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Non-standard Lagrangians and a model of branched Hamiltonians

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Abstract
Take up some examples of multi-valued structure of the governing Hamiltonians namely, the branched Hamiltonians are explored, as recently advocated by Shapere and Wilczek.These are in fact cases of switchback potential, in the continuous interpolation of discrete time dynamical systems that exhibit chaotic behaviour enabling incorporation of a canonical quasi-Hamiltonian formalism.
***Keywords**: Branched Hamiltonian, nonstandard Lagrangian, multi-valued, quantum.*

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I. Introduction

Alongside the study of standard Hamiltonians, the role of non-conventional Hamiltonians has aroused curiosity in recent times especially in their applicability to problems of nonlinear dynamics pertaining to autonomous differential equations, in shallow water-wave context and in certain types of quantum mechanical models particularly, belonging to parity-time ($\mathcal{PT}$)-symmetric schemes [1] and in their relativistic versions [2]. The subject of branched Hamiltonians [3-6] belongs to non-conventional arena of mathematical physics which can be derived from Legendre transformed Lagrangians where velocity dependence is not convex in character and leads to Riemann surface phase-space structure and to certain interesting topological issues. The existence of branched Hamiltonians which also has relevance in the continuous interpolation of discrete time dynamical systems that exhibit chaotic behaviour enabling incorporation in canonical quasi-Hamiltonian set up.

Hamiltonians that are multivalued functions of momenta present some of the not so well understood ambiguities of quantization. Branched Hamiltonians in the classical context, and their quantization, have been recently proposed by Shapere and Wilczek [3-5] and is currently an area of active interest as evidenced in a series of papers by Curtright and Zachos [7-12] and other works [13-15]. In this context a new class of innovations of the description and simulations of quantum dynamics has emerged in connection with the possible specific role of the models of physical systems. In such cases the underlying Lagrangian possesses time derivatives in excess of quadratic powers. The use of these models leads, on both classical and quantum grounds, to the necessity of a re-evaluation of the dynamical interpretation of the momentum p which, in principle, becomes a multiple function of velocity v.

In the present work, I have searched for situations where multi-valued Hamiltonians occur that result from the enforcement of the Legendre transform on the non-standard Lagrangians in the sense that the velocity dependence is not convex. Early works by Curtright and Zachos [7-12] and by Bagchi et al. [16] pointed out that multi-valued Hamiltonians arose in the continuous interpolation of discrete time dynamical systems that invariable had an underlying chaotic behaviour. The natural framework of study would then be a quasi-Hamiltonian formalism.

II. A model of branched Hamiltonians

Recently, Shapere and Wilczek considered interesting models with non-convex Lagrangians in velocity[3-5]. To demonstrate, let us consider a simple model [3, 4],

$$L = \frac{1}{4}v^4 - \frac{\kappa}{2}v^2 \tag{1}$$

For the interesting case of $\kappa > 0$, the Lagrangian is a non-convex function of velocity. Thus, the corresponding conjugate momentum is

$$p = \frac{\partial L}{\partial v} = v^3 - \kappa v = f(v) \tag{2}$$

is not monotonic in velocity, where the function $f(v)$ stands for the Legendre map. Then, making the conventional Legendre transformation gives the corresponding Hamiltonian as a function of velocity,

$$H = \frac{3}{4}v^4 - \frac{\kappa}{2}v^2 \tag{3}$$

which is a multi-valued function (with cusps) in conjugate momentum p, since each given p corresponds to one or three values of v as shown in Eq. (2).

Hence, for systems with a non-convex Lagrangian such as (1), the construction of single-valued Hamiltonian in conjugate momentum space is challenging. Related issues also arise in cosmology models [17, 18], in extensions of Einstein gravity involving topological invariants, and in theories of higher-curvature gravity [19].

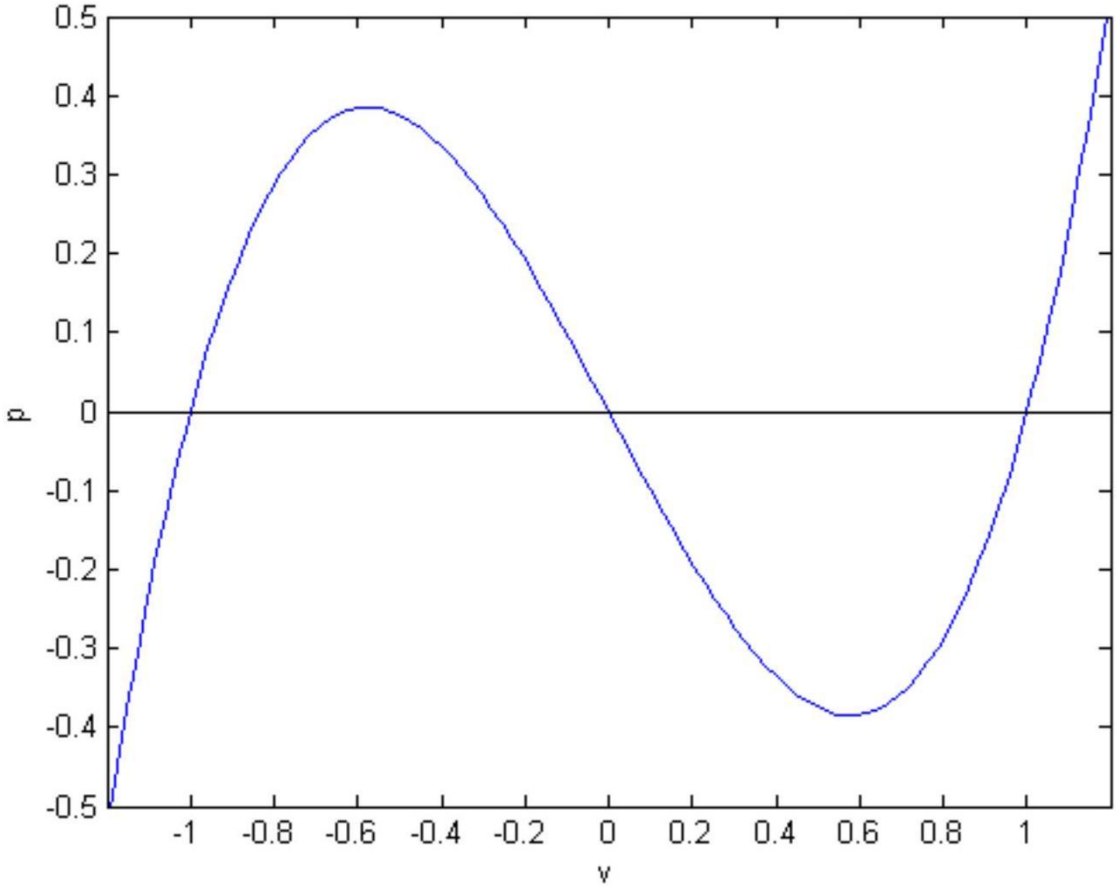

Figure 1*: The Legendre mapping function f from v to p for the non-convex Lagrangian Eq. (1) with* $\kappa > 0$.

Thus, a Legendre transformation from (x, p, H) to (x, v, L) is complicated for nonconvex $V(p)$. The resulting L is multi-valued, in general, with several branches.

Alternatively, if you start with a given single-valued $L(x, v)$, then you too will face similar complication if you are dealing with

$$L = x^2 - V(v) \tag{4}$$

instead of the usual

$$L = v^2 - V(x). \tag{5}$$

So, starting from single-valued $H(x, p)$ or starting from single-valued $L(x, v)$-either way-if the p or v dependence is non-convex then multi-valued, branched functions will arise upon Legendre transforming between Hamiltonian and Lagrangian formulations.

Several simple Lagrangian models have been considered here that lead to double valued Hamiltonian systems. Let us start with an example where the velocity dependence of L is given by a Gaussian. This example illuminates many generic features of branched Hamiltonians, in addition to its more specific peculiarities. The Gaussian model, as a quantum system cannot lead to a solution in closed form, therefore a different class of

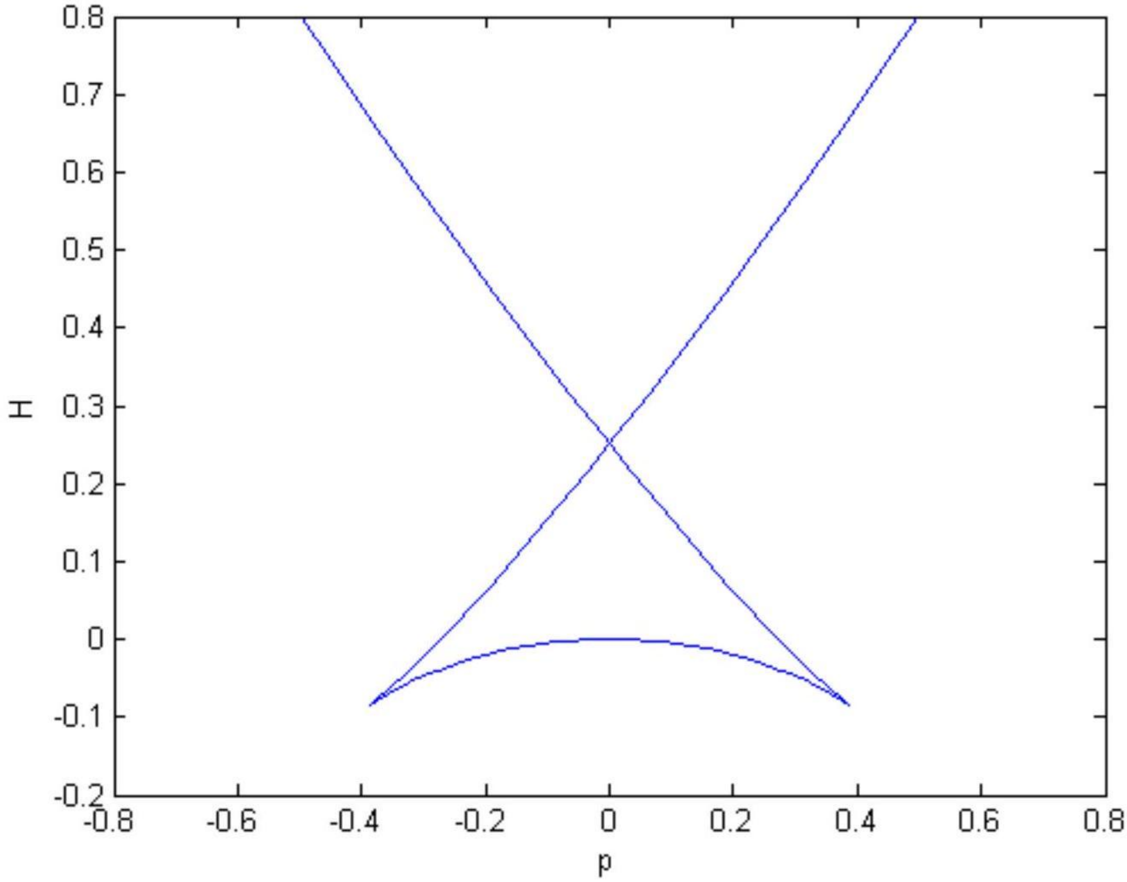

Figure 2*: The Hamiltonian Eq. (3) versus momentum which is multivalued function of p for the non-convex Lagrangian Eq. (1) with $\kappa > 0$.*

models are resorted to where analytic results can be obtained. One of this class is modified to get a pair of Hamiltonians that comprise a supersymmetric quantum mechanical system [5, 20].
Let us consider a non-convex v-dependent dimensionless Gaussian Lagrangian:

$$L(x,v) = \left(1 - \exp\left(-\frac{v^2}{2}\right)\right) - V(x) \tag{6}$$

which allows the momentum to be determined by the relation

$$p(v) = \frac{\partial L}{\partial v} = v \exp\left(-\frac{v^2}{2}\right) \tag{7}$$

Here we note that L is a union of three convex functions defined on the three v intervals $(-\infty,-1], [-1,1]$ and $[1,\infty)$. The kinetic energy of the model takes the classic shape of a fedora hat profile, when it is plotted against v (see Figure 3).
For this model, v and p always have the same sign, and clearly $-\infty \leq v \leq +\infty$. However, due to the Gaussian suppression in v, the momentum p is confined to a finite interval as given by the maximum and minimum of (7), namely, $p(v)|_{v=\pm 1} = \pm\frac{1}{\sqrt{e}}$

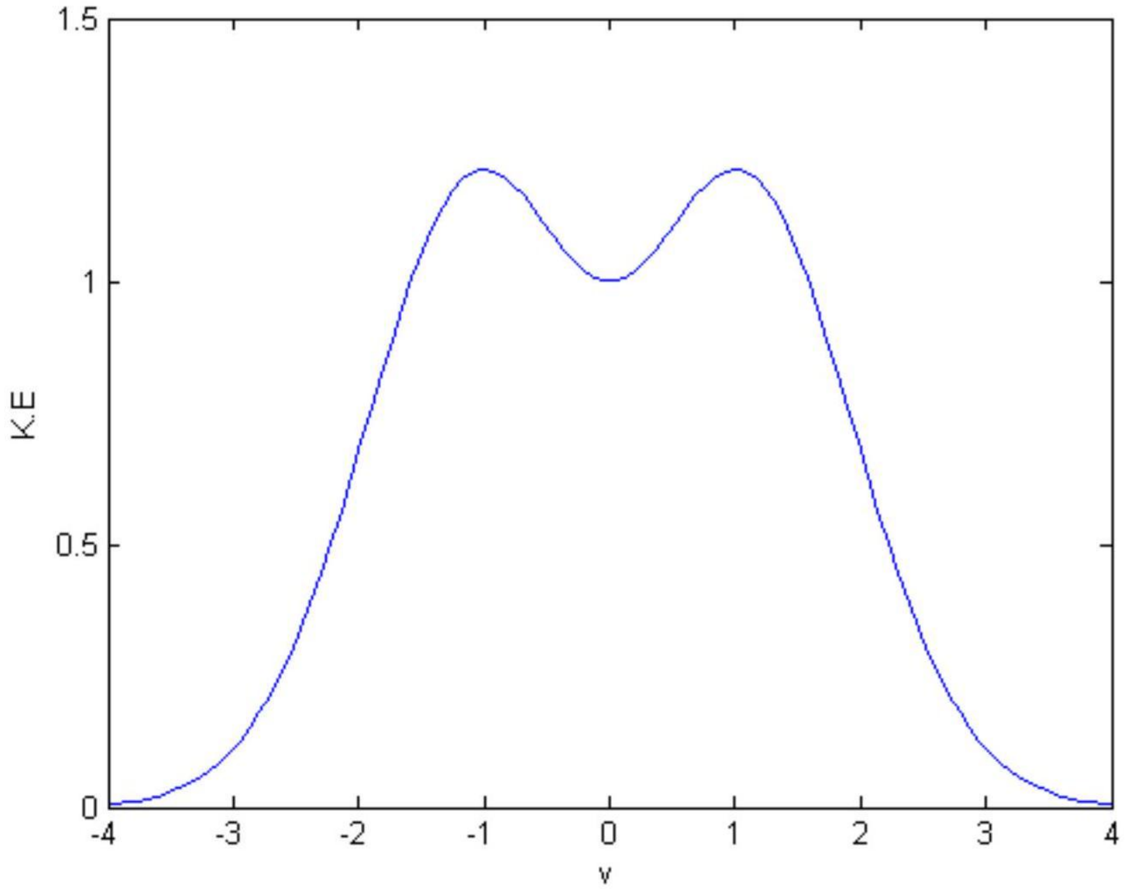

Figure 3: *Kinetic energy*, $(1 + v^2)exp\left(\frac{-v^2}{2}\right)$, *versus* v *for the Gaussian model.*

Here we get two values for H at every value of $p \in \left(-\frac{1}{\sqrt{e}}, \frac{1}{\sqrt{e}}\right)$. To get this double-valued H, we invert (7), and obtain

$$v(p) = \pm\sqrt{-\text{Lambert}W(-p^2)} \tag{8}$$

For real v it is required to return negative values for the Lambert function, with negative argument, so either the principal branch Lambert$W(0, z)$ or the lower branch Lambert$W(-1, z)$ will do, with $-\frac{1}{e} \leq z \leq 0$ i.e $-\frac{1}{e} \leq -p^2 \leq 0$. That is to say, the momentum lies in the finite interval $-\frac{1}{\sqrt{e}} \leq p \leq \frac{1}{\sqrt{e}}$. So $v(p)$ is multivalued, because of the square root and the Lambert function branches.

The solution of

$$ye^y = z \tag{9}$$

is

$$y(z) = \begin{cases} \text{Lambert } W(k, z) & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases} \tag{10}$$

where $k \in \mathbb{Z}$ and $k = 0, k = -1$ give the two real branches as shown in Figure 4

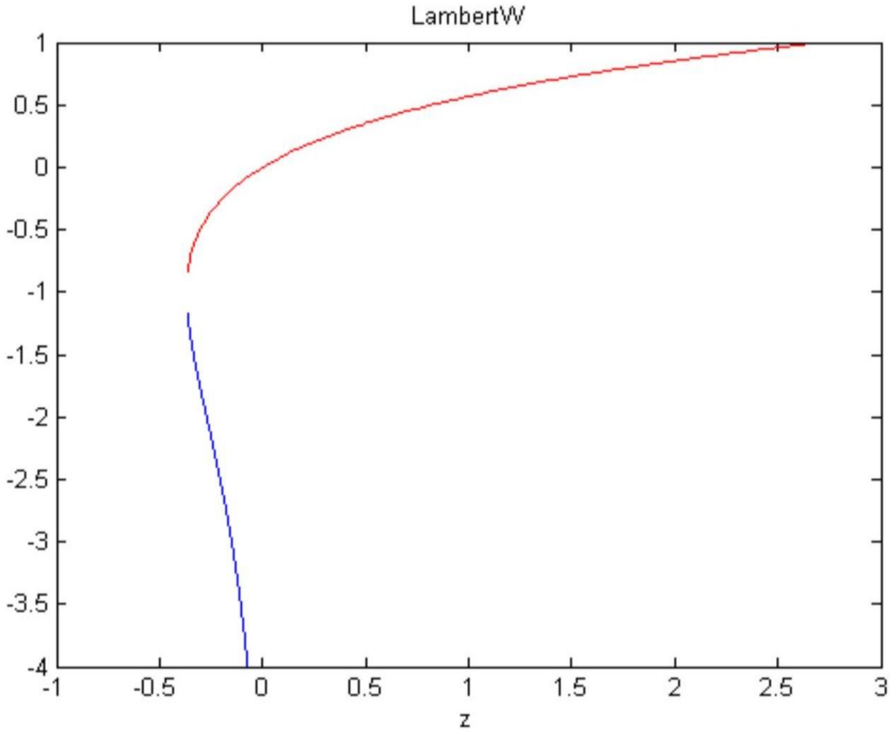

Figure 4: *Lambert* $W(0, z)$ *and Lambert* $W(-1, z)$ *in red and blue, respectively.*

The velocity dependent term is a union of three convex functions defined on $v \in (-\infty, -1], [-1,1]$ and $[1, \infty)$, as shown below in red, blue, and green respectively (see Figure 5).

Therefore, the Hamiltonian will be multi-valued.

$$H(x, v) = pv - L(x, v) \tag{11}$$

Using (7), so as a function of v

$$H(x, v) = (1 + v^2)\exp\left(\frac{-v^2}{2}\right) + V(x) - 1 \tag{12}$$

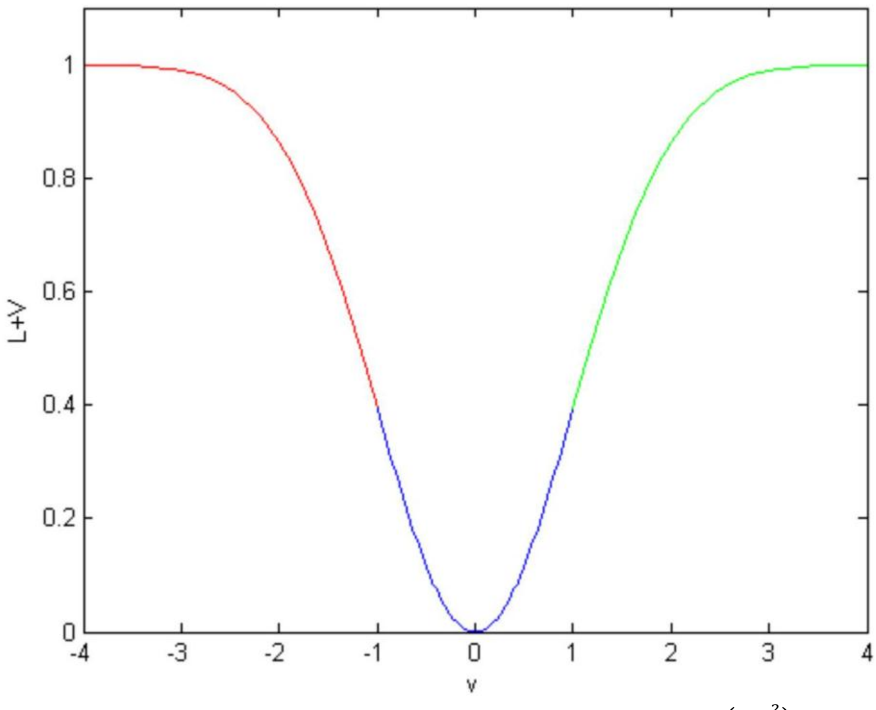

Figure 5: *The model given by* $L(x, v) + V(x) = 1 - exp\left(-\frac{v^2}{2}\right)$.

But we want $H(x,p)$. So, we need $v(p)$. Using (8), the result for $H(x,p)$ is multivalued on this momentum interval, because of the square root and the Lambert function branches (see Figure 6)

$$H = \pm p\left(\sqrt{-\text{Lambert}W(-p^2)} + \frac{1}{\sqrt{-\text{Lambert}W(-p^2)}}\right) + V(x) - 1 \qquad (13)$$

Both the square root and Lambert function have two real branches. Therefore, four values of H may be obtained at any given momentum. However, the square root and Lambert W branches are always correlated. Therefore, if we consider the $(p(v), H(x,p(v)))$ curve in parametric form on the (p,H) plane, using v as the parameter, so that the Gaussian model's Hamiltonian is only the double-valued for all $p \in (-1/\sqrt{e}, 1/\sqrt{e})$.

So $H(x,p)$ may be considered as the union of three convex functions of p: H_-, H_0 and $H_+$for $p \in [-1/\sqrt{e}, 0], [-1/\sqrt{e}, 1/\sqrt{e}]$ and $[0,/1\sqrt{e}]$, as shown below in red, black, and green, respectively. Classically, a particle switches H during its trajectory. Different branches of H control the motion at different times. Therefore,

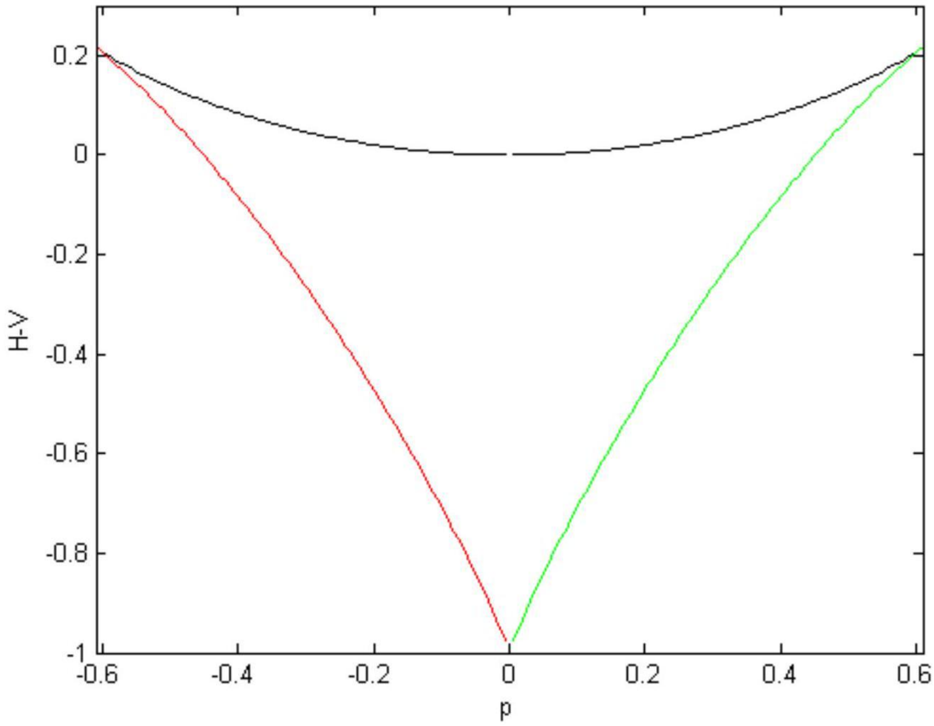

Figure 6: *The real branches oh* $H - V$ *versus* $p \in [-1/\sqrt{e}, 1/\sqrt{e}]$.

when a particle, governed by one branch of H moves on a trajectory, a classical particle generally encounters one of the Hamiltonian cusps in finite time, and then switches to be governed by another branch of H. This switching, leads trajectories to intersect and cross in the figure. This not possible for a system controlled by singlevalued Hamiltonian, as is well-known. But it is possible when different Hamiltonian branches govern the motion for the different curves that cross. A system governed by a multi-valued Hamiltonian usually does exhibit this noval feature. We have called such trajectories "quasi-Hamiltonian" flows.

III. Summary

To summarize, here I have displayed some simple non-standard Lagrangians models which, by virtue of non-convexity in their velocity dependence and showed the Hamiltonian suited for it has a non-conventional double-valued structure due to the presence of a velocity-dependent potential. So double-valued $H(x,p)$ may be considered as the union of three convex functions of p: H_-, H_0 and H_+ for $p \in [-1/\sqrt{e}, 0], [-1/\sqrt{e}, 1/\sqrt{e}]$ and $[0,/1\sqrt{e}]$, as showed in a previous Figure in red, black and green, respectively.

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