when $\tau_0 = 0$. Following a series of manipulations, a system of equations is derived, which is expressed as follows,

$$\begin{aligned} A_+^{a\dagger} + B_+^{a\dagger} &= e^{-i\varphi_-}\left(A_+^a + B_+^a\right), \\ A_+^{a\dagger} - B_+^{a\dagger} &= -e^{i\varphi_-}\left(A_+^a - B_+^a\right). \end{aligned} \tag{117}$$

The development process has yielded the following outcomes,

$$\begin{aligned} 2A_+^{a\dagger} &= e^{-i\varphi_-}(A_+^a + B_+^a) - e^{i\varphi_-}(A_+^a - B_+^a) \\ &= (\cos\varphi_- - i\sin\varphi_-)(A_+^a + B_+^a) - (\cos\varphi_- + i\sin\varphi_-)(A_+^a - B_+^a) \\ &= \cos\varphi_-(A_+^a + B_+^a) - i\sin\varphi_-(A_+^a + B_+^a) - \cos\varphi_-(A_+^a - B_+^a) - i\sin\varphi_-(A_+^a - B_+^a) \\ &= 2\cos\varphi_- B_+^a - 2i\sin\varphi_- A_+^a, \end{aligned} \tag{118}$$

either,

$$A_+^{a\dagger} = \cos\varphi_- B_+^a - i\sin\varphi_- A_+^a. \tag{119}$$

In the same way, we deduce $B_+^{a\dagger}$,

$$B_+^{a\dagger} = \cos\varphi_- A_+^a - i\sin\varphi_- B_+^a. \tag{120}$$

Using a similar process to the previous one, the operators A_+^a et B_+^a and their adjoints can be written as linear combinations of $A_+^{a\dagger}$ et $B_+^{a\dagger}$, which are defined as,

$$A_+^a = \cos\varphi_- B_+^{a\dagger} + i\sin\varphi_- A_+^{a\dagger}, \qquad B_+^a = \cos\varphi_- A_+^{a\dagger} + i\sin\varphi_- B_+^{a\dagger}. \tag{121}$$

Moreover, these operators correspond to a_+^a and $a_+^{a\dagger}$ defined above when $\rho_\pm = 1$, which implies that $\omega_\pm\tau_0 = 0$.

Similarly, we start from the equations (112) and (114), from which $A_-^{a\dagger}$ and $B_-^{a\dagger}$ will be deduced, employing the same computation method as previously outlined. Once again, we have a system of equations,

$$\begin{aligned} A_-^{a\dagger} + B_-^{a\dagger} &= e^{-i\varphi_+}\left(A_-^a + B_-^a\right), \\ A_-^{a\dagger} - B_-^{a\dagger} &= -e^{-i\varphi_+}\left(A_-^a - B_-^a\right), \end{aligned} \tag{122}$$

whose solution is,

$$A_-^{a\dagger} = \cos\varphi_+ B_-^a - i\sin\varphi_+ A_-^a, \qquad B_-^{a\dagger} = \cos\varphi_+ A_-^a - i\sin\varphi_+ B_-^a. \tag{123}$$

We can now express operators A_-^a and B_-^a and their adjoints in terms of linear combination of $A_-^{a\dagger}$ and $B_-^{a\dagger}$ as follow,

$$A_-^a = \cos\varphi_+ B_-^{a\dagger} + i\sin\varphi_+ A_-^{a\dagger}, \qquad B_-^a = \cos\varphi_+ A_-^{a\dagger} + i\sin\varphi_+ B_-^{a\dagger}. \tag{124}$$

Let's introduce the following Fock helicite combinations,

$$A_{+,\pm} = \frac{1}{\sqrt{2}}(A_+^1 \mp iA_+^2), \qquad B_{+,\pm} = \frac{1}{\sqrt{2}}(B_+^1 \pm iB_+^2),$$
$$A_{-,\pm} = \frac{1}{\sqrt{2}}(A_-^1 \mp iA_-^2), \qquad B_{-,\pm} = \frac{1}{\sqrt{2}}(B_-^1 \pm iB_-^2), \tag{125}$$

with the adjoint operators,

$$A^\dagger_{+,\pm} = \frac{1}{\sqrt{2}}\left(A_+^{1\dagger} \pm iA_+^{2\dagger}\right), \qquad B^\dagger_{+,\pm} = \frac{1}{\sqrt{2}}\left(B_+^{1\dagger} \mp iB_+^{2\dagger}\right),$$
$$A^\dagger_{-,\pm} = \frac{1}{\sqrt{2}}\left(A_-^{1\dagger} \pm iA_-^{2\dagger}\right), \qquad B^\dagger_{-,\pm} = \frac{1}{\sqrt{2}}\left(B_-^{1\dagger} \mp iB_-^{2\dagger}\right), \tag{126}$$

which are all such that,

$$[A_{+,\pm}, B_{+,\pm}] = \mathbb{I} = \left[B^\dagger_{+,\pm}, A^\dagger_{+,\pm}\right], \tag{127}$$

$$[A_{-,\pm}, B_{-,\pm}] = \mathbb{I} = \left[B^\dagger_{-,\pm}, A^\dagger_{-,\pm}\right], \tag{128}$$

as well as, on the one hand

$$A^\dagger_{+,\pm} = \cos\varphi_- B_{+,\pm} - i\sin\varphi_- A_{+,\mp}, \qquad A_{+,\pm} = \cos\varphi_- B^\dagger_{+,\pm} + i\sin\varphi_- A^\dagger_{+,\mp},$$
$$B^\dagger_{+,\pm} = \cos\varphi_- A_{+,\pm} - i\sin\varphi_- B_{+,\mp}, \qquad B_{+,\pm} = \cos\varphi_- A^\dagger_{+,\pm} + i\sin\varphi_- B^\dagger_{+,\mp}, \tag{129}$$

and the other hand,

$$A^\dagger_{-,\pm} = \cos\varphi_+ B_{-,\pm} - i\sin\varphi_+ A_{-,\mp}, \qquad A_{-,\pm} = \cos\varphi_+ B^\dagger_{-,\pm} + i\sin\varphi_+ A^\dagger_{-,\mp},$$
$$B^\dagger_{-,\pm} = \cos\varphi_+ A_{-,\pm} - i\sin\varphi_+ B_{-,\mp}, \qquad B_{-,\pm} = \cos\varphi_+ A^\dagger_{-,\pm} + i\sin\varphi_+ B^\dagger_{-,\mp}. \tag{130}$$

Expressing these operators in terms of $a_{+,\pm}$ et $a^\dagger_{+,\pm}$, we obtain

$$A_{+,\pm} = \frac{\rho_- + \rho_-^{-1}}{2}a_{+,\pm} + \frac{\rho_- - \rho_-^{-1}}{2}a^\dagger_{+,\mp}, \qquad A^\dagger_{+,\pm} = \frac{\overline{\rho}_- + \overline{\rho}_-^{-1}}{2}a^\dagger_{+,\pm} + \frac{\overline{\rho}_- - \overline{\rho}_-^{-1}}{2}a_{+,\mp},$$
$$B_{+,\pm} = \frac{\rho_- + \rho_-^{-1}}{2}a^\dagger_{+,\pm} + \frac{\rho_- - \rho_-^{-1}}{2}a_{+,\mp}, \qquad B^\dagger_{+,\pm} = \frac{\overline{\rho}_- + \overline{\rho}_-^{-1}}{2}a_{+,\pm} + \frac{\overline{\rho}_- - \overline{\rho}_-^{-1}}{2}a^\dagger_{+,\mp}. \tag{131}$$

Then, Operators involving $a_{-,\pm}$ and $a^\dagger_{-,\pm}$ yield the following results,

$$A_{-,\pm} = \frac{\rho_+ + \rho_+^{-1}}{2}a_{-,\pm} + \frac{\rho_+ - \rho_+^{-1}}{2}a^\dagger_{-,\mp}, \qquad A^\dagger_{-,\pm} = \frac{\overline{\rho}_+ + \overline{\rho}_+^{-1}}{2}a^\dagger_{-,\pm} + \frac{\overline{\rho}_+ - \overline{\rho}_+^{-1}}{2}a_{-,\mp},$$
$$B_{-,\pm} = \frac{\rho_+ + \rho_+^{-1}}{2}a^\dagger_{-,\pm} + \frac{\rho_+ - \rho_+^{-1}}{2}a_{-,\mp}, \qquad B^\dagger_{-,\pm} = \frac{\overline{\rho}_+ + \overline{\rho}_+^{-1}}{2}a_{-,\pm} + \frac{\overline{\rho}_+ - \overline{\rho}_+^{-1}}{2}a^\dagger_{-,\mp}. \tag{132}$$

The Fock algebraic relations in equations (127) and (128) are very similar to those of an ordinary Fock algebra, except that the operators $B_{\pm,\pm}$ and $A_{\pm,\pm}$ (or their adjoints) are not adjoints to each other. However, it is possible to construct a representation theory in the same way, which leads to: States that we will refer to as Fock states of type A and B.

This representation is established on the Fock void A and B, denoted by $|\Omega_A\rangle$ and $|\Omega_B\rangle$, respectively, such that

$$A_{\pm,\pm}|\Omega_A\rangle = 0, \qquad B^\dagger_{\pm,\pm}|\Omega_B\rangle = 0. \tag{133}$$

With the right choice of phases and normalisations, the inner product of these states is always normalised to unit,

$$\langle\Omega_A|\Omega_B\rangle = 1 = \langle\Omega_B|\Omega_A\rangle. \tag{134}$$

Fock states of type A are defined as,

$$\begin{aligned} |N_{+,+}, N_{+,-}, N_{-,+}N_{-,-};\Omega_A\rangle &= \frac{1}{\sqrt{N_{+,+}!N_{+,-}!N_{-,+}!N_{-,-}!}}\times \\ &\times (B_{+,+})^{N_{+,+}}(B_{+,-})^{N_{+,-}}(B_{-,+})^{N_{-,+}}(B_{-,-})^{N_{-,-}}|\Omega_A\rangle, \end{aligned} \tag{135}$$

while Fock states of type B are given by,

$$\begin{aligned} |N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-};\Omega_B\rangle &= \frac{1}{\sqrt{N_{+,+}!N_{+,-}!N_{-,+}!N_{-,-}!}}\times \\ &\times \left(A^\dagger_{+,+}\right)^{N_{+,+}}\left(A^\dagger_{+,-}\right)^{N_{+,-}}\left(A^\dagger_{-,+}\right)^{N_{-,+}}\left(A^\dagger_{-,-}\right)^{N_{-,-}}|\Omega_B\rangle, \end{aligned} \tag{136}$$

where $N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-} = 0, 1, 2, \cdots$. Indeed, given that the operators $A_{+,\pm}$, $B_{+,\pm}$, $A_{-,\pm}$, and $B_{-,\pm}$, as well as their adjoints, are linear combinations of the Fock operators $a_{+,\pm}$, $a^\dagger_{+,\pm}$, $a_{-,\pm}$ and $a^\dagger_{-,\pm}$, it is evident that either set of states, $|N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-};\Omega_A\rangle$ or $|N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-};\Omega_B\rangle$, generates the entire Hilbert space of the extended quantum system. To elaborate, each of these sets provides a basis for the given space. These two bases are dual to each other, as is immediately demonstrated. The following values are thus obtained for the overlaps of the vectors of these two bases,,

$$\begin{aligned} &\langle N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-};\Omega_A|M_{+,+}, M_{+,-}, M_{-,+}, M_{-,-};\Omega_B\rangle = \\ = \ &\delta_{N_{+,+},M_{+,+}}\delta_{N_{+,-},M_{+,-}}\delta_{N_{-,+},M_{-,+}}\delta_{N_{-,-},M_{-,-}}, \end{aligned} \tag{137}$$

$$\begin{aligned} &\langle N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-};\Omega_B|M_{+,+}, M_{+,-}, M_{-,+}, M_{-,-};\Omega_A\rangle = \\ = \ &\delta_{N_{+,+},M_{+,+}}\delta_{N_{+,-},M_{+,-}}\delta_{N_{-,+},M_{-,+}}\delta_{N_{-,-},M_{-,-}}. \end{aligned} \tag{138}$$

Consequently, the following Identity operator resolutions are hereby presented,

$$\begin{aligned} \sum_{N_{+,+},N_{+,-},N_{-,+},N_{-,-}=0}^{\infty} |N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-};\Omega_A\rangle\langle N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-};\Omega_B| = \mathbb{I}, \\ \sum_{N_{+,+},N_{+,-},N_{-,+},N_{-,-}=0}^{\infty} |N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-};\Omega_B\rangle\langle N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-};\Omega_A| = \mathbb{I}. \end{aligned} \tag{139}$$

The three sets of states, $|n_{+,+}, n_{+,-}, n_{-,+}n_{-,-};\Omega\rangle$, $|N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-};\Omega_A\rangle$ and $|N_{+,+}, N_{+,-}, N_{-,+},$ define three different bases of the extended Hilbert space. The $|n_{+,+}, n_{+,-}, n_{-,+}, n_{-,-};\Omega\rangle$ basis is self-dual since it is orthonormalised, while the other two bases are dual to each other. This duality structure is referred to as a bi-module in the mathematical literature.

It is evident that the action of operators $A_{\pm,\pm}$ and $B_{\pm,\pm}$ on Fock states of type A, as well as the action of the operators $B^\dagger_{\pm,\pm}$ and $A^\dagger_{\pm,\pm}$ on Fock states of type B, is similar to that of the ordinary Fock operators of creation and annihilation on ordinary Fock states. Particularly, the Fock states of type A, namely $|N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-};\Omega_A\rangle$ (resp., the Fock states of type B, namely $|N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-};\Omega_B\rangle$) are eigenstates of the operators $B_{\pm,\pm}A_{\pm,\pm}$ (resp., $A^\dagger_{\pm,\pm}B^\dagger_{\pm,\pm}$) with as eigenvalues, the natural numbers $N_{\pm,\pm}$.

Consider the identities (131) and (132) linking the various Fock-type operators. These relationships enable transformation expressions between the three bases, obtained using Bogoliubov transformations. The introduction of complex parameters in the " + " and " − " sectors will facilitate the following,

$$\begin{aligned} \lambda_+ &= \frac{\rho_- - \rho_-^{-1}}{\rho_- + \rho_-^{-1}}, \qquad \overline{\lambda}_+ = \frac{\overline{\rho}_- - \overline{\rho}_-^{-1}}{\overline{\rho}_- + \overline{\rho}_-^{-1}}, \\ \lambda_- &= \frac{\rho_+ - \rho_+^{-1}}{\rho_+ + \rho_+^{-1}}, \qquad \overline{\lambda}_- = \frac{\overline{\rho}_+ - \overline{\rho}_+^{-1}}{\overline{\rho}_+ + \overline{\rho}_+^{-1}}. \end{aligned} \tag{140}$$

A detailed analysis shows that the Fock voids of type A and B are given as,

$$
\begin{aligned}
|\Omega_A\rangle &= \left(\frac{2}{\rho_- + \rho_-^{-1}}\right)\left(\frac{2}{\rho_+ + \rho_+^{-1}}\right) e^{-\left(\lambda_+ a^\dagger_{+,+} a^\dagger_{+,-} + \lambda_- a^\dagger_{-,+} a^\dagger_{-,-}\right)} |\Omega\rangle, \\
|\Omega_B\rangle &= \left(\frac{2}{\overline{\rho}_- + \overline{\rho}_-^{-1}}\right)\left(\frac{2}{\overline{\rho}_+ + \overline{\rho}_+^{-1}}\right) e^{-\left(\overline{\lambda}_+ a^\dagger_{+,+} a^\dagger_{+,-} + \overline{\lambda}_- a^\dagger_{-,+} a^\dagger_{-,-}\right)} |\Omega\rangle. \qquad (141)
\end{aligned}
$$

Similarly, we have,

$$
\begin{aligned}
|\Omega_B\rangle &= N_{B_+}(\varphi_-) N_{B_-}(\varphi_+) e^{i(\tan\varphi_- B_{+,+} B_{+,-} + \tan\varphi_+ B_{-,+} B_{-,-})} |\Omega_A\rangle, \\
|\Omega_A\rangle &= N_{A_+}(\varphi_-) N_{A_-}(\varphi_+) e^{-i\left(\tan\varphi_- A^\dagger_{+,+} A^\dagger_{+,-} + \tan\varphi_+ A^\dagger_{-,+} A^\dagger_{-,-}\right)} |\Omega_B\rangle, \qquad (142)
\end{aligned}
$$

where $N_{A_+}(\varphi_-)$, $N_{A_-}(\varphi_+)$, $N_{B_+}(\varphi_-)$ et $N_{B-}(\varphi_+)$ are four normalisation factors and,

$$
\begin{aligned}
N^{-1}_{A_+}(\varphi_-) N^{-1}_{A_-}(\varphi_+) &= \langle \Omega_B | e^{-i\left(\tan\varphi_- A^\dagger_{+,+} A^\dagger_{+,-} + \tan\varphi_+ A^\dagger_{-,+} A^\dagger_{-,-}\right)} |\Omega_B\rangle, \\
N^{-1}_{B_+}(\varphi_-) N^{-1}_{B_-}(\varphi_+) &= \langle \Omega_A | e^{i(\tan\varphi_- B_{+,+} B_{+,-} + \tan\varphi_+ B_{-,+} B_{-,-})} |\Omega_A\rangle. \qquad (143)
\end{aligned}
$$

These different representations of the various Fock voids, with excitations of each other, make it possible to establish all three sets of Fock states. They thus provide complete bases for the same extended Hilbert space in which we diagonalise the quantum Hamiltonian $\hat{H}$, see (84). Finally, it should be noted that in the limit $\tau_0 \longrightarrow 0^+$, all three Fock voids become one and the same quantum state, implying that the three bases then coincide, namely the only $|n_{+,+}, n_{+,-}, n_{-,+}, n_{-,-}; \Omega\rangle$ $(n_{+,+}, n_{+,-} = 0, 1, ..., n_{-,+}, n_{-,-} = 0, 1, ...)$ states. Consequently, all three Fock voids become identical to $|\Omega\rangle$, while the following correspondences are established for the creation and annihilation operators,

$$
\begin{array}{llll}
A_{+,\pm} \to a_{+,\pm}, & B_{+,\pm} \to a^\dagger_{+,\pm}, & A^\dagger_{+,\pm} \to a^\dagger_{+,\pm}, & B^\dagger_{+,\pm} \to a_{+,\pm} \\
A_{-,\pm} \to a_{-,\pm}, & B_{-,\pm} \to a^\dagger_{-,\pm}, & A^\dagger_{-,\pm} \to a^\dagger_{-,\pm}, & B^\dagger_{-,\pm} \to a_{-,\pm}
\end{array} \qquad (144)
$$

9.2 Spectrum and energy eigenstates

In this section, the expression (84) of the extended system is to be quantified in terms of the creation and annihilation operators and diagonalised in terms of the Fock helicity operators defined above. Through the judicious application of a substitution, we derive the following result,

$$
\begin{aligned}
\hat{H} &= \hbar\frac{R_- e^{i\varphi_-}}{2i\tau_0}(B_{+,+}A_{+,+} + B_{+,-}A_{+,-} + \mathbb{I}) + \hbar\frac{R_+ e^{i\varphi_+}}{2i\tau_0}(B_{-,+}A_{-,+} + B_{-,-}A_{-,-} + \mathbb{I}) \\
&\quad - \frac{\hbar}{2i\tau_0}(B_{+,+}A_{+,+} - B_{+,-}A_{+,-} + B_{-,+}A_{-,+} - B_{-,-}A_{-,-}) + E_{KD}. \qquad (145)
\end{aligned}
$$

The Fock states of type A, $|N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-}; \Omega_A\rangle$, are the eigenstates of the operator concerned, while its adjoint $\hat{H}^\dagger \neq \hat{H}$ has as eigenstates the Fock states of type B, $|N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-}; \Omega_B\rangle$.

By adding and subtracting in (145) a contribution in $\frac{\hbar}{2i\tau_0}\mathbb{I}$, we obtain an equation of the following form,

$$
\begin{aligned}
\hat{H} &= \hbar\frac{R_- e^{i\varphi_-} - 1}{2i\tau_0} B_{+,+}A_{+,+} + \hbar\frac{R_- e^{i\varphi_-} + 1}{2i\tau_0}(B_{+,-}A_{+,-} + \mathbb{I}) \\
&\quad + \hbar\frac{R_+ e^{i\varphi_+} - 1}{2i\tau_0}(B_{-,+}A_{-,+} + \mathbb{I}) + \hbar\frac{R_+ e^{i\varphi_+} + 1}{2i\tau_0} B_{-,-}A_{-,-} + E_{KD}. \qquad (146)
\end{aligned}
$$

The quantity E_{KD} is chosen in such a way that the limit $\tau_0 \to 0^+$ exists and is non-trivial. In the context of the deformation of a single harmonic oscillator, it has been established that E_{KD} is selected as follows,

$$
E_{KD} = \left(\hbar\frac{R_+ e^{i\varphi_+} + 1}{4i\tau_0} + \Delta E_+(\omega_+, \tau_0)\right) + \left(\hbar\frac{R_- e^{i\varphi_-} + 1}{4i\tau_0} + \Delta E_-(\omega_-, \tau_0)\right), \qquad (147)
$$

where $\Delta E_{\pm}(\omega_{\pm}, \tau_0)$ are arbitrary parameters (they may even be complex for a finite value of τ_0)), and in fact are of the form,

$$\Delta E_{\pm}(\omega_{\pm}, \tau_0) = \hbar\omega_{\pm}\Delta\mathcal{E}_{\pm}(\omega_{\pm}\tau_0), \tag{148}$$

with $\Delta\mathcal{E}_{\pm}(\omega_{\pm}\tau_0)$ a function of product $(\omega_{\pm}\tau_0)$ which vanishes when this argument becomes zero, such that

$$\lim_{\tau_0 \longrightarrow 0^+} \Delta\mathcal{E}_{\pm}(\omega_{\pm}\tau_0) = 0. \tag{149}$$

Given the choice of subtraction constant E_{KD}, the complex energy spectrum of the system, for a value of $\tau_0 > 0$, is given by,

$$\begin{aligned} & \hat{H}|N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-}; \Omega_A\rangle = \\ = \;& E_{KD}(N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-})|N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-}; \Omega_A\rangle, \end{aligned} \tag{150}$$

$$\begin{aligned} & \hat{H}^{\dagger}|N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-}; \Omega_B\rangle = \\ = \;& E^{*}_{KD}(N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-})|N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-}; \Omega_B\rangle, \end{aligned} \tag{151}$$

with,

$$\begin{aligned} & E_{KD}(N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-}) = \\ = \;& \hbar\frac{R_- e^{i\varphi_-} - 1}{2i\tau_0}N_{+,+} + \hbar\frac{R_- e^{i\varphi_-} + 1}{2i\tau_0}(N_{+,-} + 1) \\ & + \hbar\frac{R_+ e^{i\varphi_+} - 1}{2i\tau_0}(N_{-,+} + 1) + \hbar\frac{R_+ e^{i\varphi_+} + 1}{2i\tau_0}N_{-,-} + \Delta\mathcal{E}_{\pm}. \end{aligned} \tag{152}$$

10 Limit $\tau_0 \to 0^+$.

10.1 Energy spectrum of the extended system

Let's first consider the development of the functions in (152) to first order, where the argument of each function must be positive or zero. This will facilitate the integration of the deformation term and the angular frequencies, thereby enabling the accurate formulation of the energy spectrum of the deformed system. Then we have

$$R^2_{\pm} e^{2i\varphi_{\pm}} = 1 + 4i\omega_{\pm}\tau_0, \qquad R_{\pm} e^{i\varphi_{\pm}} \simeq 1 + 2i\omega_{\pm}\tau_0 + \cdots \tag{153}$$

after, computations, we have,

$$\frac{R_{\pm} e^{i\varphi_{\pm}} + 1}{2i\tau_0} \simeq \left(-\frac{i}{\tau_0} + \omega_{\pm}\right) + \cdots, \quad \frac{R_{\pm} e^{i\varphi_{\pm}} - 1}{2i\tau_0} \simeq \omega_{\pm} + \cdots. \tag{154}$$

The following function developments are to be used for each helicity sector, right and left respectively. Subsequently, these function developments will be substituted into the expression (152) in order to identify the energy of the extended system. The resulting expression is as follows,

$$\begin{aligned} & E_{KD}(N_{+,+}, N_{+,-}, N_{-,+}, N_{-,-}) \simeq \hbar\omega_- N_{+,+} + \hbar\left(-\frac{i}{\tau_0} + +\omega_-\right)(N_{+,-} + 1) + \\ & + \hbar\omega_+(N_{-,+} + 1) + \hbar\left(-\frac{i}{\tau_0} + \omega_+\right)N_{-,-} + \Delta\mathcal{E}_{\pm}. \end{aligned} \tag{155}$$

In the limit $\tau_0 \to 0^+$, only the lowest Landau sectors for the two decoupled harmonic oscillators with $N_{+,-} = 0$ and $N_{-,-} = 0$ maintain finite energy values, namely the states $|N_{+,+}, N_{+,-} = 0, N_{-,+}, N_{-,-} = 0; \Omega_A\rangle \to |n_{+,+} = N_{+,+}, n_{+,-} = 0, n_{-,+} = N_{-,+}, n_{-,-} = 0; \Omega_{KD}\rangle$ for $\hat{H}$, and $|N_{+,+}, N_{+,-} = 0, N_{-,+}, N_{-,-} = 0; \Omega_B\rangle \to |n_{+,+} = N_{+,+}, n_{+,-} = 0, n_{-,+} = N_{-,+}, n_{-,-} = 0; \Omega_{KD}\rangle$ for $\hat{H}^{\dagger}$, with $N_{+,+}, N_{-,+} = 0, 1 \ldots$.

A choice for E_K that makes the limit consistent is equivalent to the following value being taken,

$$E_{KD} \overset{\tau_0 \to 0^+}{\simeq} \frac{\hbar}{i\tau_0} + \frac{1}{2}\hbar(\omega_+ + \omega_-) + \cdots = \frac{\hbar}{i\tau_0} + \hbar\omega + \cdots. \tag{156}$$

Substituting the expression (156) into the equation (155) gives

$$\lim_{\tau_0 \longrightarrow 0^+} E_{KD}(N_{+,+}, N_{+,-} = 0, N_{-,+}, N_{-,-} = 0) \simeq \hbar\omega_- N_{+,+} + \hbar\omega_+ N_{-,+} + \hbar\omega. \tag{157}$$

It is evident that, within the context of this extended system, the Landau case corresponds to the scenario where $\omega_0 = 0$. This observation subsequently implies the following equation $\omega = \frac{1}{2}\omega_c$. This result enables us to express the expression (157) as follows,

$$\lim_{\tau_0 \to 0^+} E_{KD}(N_{+,+}, N_{+,-} = 0, N_{-,+}, N_{-,-} = 0) \simeq \hbar\omega_c \left(N_{-,+} + \frac{1}{2} \right), \tag{158}$$

where ω_c is the cyclotron frequency of Landau problem.

In this expression, the real energy spectrum of the harmonic oscillator is recognised, where the pure Landau problem is considered, with no interaction other than with the magnetic field, including its vacuum quantum energy. The corresponding energy eigenstates are the Fock states$|n_{+,+} = N_{+,+}, n_{+,-} = 0, n_{-,+} = N_{-,+}, n_{-,-} = 0; \Omega_{KD}\rangle$. Therefore, the subspace of the extended Hilbert space of the extended system, generated by the lowest Landau sector, in the limit $\tau_0 = 0$, is determined to be the Hilbert space of the original quantum system in this Landau problem.

10.2 Energy spectrum in the limits $m \to 0$ and $\tau_0 \to 0$.

In this section, the two limits in the full spectrum of the Klauder-Daubechies deformation of the Landau problem are to be considered, with all quantities in which a mass contribution appears being taken into account. Use the following expressions as a starting point,

$$\omega_\pm = \omega \pm \frac{1}{2}\omega_c, \tag{159}$$

the $m \to 0$ limit for each of the angular frequencies will be performed from this data, according to the + or − mode. The results obtained for each of these angular frequencies in the $m \to 0$ limit are presented below, without the details of the process. In this limit, the following is true,

$$\lim_{m\to 0} \omega_+ = +\infty, \qquad \lim_{m\to 0} \omega_- = \frac{k_0}{B}. \tag{160}$$

It is important to introduce the following parameters,

$$R_0^2 e^{2i\varphi_0} = 1 + 4i\frac{k_0}{B}\tau_0. \tag{161}$$

The expression of the complete spectrum of the Klauder-Daubechies deformation of the Landau problem (152) in the limit $m \to 0$ with a finite value of τ_0 becomes, for states which maintain a finite value of their energy,

$$E(N_{+,+}, N_{+,-}, N_{-,+} = 0, N_{-,-} = 0) \simeq \hbar\frac{R_0 e^{i\varphi_0} + 1}{2i\tau_0} N_{+,-} + \hbar\frac{R_0 e^{i\varphi_0} - 1}{2i\tau_0} \left(N_{+,+} + \frac{1}{2} \right). \tag{162}$$

Then in the limit $\tau_0 \to 0$ the energy spectrum is,

$$E(N_{+,+}, N_{+,-}, N_{-,+} = 0, N_{-,-} = 0) \overset{\tau_0 \to 0^+}{\simeq} \frac{\hbar}{i\tau_0}\left(1 + i\frac{k_0}{B}\tau_0 + \cdots \right) N_{+,-} + \hbar\frac{k_0}{B}\left(N_{+,+} + \frac{1}{2} \right).$$

In the following step, the limit $\tau_0 \to 0$ on the quantum spectrum in (10.2) will be taken, which will result in the existence of finite energy when $N_{+,-} = 0$, as expressed in the following form,

$$\lim_{\tau_0 \to 0^+} E(N_{+,+}, N_{+,-} = 0, N_{-,+} = 0, N_{-,-} = 0) = \hbar\frac{k_0}{B}\left(N_{+,+} + \frac{1}{2} \right). \tag{163}$$

11 Non-commutative geometry in the Moyal plane.

The Landau problem with the Klauder-Daubechies deformation, which can lead to non-commutative geometry, can be treated in the same way as the ordinary Landau model. The complete Klauder-Daubechies system of the Landau problem, as described in this work, can be extended to commutator Algebras. This is a Landau problem in the presence of a spherically symmetric harmonic potential with its two degrees of freedom, coupled to an external field with velocity-dependent coupling. To achieve this objective, it is necessary to consider the limits $m \to 0$ and τ_0 in the two possible inverse orders.

It can thus be concluded that the two limits result in the expression (67) being written in the following form,

$$\begin{aligned} L_{KD} &= \frac{1}{2}(\dot{x}_1 p_1 - x_1\dot{p}_1) + \frac{1}{2}(\dot{x}_2 p_2 - x_2\dot{p}_2) - \\ &- \left(\frac{1}{2}k_0(x_1^2 + x_2^2) + \frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{8m}B^2(x_1^2 + x_2^2) - \frac{1}{2m}B(x_1 p_2 - x_2 p_1) + {}^c E_{KD} \right) + \\ &+ \frac{1}{2}i\tau_0 \left(\frac{2}{B}(\dot{p}_1^2 + \dot{p}_2^2) + \frac{1}{2}B(\dot{x}_1^2 + \dot{x}_2^2) \right), \end{aligned} \tag{164}$$

or

$$L_{KD} = \frac{1}{2}(\dot{x}_1 p_1 - x_1\dot{p}_1) + \frac{1}{2}(\dot{x}_2 p_2 - x_2\dot{p}_2) -$$

$$-\left(\frac{1}{2m}\left[\left(\frac{1}{2}Bx_2+p_1\right)^2+\left(\frac{1}{2}Bx_1-p_2\right)^2\right]+\frac{1}{2}k_0(x_1^2+x_2^2)+{}^cE_{KD}\right)+$$
$$+\frac{1}{2}i\tau_0\left(\frac{2}{B}(\dot{p}_1^2+\dot{p}_2^2)+\frac{1}{2}B(\dot{x}_1^2+\dot{x}_2^2)\right). \tag{165}$$

In order to surmount the singularity problem that manifests in (165) when $m \to 0$ for the configurations of (x_i, p_j) with $i, j = 1, 2$, which would be generalised, it is necessary to consider a subspace of the configurations in this phase space following the singularities. It is sufficient to consider the restriction of configurations (x_i, p_j) for which the terms multiplied by $\frac{1}{m}$ are zero. This amounts to working in the space defined by these two conditions,

$$p_1+\frac{1}{2}Bx_2=0, \qquad p_2-\frac{1}{2}Bx_1=0. \tag{166}$$

Consequently, the phase space is reduced and once again becomes two-dimensional space, for which two coordinates, for examplex_i, can be chosen. It is imperative to make the following substitution,

$$p_1=-\frac{1}{2}Bx_2, \qquad p_2=\frac{1}{2}Bx_1, \tag{167}$$

in (165) before taking the limit $m \to 0$ and τ_0 finite. We obtain then,

$$L_{KD}^{NC}=\frac{1}{2}i\tau_0B\dot{x}_i^2-\frac{1}{2}B\epsilon_{ij}\dot{x}_ix_j-\frac{1}{2}k_0x_i^2-{}^cE_{KD}^{NC} \tag{168}$$

In the next section, we will quantify this result in order to solve the quantum spectrum of this massless particle system. The expression (164) is to be compared with the form of the first-order Hamiltonian action below,

$$\frac{1}{2}(\dot{q}p-q\dot{p})-H(q,p). \tag{169}$$

The following identifications are to be made between the phase space coordinates of the deformed classical system

$$x_1 \to q, \qquad -Bx_2 \to p, \tag{170}$$

in which (q, p) define canonically conjugate phase space variables with $[\hat{q}, \hat{p}] = i\hbar 1$ as quantum commutator. It can be deduced that the commutator of the non-commutative Heisenberg algebra for the two coordinates of the Euclidean plane is as follows,

$$[\hat{x}_1,\hat{x}_2]=-i\frac{\hbar}{B}. \tag{171}$$

We then obtain a quantum system generating a non-commutative geometry in the Cartesian coordinates of the plane, due to the singularities observed in the variables $Q_\pm$ and $P_\pm$ which define a canonical transformation.

12 Klauder–Daubechies quantum non-commutative plane

In this section, the result obtained for the limit $m \to 0$ and τ_0 finite will be quantified. This is the expression (168) in the presence of a spherically symmetric harmonic potential, which will be analysed in order to determine the quantum energy spectrum of the system. The expression (168) is written, after rearranging the terms, in the form of the Lagrange function L_{KD}^{NC},

$$L_{KD}^{NC}=\frac{1}{2}i\tau_0B\dot{x}_i^2-\frac{1}{2}B\epsilon_{ij}\dot{x}_ix_j-\frac{1}{2}k_0x_i^2-{}^cE_{KD}^{NC}, \qquad {}^cE_{KD}^{NC}=\lim_{m\to 0}{}^cE_{KD} \tag{172}$$

12.1 Classical Hamiltonian

In order to construct the Hamiltonian description of the system, it is first necessary to define the conjugate momentum π_i at the degrees of freedom x_i by the following expressions,

$$\pi_i = i\tau_0 B\dot{x}_i - \frac{1}{2}B\epsilon_{ij}x_j, \qquad \dot{x}_i = \frac{1}{i\tau_0 B}\left[\pi_i + \frac{1}{2}B\epsilon_{ij}x_j\right], \tag{173}$$

leading to the canonical Hamiltonian,

$$\begin{aligned} H &= \frac{1}{2i\tau_0 B}\left(\pi_i + \frac{1}{2}B\epsilon_{ij}x_j\right)^2 + \frac{1}{2}k_0 x_i^2 + {}^cE_{KD}^{NC} \\ &= \frac{1}{2i\tau_0 B}\pi_i^2 + \frac{1}{2i\tau_0 B}\left(\frac{1}{4}B^2 + i\tau_0 Bk_0\right)x_i^2 - \frac{1}{2i\tau_0}\epsilon_{ij}x_i\pi_j + {}^cE_{KD}^{NC} \\ &= \frac{1}{2i\tau_0 B}\left[\pi_i^2 + \frac{1}{4}B^2\left(1 + 4i\tau_0\frac{k_0}{B}x_i^2\right)\right] - \frac{1}{2i\tau_0}\epsilon_{ij}x_i\pi_j + {}^cE_{KD}^{NC}. \end{aligned} \tag{174}$$

The canonical quantization of the system is derived from the following Heisenberg algebra,

$$[\hat{x}_i, \hat{\pi}_j] = i\hbar\delta_{ij}\mathbb{I}, \tag{175}$$

Thus, the Hamiltonian (?) becomes an operator and is written explicitly as follows,

$$\begin{aligned} \hat{H} &= \frac{1}{2i\tau_0}\left[\hat{\pi}_1^2 + \hat{\pi}_2^2 + \frac{1}{4}B^2\left(1 + 4i\tau_0\frac{k_0}{B}\right)\left(\hat{x}_1^2 + \hat{x}_2^2\right)\right] - \\ &\quad - \frac{1}{2i\tau_0}\left(\hat{x}_1\hat{\pi}_2 - \hat{x}_2\hat{\pi}_1\right) + {}^qE_{KD}^{NC}. \end{aligned} \tag{176}$$

12.2 Quantification of the system

However, the Hamiltonian in (176) will be canonically quantified using operators $\bar{\alpha}_i$ and $\bar{\alpha}_i^\dagger$. In order to undertake this task, it appears to be a logical progression to initiate a Fock algebra associated with this choice of degrees of freedom and ordinary Fock states, namely,

$$\bar{\alpha}_i = \frac{1}{\sqrt{\hbar B}}\left(\frac{1}{2}B\hat{x} + i\hat{\pi}_i\right), \qquad \bar{\alpha}_i^\dagger = \frac{1}{\sqrt{\hbar B}}\left(\frac{1}{2}B\hat{x}_i - i\hat{\pi}_i\right), \tag{177}$$

which define the tensor product of two Fock algebras,

$$\left[\bar{\alpha}_i, \bar{\alpha}_j^\dagger\right] = \delta_{ij}\mathbb{I}. \tag{178}$$

Conversely, the operators $\hat{x}_i$ and $\hat{\pi}_i$ are defined as follows,

$$\hat{x}_i = \sqrt{\frac{\hbar}{B}}\left(\bar{\alpha}_i + \bar{\alpha}_i^\dagger\right), \quad \hat{\pi}_i = -\frac{i}{2}\sqrt{\hbar B}\left(\bar{\alpha}_i - \bar{\alpha}_i^\dagger\right), \tag{179}$$

Then consider the following helicity Fock operators,

$$\bar{\alpha}_\pm = \frac{1}{\sqrt{2}}\left(\bar{\alpha}_1 \mp i\bar{\alpha}_2\right), \qquad \bar{\alpha}_\pm^\dagger = \frac{1}{\sqrt{2}}\left(\bar{\alpha}_1^\dagger \pm i\bar{\alpha}_2^\dagger\right), \tag{180}$$

such that,

$$[\bar{\alpha}_\pm, \bar{\alpha}_\pm^\dagger] = \mathbb{I}, \qquad [\bar{\alpha}_\pm, \bar{\alpha}_\mp^\dagger] = 0. \tag{181}$$

From the expressions (179) and (180), we can explicitly deduce the following inverse relationships,

$$\hat{x}_1 = \sqrt{\frac{\hbar}{2B}}\left(\bar{\alpha}_+ + \bar{\alpha}_- + \bar{\alpha}_+^\dagger + \bar{\alpha}_-^\dagger\right), \quad \hat{x}_2 = i\sqrt{\frac{\hbar}{2B}}\left(\bar{\alpha}_+ - \bar{\alpha}_- - \bar{\alpha}_+^\dagger + \bar{\alpha}_-^\dagger\right),$$
$$\hat{\pi}_1 = -\frac{i}{2}\sqrt{\frac{\hbar B}{2}}\left(\bar{\alpha}_+ + \bar{\alpha}_- - \bar{\alpha}_+^\dagger - \bar{\alpha}_-^\dagger\right), \quad \hat{\pi}_2 = \frac{i}{2}\sqrt{\frac{\hbar B}{2}}\left(\bar{\alpha}_+ - \bar{\alpha}_- + \bar{\alpha}_+^\dagger - \bar{\alpha}_-^\dagger\right). \tag{182}$$

The construction of an abstract representation of the algebraic structures under consideration is a prerequisite for the subsequent consideration of a normalised Fock vacuum, $\mid \Omega_{KD}^{NC}\rangle$, for helicity Fock operators,

$$\bar{\alpha}_\pm \mid \Omega_{KD}^{NC}\rangle = 0, \qquad \langle \Omega_{KD}^{NC} \mid \Omega_{KD}^{NC}\rangle = 1, \tag{183}$$

with the following orthonormalised states of the Hilbert space of the quantum system,

$$\mid k_+, k_-; \Omega_{KD}^{NC}\rangle = \frac{1}{\sqrt{k_+! k_-!}}\left(\bar{\alpha}_+^\dagger\right)^{k_+}\left(\bar{\alpha}_-^\dagger\right)^{k_-} \mid \Omega_{KD}^{NC}\rangle, \tag{184}$$

$$\langle k_+, k_-; \Omega_{KD}^{NC} \mid j_+, j_-; \Omega_{KD}^{NC}\rangle = \delta_{k_+, j_+}\delta_{k_-, j_-}, \tag{185}$$

where $k_+, k_- = 0, 1, 2, \cdots$, and the operator identity,

$$\sum_{k_+, k_- = 0}^{\infty} \mid k_+, k_-; \Omega_{KD}^{NC}\rangle\langle k_+, k_-; \Omega_{KD}^{NC} \mid = \mathbb{I}. \tag{186}$$

The projector on the Landau sector is thus expressed as follows,

$$\mathbb{P}_{KD}^{NC} = \sum_{k_+ = 0}^{\infty} \mid k_+, k_- = 0; \Omega_{KD}^{NC}\rangle\langle k_+, k_- = 0; \Omega_{KD}^{NC} \mid . \tag{187}$$

The Cartesian coordinates of the plane projected onto this sector of Hilbert space are then given by,

$$\bar{x}_1 = \sqrt{\frac{\hbar}{2B}}\left(\bar{\alpha}_+ + \bar{\alpha}_+^\dagger\right), \quad \bar{x}_2 = i\sqrt{\frac{\hbar}{2B}}\left(\bar{\alpha}_+ - \bar{\alpha}_+^\dagger\right), \tag{188}$$

and obey the following commutation relationship.

$$[\bar{x}_1, \bar{x}_2] = -i\frac{\hbar}{B}\mathbb{P}_{KD}^{NC}. \tag{189}$$

Once again, this reproduces the non-commutative Euclidean plane.

12.3 Diagonalisation of the Hamiltonian

The projected sub-space of the Heisenberg algebra of the quantum system, despite being situated within the Hilbert space of the $\hat{x}_i$ position operators of the phase space, no longer exhibits mutual commutation amongst these operators. For the extended system, after the limit $m = 0$, but with a finite value of τ_0, it is useful to introduce linear combinations in order to diagonalise the Hamiltonian into (176), according to the following relations,

$$\rho_0^4 = R_0^2 e^{2i\tau_0} = 1 + 4i\tau_0\frac{k_0}{B}, \quad R_0 > 0, \quad \rho_0 = \sqrt{R_0}e^{\frac{1}{2}i\varphi_0}. \tag{190}$$

Let's define the operators below

$$\alpha_i = \frac{1}{\sqrt{\hbar B}}\left(\frac{1}{2}B\rho_0\hat{x}_i + \frac{i}{\rho_0}\hat{\pi}_i\right), \quad \beta_i = \frac{1}{\sqrt{\hbar B}}\left(\frac{1}{2}B\rho_0\hat{x}_i - \frac{i}{\rho_0}\hat{\pi}_i\right), \tag{191}$$

While their algebra is expressed as,

$$[\alpha_i, \beta_j] = \delta_{ij}\mathbb{I} = \left[\beta_i^\dagger, \alpha_j^\dagger\right]. \tag{192}$$

The helicity operators are chosen from the form,

$$\alpha_\pm = \frac{1}{\sqrt{2}}(\alpha_1 \mp i\alpha_2), \quad \beta_\pm = \frac{1}{\sqrt{2}}(\beta_1 \pm i\beta_2), \tag{193}$$

as well as for adjoint operators,

$$\alpha_\pm^\dagger = \frac{1}{\sqrt{2}}(\alpha_1^\dagger \pm i\alpha_2^\dagger), \quad \beta_\pm^\dagger = \frac{1}{\sqrt{2}}(\beta_1^\dagger \mp i\beta_2^\dagger). \tag{194}$$

The corresponding algebraic structures can be written as,

$$[\alpha_\pm, \beta_\pm] = \mathbb{I} = \left[\beta_\pm^\dagger, \alpha_\pm^\dagger\right] \tag{195}$$

To determine the phase relationships between these operators and their adjoints, consider,

$$\alpha_i^\dagger = \frac{1}{\sqrt{\hbar B}}\left(\frac{1}{2}B\bar{\rho}_0\hat{x}_i - \frac{i}{\bar{\rho}_0}\hat{\pi}_i\right), \quad \beta_i^\dagger = \frac{1}{\sqrt{\hbar B}}\left(\frac{1}{2}B\bar{\rho}_0\hat{x}_i + \frac{i}{\bar{\rho}_0}\hat{\pi}_i\right). \tag{196}$$

From the expressions (191) and (196), we have,

$$\hat{x}_i = \frac{\sqrt{\hbar B}}{B\rho_0}(\alpha_i + \beta_i), \quad \hat{\pi}_i = -\frac{1}{2}i\rho_0\sqrt{\hbar B}(\alpha_i - \beta_i), \tag{197}$$

$$\hat{x}_i = \frac{\sqrt{\hbar B}}{B\bar{\rho}_0}(\alpha_i^\dagger + \beta_i^\dagger), \quad \hat{\pi}_i = \frac{1}{2}i\bar{\rho}_0\sqrt{\hbar B}(\alpha_i^\dagger - \beta_i^\dagger). \tag{198}$$

In the following sections, the operators $\alpha_i^\dagger$ and $\beta_i^\dagger$ will be expressed in terms of circular functions. To achieve this, we will substitute (197) into (196) and subsequently develop. This will enable the remainder of the demonstration to be obtained

$$\begin{aligned}\alpha_i^\dagger &= \frac{1}{\sqrt{\hbar B}}\left(\frac{1}{2}B\rho_0 e^{i\varphi_0}\hat{x}_i - \frac{i}{\rho_0}e^{i\varphi_0}\hat{\pi}_i\right)\\ &= \frac{1}{2}e^{-i\varphi_0}(\alpha_i + \beta_i) - \frac{1}{2}e^{i\varphi_0}(\alpha_i - \beta_i)\\ &= \cos\varphi_0\beta_i - i\sin\varphi_0\alpha_i,\end{aligned} \tag{199}$$

and the same applies to ,

$$\begin{aligned}\beta_i^\dagger &= \frac{1}{\sqrt{\hbar B}}\left(\frac{1}{2}B\rho_0 e^{-i\varphi_0}\hat{x}_i + \frac{i}{\rho_0}e^{i\varphi_0}\hat{\pi}_i\right)\\ &= \frac{1}{2}e^{-i\varphi_0}(\alpha_i + \beta_i) + \frac{1}{2}e^{i\varphi_0}(\alpha_i - \beta_i)\\ &= \cos\varphi_0\alpha_i - i\sin\varphi_0\beta_i.\end{aligned} \tag{200}$$

Thus it appears that the helicity operators $\alpha_\pm^\dagger$ and $\beta_\pm^\dagger$ can again be expressed in terms of circular functions. This is done by substituting (199), (200) and (193) into (194), leading to

$$\begin{aligned} \alpha_\pm^\dagger &= \frac{1}{\sqrt{2}}\left(\alpha_1^\dagger \pm i\alpha_2^\dagger\right) \\ &= \cos\varphi_0\beta_\pm - i\sin\varphi_0\alpha_\mp, \end{aligned} \tag{201}$$

and

$$\begin{aligned} \beta_\pm^\dagger &= \frac{1}{\sqrt{2}}\left(\beta_1^\dagger \mp i\beta_2^\dagger\right) \\ &= \cos\varphi_0\alpha_\pm - i\sin\varphi_0\beta_\mp. \end{aligned} \tag{202}$$

Similarly to the previous procedure, we obtain the following constructions,

$$\alpha_i = \cos\varphi_0\beta_i^\dagger + i\sin\varphi_0\alpha_i^\dagger, \qquad \beta_i = \cos\varphi_0\alpha_i^\dagger + i\sin\varphi_0\beta_i^\dagger, \tag{203}$$

as well as

$$\alpha_\pm = \cos\varphi_0\beta_\pm^\dagger + i\sin\varphi_0\alpha_\mp^\dagger, \qquad \beta_\pm = \cos\varphi_0\alpha_\pm^\dagger + i\sin\varphi_0\beta_\mp^\dagger. \tag{204}$$

Note that it is also possible to formulate these constructions using the $\bar{\alpha}_\pm$ and $\bar{\alpha}_\pm^\dagger$ operators. In this way, we obtain,

$$\begin{aligned} \alpha_\pm &= \frac{\rho_0+\rho_0^{-1}}{2}\bar{\alpha}_\pm + \frac{\rho_0-\rho_0^{-1}}{2}\bar{\alpha}_\mp^\dagger, \qquad & \alpha_\pm^\dagger &= \frac{\bar{\rho}_0+\bar{\rho}_0^{-1}}{2}\bar{\alpha}_\pm^\dagger + \frac{\bar{\rho}_0-\bar{\rho}_0^{-1}}{2}\bar{\alpha}_\mp, \\ \beta_\pm &= \frac{\rho_0+\rho_0^{-1}}{2}\bar{\alpha}_\pm^\dagger + \frac{\rho_0-\rho_0^{-1}}{2}\bar{\alpha}_\mp, \qquad & \beta_\pm^\dagger &= \frac{\bar{\rho}_0+\bar{\rho}_0^{-1}}{2}\bar{\alpha}_\pm + \frac{\bar{\rho}_0-\bar{\rho}_0^{-1}}{2}\bar{\alpha}_\mp^\dagger. \end{aligned} \tag{205}$$

We therefore have Fock void of types α and β, respectively, of the following form

$$\alpha_\pm \mid \Omega_\alpha\rangle = 0, \quad \beta_\pm^\dagger \mid \Omega_\beta\rangle = 0. \tag{206}$$

With the right choice of phases and normalisations, it is always possible to write these states as,

$$\langle\Omega_\alpha \mid \Omega_\beta\rangle = 1 = \langle\Omega_\beta \mid \Omega_\alpha\rangle. \tag{207}$$

The Fock states of type X are then defined by

$$\mid K_+, K_-; \Omega_\alpha\rangle = \frac{1}{\sqrt{K_+!K_-!}}\beta_+^{K_+}\beta_-^{K_-} \mid \Omega_\alpha\rangle, \tag{208}$$

However, for Fock states of type β,

$$\mid K_+, K_-; \Omega_\beta\rangle = \frac{1}{\sqrt{K_+!K_-!}}\left(\alpha_+^\dagger\right)^{K_+}\left(\alpha_-^\dagger\right)^{K_-} \mid \Omega_\beta\rangle, \tag{209}$$

where K_+,$K_- = 0, 1, 2, \ldots$. Since the operators $\alpha_\pm$, $\beta_\pm$ and their adjoints are linear combinations of the Fock operators $\bar{\alpha}_\pm$ and $\bar{\alpha}_\pm^\dagger$, it is clear that one or the other of these sets of states, $\mid K_+, K_-; \Omega_\alpha\rangle$ or $\mid K_+, K_-; \Omega_\beta\rangle$, generates the entire Hilbert space of the quantum system. More specifically, each of these two sets provides a basis for this space, and these two bases are actually dual to one another,

$$\langle K_+, K_-; \Omega_\alpha \mid J_+, J_-; \Omega_\beta\rangle = \delta_{K_+,J_+}\delta_{K_-,J_-} = \langle K_+, K_-; \Omega_\beta \mid J_+, J_-; \Omega_\alpha\rangle. \tag{210}$$

We therefore obtain the following resolutions of the identity operator ,

$$\sum_{K_+,K_-=0}^{\infty} \mid K_+,K_-;\Omega_\alpha\rangle\langle K_+,K_-;\Omega_\beta \mid = \mathbb{I} = \sum_{K_+,K_-=0}^{\infty} \mid K_+,K_-;\Omega_\beta\rangle\langle K_+,K_-;\Omega_\alpha \mid . \tag{211}$$

In other words, the three sets of states, $\mid k_+,k_-;\Omega_{KD}^{NC}\rangle$, $\mid K_+,K_-;\Omega_\alpha\rangle$ and $\mid K_+,K_-;\Omega_\beta\rangle$, define three distinct bases of the same extended Hilbert space. The $\mid k_+,k_-;\Omega_{KD}^{NC}\rangle$ basis is self-dual, i.e. orthonormalised, while the other two bases are dual to each other. It is important to note that the action of the operators $\alpha_\pm$ and $\beta_\pm$ on Fock states of type α, on the one hand, and of the operators $\alpha_\pm^\dagger$ and $\beta_\pm^\dagger$ on Fock states of type β, on the other hand, is exactly that of ordinary Fock operators of creation and annihilation, respectively, on ordinary Fock states. In particular, the Fock states α, $\mid K_+,K_-;\Omega_\alpha\rangle$ (resp., Fock states β, $\mid K_+,K_-;\Omega_\beta\rangle$) are eigenstates of the operators $\alpha_\pm\beta_\pm$ (resp., $\alpha_\pm^\dagger\beta_\pm^\dagger$) with $K_\pm$ eigenvalues. It is evident that the identities relating the Fock operators between these three different bases are obtained by means of Bogoliubov transformations with the following complex parameters,

$$\lambda_0 = \frac{\rho_0-\rho_0^{-1}}{\rho_0+\rho_0^{-1}}, \quad \lambda_0 = \frac{\bar\rho_0-\bar\rho_0^{-1}}{\bar\rho_0+\bar\rho_0^{-1}}. \tag{212}$$

A detailed analysis shows that the Fock voids of type α and β are given as follows,

$$\mid \Omega_\alpha\rangle = \left(\frac{2}{\rho_0+\rho_0^{-1}}\right) e^{\lambda_0\bar\alpha_+^\dagger\bar\alpha_-^\dagger} \mid \Omega_{KD}^{NC}\rangle, \quad \mid \Omega_\beta\rangle = \left(\frac{2}{\bar\rho_0+\bar\rho_0^{-1}}\right) e^{-\bar\lambda_0\bar\alpha_+^\dagger\bar\alpha_-^\dagger} \mid \Omega_{KD}^{NC}\rangle, \tag{213}$$

and similarly

$$\mid \Omega_\beta\rangle = N_\beta(\varphi_0) e^{i\tan\varphi_0\beta_+\beta_-} \mid \Omega_\alpha\rangle, \quad \mid \Omega_\alpha\rangle = N_\alpha(\varphi_0) e^{-i\tan\varphi_0\alpha_+^\dagger\alpha_-^\dagger} \mid \Omega_\beta\rangle, \tag{214}$$

$N_\alpha(\varphi_0)$ and $N_\beta(\varphi_0)$ are two normalisation factors whose explicit evaluation is not required here,

$$N_\alpha^{-1}(\varphi_0) = \langle \Omega_\beta \mid e^{-i\tan\varphi_0\alpha_+^\dagger\alpha_-^\dagger} \mid \Omega_\beta\rangle, \quad N_\beta^{-1} = \langle \Omega_\alpha \mid e^{i\tan\varphi_0\beta_+\beta_-} \mid \Omega_\alpha\rangle. \tag{215}$$

It has been demonstrated that the various helicity Fock vacuum representations, which relate to coherent states in another basis, establish that all three sets of Fock states provide complete bases within the same extended Hilbert space. This enables the diagonalization of the quantum Hamiltonian $\hat H$.

It should be noted that, in the limit where $\tau_0 \to 0$, all three sets of Fock states reduce to a single set of basis vectors. The basis vectors in question are the states $\mid k_+,k_-;\Omega_{KD}^{NC}\rangle$ $(k_+,k_- = 0,1,\ldots)$, since all Fock vacuum states become identical to $\mid \Omega\rangle$, and we have the correspondences for the creation and annihilation operators,

$$\alpha_\pm \to \bar\alpha_\pm, \qquad \beta_\pm \to \bar\alpha_\pm^\dagger, \qquad \alpha_\pm^\dagger \to \bar\alpha_\pm^\dagger, \qquad \beta_\pm^\dagger \to \bar\alpha_\pm. \tag{216}$$

12.4 The Quantum Hamiltonian.

The terms contributing to the construction of the Hamiltonian operator will first be determined, and then substituted in turn in $hatH$ to find the appropriate form of this operator. We thus obtain,

$$\hat\pi_i^2 + \frac{1}{4}B^2\rho_0^4\hat x_i^2 = -\frac{1}{4}\rho_0^2\hbar B\left[(\alpha_1-\beta_1)^2+(\alpha_2-\beta_2)^2\right] + \frac{1}{4}B^2\rho_0^4\frac{\hbar B}{B^2\rho_0^2}\left[(\alpha_1+\beta_1)^2+(\alpha_2+\beta_2)^2\right]$$

$$
\begin{aligned}
&= \frac{1}{2}\hbar\rho_0^2 B\left[\alpha_1\beta_1 + \beta_1\alpha_1 + \alpha_2\beta_2 + \beta_2\alpha_2\right] \\
&= \frac{1}{2}\hbar\rho_0^2 B\left[\alpha_+\beta_+ + \alpha_-\beta_- + \beta_+\alpha_+ + \beta_-\alpha_-\right] \\
&= \hbar\rho_0^2 B(\beta_+\alpha_+ + \beta_-\alpha_- + 1),
\end{aligned}
\tag{217}
$$

as well as,

$$
\begin{aligned}
\hat{x}_1\hat{\pi}_2 - \hat{x}_2\hat{\pi}_1 &= -\frac{1}{2}i\hbar[(\alpha_1+\beta_1)(\alpha_2-\beta_2) - (\alpha_2+\beta_2)(\alpha_1-\beta_1)] \\
&= -i\hbar[\beta_1\alpha_2 - \beta_2\alpha_1] \\
&= \frac{1}{2}\hbar[(\beta_+ + \beta_-)(\alpha_+ - \alpha_-) + (\beta_+ - \beta_-)(\alpha_+ + \alpha_-) \\
&= \hbar[\beta_+\alpha_+ - \beta_-\alpha_-].
\end{aligned}
\tag{218}
$$

Subsequently, we are able to obtain highly explicit and detailed results for each term calculated in $\hat{H}$, in order to ascertain the energy spectrum required.

12.5 Energy spectrum.

In the context of the previous representation theory of Hilbert space, it is possible to diagonalise the Hamiltonian (176) of the extended system, a process which is relatively straightforward. Using the operators $\beta_\pm$ and $\alpha_\pm$, we obtain the following result,

$$
\begin{aligned}
\hat{H} &= \frac{\hbar\rho_0^2}{2i\tau_0}\left[\beta_+\alpha_+ + \beta_-\alpha_- + 1\right] - \frac{\hbar}{2i\tau_0}\left[\beta_+\alpha_+ - \beta_-\alpha_-\right] + {}^qE_{KD}^{NC} \\
&= \frac{\hbar}{2i\tau_0}\left(\rho_0^2 - 1\right)\left(\beta_+\alpha_+ + \frac{1}{2}\right) + \frac{\hbar}{2i\tau_0}\left(\rho_0^2+1\right)\left(\beta_-\alpha_- + \frac{1}{2}\right) + {}^qE_{KD}^{NC}
\end{aligned}
\tag{219}
$$

It is evident that the Fock states of type α, $\mid K_+, K_-; \Omega_\alpha\rangle$, correspond to the eigenstates of the operator, while those of its adjoint, $\hat{H}^\dagger \neq \hat{H}$, are the Fock states of type β, i.e. $\mid K_+, K_-; \Omega_\beta\rangle$. Moreover, it is imperative that the subtraction constant is adjusted according to the following procedure,

$$
{}^qE_{KD}^{NC} = \frac{\hbar}{4i\tau_0}\left(\rho_0^2+1\right) + \Delta E_{KD}^{NC}, \quad \lim_{\tau_0 \longrightarrow 0} \Delta E_{KD}^{NC} = \Delta\bar{E}_{NC}. \tag{220}
$$

12.6 The limit $\tau_0 \to 0$

We have the following quantity,

$$
\frac{R_0 e^{i\varphi_0} + 1}{2i\tau_0}, \tag{221}
$$

which in the limit $\tau_0 \to 0$ the behaviour is as follows

$$
\frac{R_0 e^{i\varphi_0} + 1}{2i\tau_0} \overset{\tau_0 \longrightarrow 0}{\simeq} \frac{1}{i\tau_0} + \frac{k_0}{B} + \cdots . \tag{222}
$$

It is imperative to reiterate that it is the scale of $\frac{1}{i\tau_0}$ that determines the contribution of this gap, which diverges in the limit. It is evident that only the Landau sector with $K_- = 0$ maintains zero energy in this limit. Furthermore, in a given Landau sector, the gap between states is determined by the second quantity,

$$
\frac{R_0 e^{i\varphi_0} - 1}{2i\tau_0}, \tag{223}
$$

which in the limit $\tau_0 \to 0$ the behaviour is as follows

$$\frac{R_0 e^{i\varphi_0} - 1}{2i\tau_0} \overset{\tau_0 \longrightarrow 0}{\simeq} \frac{k_0}{B} + \cdots . \tag{224}$$

These quantities, taken in the limit $\tau_0 \to 0$, will be substituted in $\hat{H}$ below to keep the approximate expression of the Hamiltonian operator,

$$\hat{H} = \frac{\hbar}{2i\tau_0}\left(\rho_0^2 - 1\right)\left(\beta_+\alpha_+ + \frac{1}{2}\right) + \frac{\hbar}{2i\tau_0}\left(\rho_0^2 + 1\right)\beta_-\alpha_- + \Delta E_{KD}^{NC}. \tag{225}$$

In the limit $\tau_0 \to 0$, the projector is

$$\sum_{K_+=0}^{\infty} \mid K_+, K_- = 0; \Omega_\alpha\rangle\langle K_+, K_- = 0; \Omega_\alpha \mid \equiv \mathbb{P}, \tag{226}$$

leading to the energy spectrum given by,

$$\lim_{\tau_0 \to 0} \bar{H} = \hbar\frac{k_0}{B}\left(\bar{\alpha}_+^\dagger \bar{\alpha}_+ + \frac{1}{2}\right) + \Delta\bar{E}_{KD}^{NC}. \tag{227}$$

Finally, we will take the limits $m \to 0$ and $\tau_0 \to 0$ in order to deduce the algebraic structures of non-commutative geometry. It is clear that by taking these limits in expression (172) we obtain the result in the following form,

$$[\hat{x}_1, \hat{x}_2] = -i\frac{\hbar}{B}. \tag{228}$$

The result obtained in this study is analogous to the switch observed in the complete Landau KD system in the limit $m \to 0$. In the system under consideration, this switch is replicated in two distinct limit orders: $m \to 0$ and $\tau_0 \to 0$. This property can be discerned in both quantum systems and their Klauder-Daubechies deformation, as well as in their classical counterparts in the $\hbar = 0$ limit.

13 Conclusions

In this investigation, we have adopted an approach based on the formalism of canonical quantisation in terms of operators, rather than an approach involving the use of the functional integral. This operator and canonical quantisation approach is particularly pertinent to the type of system under consideration, namely the generalised Landau problem in the presence of a harmonic potential and its various deformations, including that proposed by Klauder. This approach, using operators and canonical quantification, is particularly pertinent to the system under study, namely the generalised Landau problem in the presence of a spherically symmetric harmonic potential and its various deformations, including the negative one proposed by Klauder. The latter is, in fact, a pure positive imaginary mass playing the role of a regularisation parameter and determined in terms of a new fundamental time scale, τ_0. In the limit where this time scale cancels out, the usual quantum formalism and dynamics are recovered.

The technique of canonical quantisation of a distorted formulation of the dynamics of the original system in terms of time scale τ_0 is equivalent to the construction of the Klauder-Daubechies functional integral of the path integral in phase space and based on canonical coherent states. This canonical formulation has a regularisation parameter proportional to a new time scale $\tau_0 > 0$. In the limit where this regularisation parameter cancels out, the correct quantum dynamics of the system is reproduced. The above method has been explicitly applied to the specific case of the harmonic oscillator. This approach is different from and complementary to the Klauder-Daubechies method for quantum dynamics. This approach makes it possible to recover the original system with extended dynamics whose configuration space is the phase space of the original system. The phase space and its symplectic geometry are then possessed by the configuration space, as is a Riemannian metric with the identical volume form. This structure is associated with an additional Brownian motion component to the quantum dynamics of the original system. Indeed, when the regularisation of Brownian motion is zero, the original quantum system persists. This formulation has several advantages, including covariance under general canonical transformations in phase space, which can be used effectively with the development of new non-perturbative quantization techniques. This regularisation dynamics is now applied to the generalised Landau problem in the presence of a spherically symmetric harmonic potential in phase space, with a purely positive imaginary mass $i\tau_0$. This Klauder-Daubechies construction resonates with recent developments in quantum mechanics and its non-commutative geometry, which are inspired precisely by the Landau problem, where the mass parameter is taken to cancel out. The canonical formulation of the operators should also make it possible to extend the Klauder-Daubechies construction to systems with more than two degrees of freedom. The first case of interest is then the Landau problem itself and its non-commutative geometry associated with the Moyal plane in the two possible inverse limit orders. This analysis of the Klauder-Daubechies deformation of the Landau harmonic problem reveals a non-commutativity diagram. Indeed, if we consider the limits $m \to 0$ and $\tau_0 \to 0$ as the quantum energy spectrum of the complete Landau system, including its Klauder-Daubechies deformation, we obtain the same quantum energy spectrum as that of the system obtained in the limit $m \to 0$. However, it is worth mentioning that the quantum theory of relativistic fields, with its short-range divergences in perturbation theory, provides another eloquent example. Indeed, the operator technique has been demonstrated to be a particularly suitable tool for preserving the finite value of τ_0 everywhere. This approach is arguably a physically significant choice in the context of deformations of quantum algebraic structures, and it also facilitates the identification of novel devices. In this respect, a finite τ_0 provides a new type of short-distance regularisation in local quantum field theory, with all the short-distance divergences. In the context of the latter, the status of initial cosmological singularities, where the question of black hole radiation energies remains unresolved, could be addressed in the Klauder-Daubechies context for quantum dynamics.

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