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The Riemann Curvature Tensor As A Function Of The Electromagnetic Field

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Abstract:
The torsion tensor field Ω^i_{jk} determines the Riemann curvature tensor R^i_{jkl} and consecuently as a function of the Electromagnetic Field.
***Key Word**: Christoffel Symbols, Cartan Tensor, Riemann tensor, Transporter, Covariant Derivative.*

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I. Introduction

The affine connection or parallel transporter $L^i_{jk} = \Gamma^i_{jk} + \Omega^i{}_{jk}$, where Γ^i_{jk} are the Christoffel Symbols and Ω^i_{jk} is the Cartan Tensor, allows us to construct the covariant derivative of a contravariant parallel vector field A^i and a covariant paralell vector field A_i as:

$$A^i_{|j} = \frac{\partial A^i}{\partial x^j} + L^i_{jk} A^k = 0 \qquad \text{(1.1)}$$

$$A_{i|j} = \frac{\partial A_i}{\partial x^j} - L^k_{ij} A_k = 0 \qquad \text{(1.2)}$$

Where the solid followed by an index means Covariant derivative with respect to the affine connection.
For a parallel covariant four-velocity u_i, we have:

$$u_{i|j} = \frac{\partial u_i}{\partial x^j} - L^k_{ij} u_k = 0 \qquad \text{(1.3)}$$

If $x^i = x^i(s)$ and $u_i = u_i(x)$

$$\frac{du_i}{ds} = \frac{\partial u_i}{\partial x^j}\frac{dx^j}{ds} = \frac{\partial u_i}{\partial x^j} u^j \qquad \text{(1.4)}$$

Multiplying (1.3) by u^j, we obtain:

$$\frac{\partial u_i}{\partial x^j} u^j - L^i_{jk} u_k u^j = 0 \qquad \text{(1.5)}$$

Or, using the affine connection:

$$\frac{du_i}{ds} = \Gamma^k_{ij} u_k u^j + \Omega^k_{ij} u_k u^j \qquad \text{(1.6)}$$

It is well known that the equation of a charged particle in the presence of both Gravitational and Electromagnetic Fields is a generalization of the H. A. Lorentz equation of the Electrodynamics to Curvilinear coordinates [1]:

$$\frac{du_i}{ds} = \frac{e}{mc^2} f_{ij} u^j + \Gamma^k_{ij} u_k u^j \qquad \text{(1.7)}$$

Comparing (1.6) with (1.7), we have:

$$u_k \Omega^k_{ij} = \frac{e}{mc^2} f_{ij} \qquad \text{(1.8)}$$

Where f_{ij} is the Electromagnetic field.
Cartan Torsion Tensor is: [2], [3]

$$\Omega^k_{ij} = -\frac{e}{mc^2} u^k f_{ij} \qquad \text{(1.9)}$$

This connection between the torsion tensor and the Electromagnetic field is very important in the construction of Maxwell´s Equations and Riemann Curvature Tensor.

In Section II we find Riemann curvature Tensor as a function of the electromagnetic field. In section III we construct Maxwell´s equations. In section IV we give Somme conclusions.

II. The Riemann Curvature Tensor

The condition of integrability of the equation:

$$\frac{\partial u_i}{\partial x^j} - L^k_{ij}\, u_k = 0 \qquad \text{(2.1)}$$

Is:

$$u_i\, L^i_{jkl} = 0 \qquad \text{(2.2)}$$

Or simply as:

$$L^i_{jkl} = 0 \qquad \text{(2.3)}$$

Where:

$$L^i_{jkl} = \frac{\partial L^i_{jl}}{\partial x^k} - \frac{\partial L^i_{jk}}{\partial x^j} + L^i_{hk} L^h_{jl} - L^i_{hl}\, L^h_{jk} \qquad \text{(2.4)}$$

If we substitute $L^i_{jk} = \Gamma^i_{jk} + \Omega^i_{jk}$ in (2.4), we have:

$$L^i_{jkl} = R^i_{jkl} + \Omega^i_{jkl} \qquad \text{(2.5)}$$

Where:

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^i_{hk}\ \Gamma^h_{jl} - \Gamma^i_{hl}\ \Gamma^h_{jk} \qquad \text{(2.6)}$$

is the Riemann tensor and the Cartan tensor [4] is:

$$\Omega^i_{jkl} = \Omega^i_{jl\,|\,k} - \Omega^i_{jk\,|\,l} - \Omega^i_{hk}\Omega^h_{jl} + \Omega^i_{hl}\Omega^h_{jk} - 2\Omega^h_{kl}\Omega^i_{jh} \qquad \text{(2.7)}$$

Where the solid means covariant derivative whit respect to the affine connection.

Using Einstein´s condition [5]:

$$\Omega^h_{jk}\Omega^i_{lh} + \Omega^h_{kl}\Omega^i_{jh} + \Omega^h_{lj}\Omega^i_{kh} = 0 \qquad \text{(2.8)}$$

in (2.7), we have:

$$\Omega^i_{jkl} = \Omega^i_{jl\,|\,k} - \Omega^i_{jk\,|\,l} - \Omega^h_{kl}\,\Omega^i_{jh} \qquad \text{(2.9)}$$

Now, parallelism condition implies:

$$L^i_{jkl} = 0 \qquad \text{(2.10)}$$

Or:

$$R^i_{jkl} + \Omega^i_{jkl} = 0 \qquad \text{(2.11)}$$

And from the beautiful property of the Riemann tensor:

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0 \qquad \text{(2.12)}$$

We have:

$$\Omega^i_{jkl} + \Omega^i_{klj} + \Omega^i_{ljk} = 0 \qquad \text{(2.13)}$$

And from (2.9) and Einstein condition (2.8), we have:

$$\Omega^i_{jl\,|\,k} + \Omega^i_{lk\,|\,j} + \Omega^i_{kj\,|\,l} = 0 \qquad \text{(2.14)}$$

From (2.9) and (2.14):

$$\Omega^i_{jkl} = \Omega^i_{kl\,|\,j} - \Omega^h_{kl}\,\Omega^i_{jh} \qquad \text{(2.15)}$$

And from (2.11), the Riemann tensor is:

$$R^i_{jkl} = \Omega^i_{lk\,|\,j} + \Omega^h_{kl}\Omega^i_{jh} \qquad \text{(2.16)}$$

This expression shows that the Riemann tensor is determined by the Cartan torsion tensor.

Now, from Bianchi identity:

$$R^i_{jkl;n} + R^i_{jln;k} + R^i_{jnk;l} = 0 \qquad \text{(2.17)}$$

Or, Because $g_{ij;k} = 0$:

$$R_{ijkl;n} + R_{ijln;k} + R_{ijnk;l} = 0 \qquad \text{(2.18)}$$

Multiplication by $g^{il}\, g^{jk}$ and taking in account the anti-symmetry of the Riemann tensor, we have:

$$\left(R^k_n - \tfrac{1}{2} g^k_n\, R\right)_{;k} = 0 \qquad \text{(2.19)}$$

The Tensor $E^i_j = R^i_j - \frac{1}{2} g^i_j R$ is the Einstein Tensor.

Now, the contraction of the indices i and l in (2.16) gives Ricci Tensor:

$$R_{jk} = \Omega_{k\,|\,j} + \Omega^h_{kn}\Omega^n_{jh} \qquad \text{(2.20)}$$

Where: $\Omega^n_{nk} = \Omega_k$.

If:

$$\Omega_{k\,|\,j} = \Omega^h_{nk}\,\Omega^n_{jh} \qquad \text{(2.21)}$$

$$R_{jk} = 0 \qquad \text{(2.22)}$$

And we have Einstein´s equations of the Gravitational Field in empty space without any masses.

These equations were solved by K. Schwarzschild and then used by Mr. Nassim Haramein on his very interesting article ¨The Schwarzschild Proton¨.

Substitution of (1.9) in (2.16) gives:

$$R^i_{jkl} = \frac{e}{mc^2} u^i f_{kl|j} + (\frac{e}{mc^2})^2 u^h u^i f_{jh} f_{kl} \quad \text{.... (2.23)}$$

And we find this very important relation between Riemann Curvature Tensor and the electromagnetic Field.

III. Maxwell´S Equations

From (2.14), (2.8) and (1.8), we have:

$$\frac{\partial f_{jl}}{\partial x^k} + \frac{\partial f_{lk}}{\partial x^j} + \frac{\partial f_{kj}}{\partial x^l} = 0 \quad \text{.... (3.1)}$$

Which correspond to

$$\nabla \cdot \mathbb{H} = 0 \ , \ \nabla \times E = -\frac{1}{c}\frac{\partial \mathbb{H}}{\partial t} \quad \text{.... (3.2)}$$

The second pair of Maxwell´s equations follow from (2.16):

$$\Omega^i_{lk|j} = R^i_{jkl} - \Omega^h_{kl}\Omega^i_{jh} \quad \text{.... (3.3)}$$

Multiplication by u_i gives:

$$\frac{e}{mc^2} f_{lk|j} = u_i R^i_{jkl} - \frac{e}{mc^2}\Omega^h_{kl} f_{jh} \quad \text{.... (3.4)}$$

Developing the covariant derivative whit respect to the affine connection and making l equal to j, we have:

$$\frac{\partial f_{kj}}{\partial x^j} = \Gamma^h_{kj} f_{hj} + \Gamma^i_{jj} f_{kh} - \frac{mc^2}{e} u_i R^i_{jkj} \quad \text{.... (3.5)}$$

These equations reduce to:

$$\frac{\partial f_{kj}}{\partial x^j} = 0 \ \text{ if } \Gamma^i_{jk} = 0$$

Which correspond to:

$$\nabla \cdot E = 0 \ ; \ \nabla \times \mathbb{H} = \frac{1}{c}\frac{\partial E}{\partial t} \quad \text{.... (3.6)}$$

Now, the meaning of Einstein condition follows from (2.8).

Multiplication by u_i gives:

$$\Omega^h_{jk} f_{lh} + \Omega^h_{kl} f_{jh} + \Omega^h_{lj} f_{kh} = 0 \quad \text{.... (3.7)}$$

And from (1.9):

$$u^h\{ f_{jk} f_{lh} + f_{kl} f_{jh} + f_{lj} f_{kh}\} = 0 \quad \text{.... (3.8)}$$

Or:

$$f_{jk} f_{lh} + f_{kl} f_{jh} + f_{lj} f_{kh} = 0 \quad \text{.... (3.9)}$$

Let us make j=2, k=3, l=1, h=4 in (3.9) we have:

$$f_{23} f_{14} + f_{31} f_{24} + f_{12}\, f_{34} = 0$$

$$H_x\,(-iE_x) + H_y(-iE_y) + H_z(-iE_z) = 0$$

$$E \cdot \mathbb{H} = 0 \quad \text{.... (3.10)}$$

Einstein´s condition is nothing other than the perpendicularity of the Electric and Magnetic fields.

IV. Conclusions

1. Equation (2.16) shows that Riemann tensor is a function of Cartan´s torsion tensor.
2. Equation (2.23) shows that Gravitation is generated by the Electromagnetic Field.
3. Einstein condition, which we use in this paper repeatedly has a very simple interpretation, it says that the electromagnetic fields are perpendicular.
4. The second pair of Maxwell equations are modified in presence of the gravitational field.
5. It is interesting that in this context we can recover Einstein’s famous field equations (2.22).

References

[1]. L.D. Landau And E.M. Lifshitz; The Classica Theory Of Fields; Addison -Wesley Publishing Company, Inc.
[2]. Alcantara Montes Samuel And Velazquez Arcos Juan Manuel; Iosr Journal Of Applied Physics Vol. 13 (May -June) Pp39-41, 2021.
[3]. S. Capozziello And C. Stornaiolo; Annales De La Foundation De Broglie, Vol.32 ,2-3, 2007
[4]. L.F. Eisenhart; Continuous Group Of Transformation. Dover Publication Inc. Ny.
[5]. L.F. Eisenhart; American Mathematical Society Colloquium Publication Vol. Viii 1929, Ii, P,5.